The topological degree and fixed points of DC-mappings

by

W. Kryszewski and **B. Przeradzki** (Łódź)

Abstract. In this paper the class of DC-mapping defined by B. Nowak [11] is examined. For this class, a definition of degree and its properties are given, without assuming the regularity of filtration. Several approximative fixed point theorems are proved as well.

Introduction. This paper is devoted to further investigations of a new class of nonlinear operators — the so-called DC-mappings — introduced in 1981 by B. Nowak in his dissertation [11]. These maps, defined on normed spaces endowed with filtrations by finite-dimensional linear subspaces, possess properties which allow us to obtain some results concerning the existence of approximate solutions of certain equations.

The class of DC-mappings includes the family of Leray–Schauder operators (i.e., maps of the form "the identity + compact mapping"). Unlike Leray–Schauder operators, form a module over the ring of continuous and bounded scalar functions.

Below, we develop a theory of the degree of DC-mappings which generalizes the Leray–Schauder degree [8]. We omit Nowak's assumption of the regularity of filtration by using a certain approximation lemma.

Attempts to extend the Leray–Schauder theory have been made by many authors, see a survey by W. V. Petryshyn [13] on A-proper mappings and an article by R. D. Nussbaum [12] on condensing maps. These extensions, however, concern proper maps (continuous A-proper maps on bounded domains are proper [13], 1.1 C), while there are examples of DC-mappings which are not proper (see [11]). Recently, one of the present authors has shown that a DC-mapping is A-proper iff it is proper (see [6]). Moreover, there are continuous A-proper maps which are not DC-mappings.

Finally, we should mention that DJ-mappings (i.e., uniformly continuous DC-maps) were examined in [14] and multivalued mappings of this type were studied in [7].

1. Preliminaries. For a subset A of a topological space, we denote its closure by $\bar{A}$ or by $\mathrm{cl}A$, and its boundary by $\mathrm{Fr}A$.

By a metric space with filtration we shall mean a pair $(X, (X_n)_{n=1}^{\infty})$ composed of a metric space X and an increasing sequence $(X_n)_{n=1}^{\infty}$ of its closed subsets such that

$$\operatorname{cl} \bigcup_{n=1}^{\infty} X_n = X. \tag{1.1}$$

Let $(X, (X_n)_{n=1}^{\infty})$ and $(Y, (Y_n)_{n=1}^{\infty})$ be two metric spaces with filtrations. A mapping $f\colon X \to Y$ for which there exists a positive integer n_0 such that

$$f(X_n) \subset Y_n \tag{1.2}$$

for $n \geqslant n_0$ is said to be an F-mapping. If, instead of (1.2)

$$\lim_{n\to\infty} \sup_{x\in X_n} d_Y(f(x), Y_n) = 0, \tag{1.3}$$

then f is said to be a D-mapping.

Obviously, any F-mapping is also a D-mapping.

A continuous mapping satisfying (1.2) or (1.3) will be called an FC-mapping or a DC-mapping, respectively.

Let E be a normed space, and let $(E_n)_{n=1}^{\infty}$ be its filtration consisting of finite-dimensional linear subspaces of E (in the sequel, when speaking about a normed space, we shall consider only such filtrations). Let Ω be an arbitrary open bounded subset of E. It is easy to verify that $(\overline{\Omega}, (\overline{\Omega}_n)_{n=1}^{\infty})$ and $(\operatorname{Fr}\Omega, (\operatorname{Fr}\Omega_n)_{n=1}^{\infty})$, where $\Omega_n = \Omega \cap E_n$, are metric spaces with filtrations (comp. Lemma 4.1, [7]).

After Nowak we quote

Proposition 1.1. (i) *The space of all* DC-*mappings defined on a metric space X with filtration, taking values in a normed space E with filtration, is a module over the ring of continuous and bounded scalar functions on X and is closed in the set of all continuous maps $X \to E$ (uniform convergence topology).*

(ii) *Let X, Y, Z be metric spaces with filtrations and let $f\colon X \to Y$, $g\colon Y \to Z$ be* DC-*maps. If g is uniformly continuous, then $g\circ f$ is a* DC-*mapping.*

Proposition 1.2. *Let Ω be an open bounded subset of a normed space E with filtration, and let $f\colon \overline{\Omega} \to E$ be compact. Then f and $I-f$ are* DC-*mappings (I stands for the identity map).*

2. FC-approximation lemma. The following theorem will be basic for the definition of the degree of a DC-mapping in the next section.

Theorem 2.1. *Let $(X, (X_n))$ be a metric space and $(E, (E_n))$ — a normed space with filtrations. For an arbitrary $\varepsilon > 0$ and any* DC-*mapping $f\colon X \to E$, there exists an* FC-*mapping $f_\varepsilon\colon X \to E$ such that*

$$\|f(y) - f_\varepsilon(y)\| < \varepsilon \tag{2.1}$$

for all $y \in X$.

Proof. For any $x \in X$, we put $n(x) = \inf\{n\colon B(f(x), \varepsilon) \cap E_n \neq \emptyset\}$ and take $a(x) \in B(f(x), \varepsilon) \cap E_{n(x)}$ where $B(y, r) = \{z \in E\colon \|z-y\| < r\}$. Denote by

$$U(x) = f^{-1}\big(B(f(x), \varepsilon/2) \cap B(a(x), \varepsilon)\big)$$

a neighbourhood of x and consider a locally finite partition of unity $\{\lambda_x\colon x \in X\}$ inscribed into the open cover $\{U(x)\colon x \in X\}$ of X. We define a mapping $f_\varepsilon\colon X \to E$ by the formula

$$f_\varepsilon(y) = \sum_{x\in X} \lambda_x(y)\cdot a(x), \quad y \in X. \tag{2.2}$$

The continuity of f_ε is obvious. We shall show that f_ε is an FC-mapping. Indeed, $\lambda_x(y) \neq 0$ implies that

$$\|f(y) - f(x)\| < \varepsilon/2, \tag{2.3}$$

$$\|f(y) - a(x)\| < \varepsilon. \tag{2.4}$$

By the definition of DC-mapping, there exists an n_0 such that, for $n \geqslant n_0$ and $y \in X_n$,

$$B(f(y), \varepsilon/2) \cap E_n \neq \emptyset. \tag{2.5}$$

If, for such n and y, $\lambda_x(y) \neq 0$, then, by (2.3), $B(f(x), \varepsilon) \cap E_n \neq \emptyset$. Hence $n \geqslant n(x)$ and $a(x) \in E_n$. So $f_\varepsilon(X_n) \subset E_n$ for $n \geqslant n_0$.

Inequality (2.1) is a simple consequence of (2.4). ■

One can give a proof of this theorem, based on Michael's selection theorem [9], under the assumption of the completeness of E. Then the FC-approximation lemma and the degree theory can be generalized to the case of a complete linear metric locally convex space.

3. The topological degree of DC-mapping. We are now ready to introduce the notion of a topological degree for the class of DC-mappings defined on a bounded domain in a normed space with filtration. The resulting degree will have all standard properties and, in some sense, will constitute a generalization of the well-known Leray–Schauder degree of compact vector fields.

Let $(E, (E_n)_{n=1}^{\infty})$ and $(F, (F_n)_{n=1}^{\infty})$ be two normed spaces with filtrations, such that

$$\dim E_n = \dim F_n$$

for $n = 1, 2, \ldots$ On each E_n and F_n we fix certain orientations. Let Ω be an open bounded subset of E, let $f\colon \overline{\Omega} \to F$ be a DC-mapping and let $y \in F\setminus f(\operatorname{Fr}\Omega)$. For $\varepsilon = d_F(y, f(\operatorname{Fr}\Omega)) > 0$, choose an FC-mapping g which is an $\varepsilon/2$-approximation of f, i.e., $\|f(x) - g(x)\| < \varepsilon/2$ for all $x \in \overline{\Omega}$ (see Th. 2.1) and a positive integer n_0 such that

$$g(\overline{\Omega}_n) \subset F_n \tag{3.1}$$

and

$$d_F(y, F_n) < \varepsilon/2 \tag{3.2}$$

for all $n \geq n_0$ ($\Omega_n = \Omega \cap E_n$). According to (3.2), there exists a sequence $(y_n)_{n=n_0}^{\infty}$ such that $y_n \in F_n$ and $\|y_n - y\| < \varepsilon/2$. Then, it is easily seen that

$$d_F(y_n, g(\mathrm{Fr}\,\Omega_n)) > 0 .$$

Having all these, we are able to define an integer

$$s_n = \deg(\Phi_n \circ (g|\overline{\Omega}_n), \Omega_n, \Phi_n(y_n)) \tag{3.3}$$

for $n \geq n_0$, where $\Phi_n\colon F_n \to E_n$ is an arbitrary linear isomorphism which preserves orientations (the symbol deg denotes here the Brouwer degree in finite-dimensional spaces, see [9], def. 1.3.2). In addition, put $s_n = 0$ for $n < n_0$.

Introducing a group $\mathscr{G} = \prod_{n=1}^{\infty} Z / \bigoplus_{n=1}^{\infty} Z$ and the canonical homomorphism $\nu\colon \prod_{n=1}^{\infty} Z \to \mathscr{G}$, we define the topological degree of the DC-mapping f on Ω at the point y by the formula

$$\mathrm{Deg}(f, \Omega, y) = \nu\big((s_n)_{n=1}^{\infty}\big) . \tag{3.4}$$

The degree is well defined, for it does not depend on the choice of the mapping g, the sequence $(y_n)_{n=n_0}^{\infty}$ and the family of linear orientation-preserving isomorphism $\Phi_n\colon F_n \to E_n$. Indeed, consider another FC-mapping g' such that $g'(\overline{\Omega}_n) \subset F_n$ for $n \geq n_0'$ and $\|f(x) - g'(x)\| < \varepsilon/2$ for all $x \in \overline{\Omega}$, and another sequence $(y_n')_{n=n_0'}^{\infty}$, satisfying the following condition $\|y_n' - y\| < \varepsilon/2$ for $n \geq n_0'$. Define the family $g_t\colon \overline{\Omega} \to F$, $t \in \langle 0, 1\rangle$, by the formula $g_t(x) = tg(x) + (1-t)g'(x)$ and, for $n \geq \max(n_0, n_0')$, the sequence of functions $y_n\colon \langle 0, 1\rangle \to F_n$ — by the formula $y_n(t) = ty_n + (1-t)y_n'$. Then, for any such n, $y_n(t) \notin g_t(\mathrm{Fr}\,\Omega_n)$. Therefore, by Proposition 1.4.3, [10], $\deg(\Phi_n \circ (g_t|\overline{\Omega}_n), \Omega_n, \Phi_n(y_n(t)))$ does not depend on $t \in \langle 0, 1\rangle$. The independence of the choice of Φ_n is self-evident.

Remark. Assume that $E = F$ and $E_n = F_n$. Then, in (3.3) Φ_n can be omitted. Nowak [11] considered the case where the filtration (E_n) is regular, i.e., there are linear projections $P_n\colon E \to E_n$ with $\sup\{\|P_n\|\colon n \in N\} < \infty$; he then defined a degree alternatively as

$$\nu\big((\deg(P_n f|\overline{\Omega}_n, \Omega_n, P_n y))_{n=1}^{\infty}\big) \in \mathscr{G} .$$

Applying a similar technique, one can prove that in this case the two definitions are equivalent.

Notice that Nowak's definition is quite similar to that of Browder and Petryshyn [3] for A-proper mappings. If $f\colon \overline{\Omega} \to E$ is A-proper map, they defined $D(f, \Omega, y)$ as the set of limit points of (s_n) where $s_n = \deg(P_n f|\overline{\Omega}_n, \Omega_n, P_n y)$. Nowak's degree gives more information about the homotopy class of f than D does, for there may exist sequences (s_n) and (s_n') with $\nu((s_n)) \neq \nu((s_n'))$ but with the same set of limiting points.

4. Properties of the degree. The degree introduced above has similar properties to those of Brouwer and Leray–Schauder.

Let E, F, E_n, F_n, Ω satisfy the assumptions of Section 3.

PROPOSITION 4.1. (i) (Homotopy invariance). *If $h\colon \overline{\Omega} \times \langle 0, 1\rangle \to F$ is a DC-mapping relative to the filtration $(\overline{\Omega}_n \times \langle 0, 1\rangle)_{n=1}^{\infty}$ in $\overline{\Omega} \times \langle 0, 1\rangle$, and $y \notin \mathrm{cl}\big(h(\mathrm{Fr}\,\Omega \times \langle 0, 1\rangle)\big)$, then*

$$\mathrm{Deg}(h(\cdot, 0), \Omega, y) = \mathrm{Deg}(h(\cdot, 1), \Omega, y) .$$

(ii) *$\mathrm{Deg}(f, \Omega, y)$ is uniquely determined by the values of the DC-mapping $f\colon \overline{\Omega} \to F$ on $\mathrm{Fr}\,\Omega$.*

(iii) *If $f\colon \overline{\Omega} \to F$ is a DC-mapping and y, y' belong to the same component of the set $F \setminus \overline{f(\mathrm{Fr}\,\Omega)}$, then*

$$\mathrm{Deg}(f, \Omega, y) = \mathrm{Deg}(f, \Omega, y') .$$

(iv) *If $y \in \Omega$, then $\mathrm{Deg}(I, \Omega, y) = \mathbf{1}$ where I denotes the identity mapping and $\mathbf{1} = \nu((s_n))$ with $s_n = 1$ for $n = 1, 2, \ldots$*

Proof. (i) Let $\varepsilon = d_F(y, h(\mathrm{Fr}\,\Omega \times \langle 0, 1\rangle))$, and let g be an $\varepsilon/2$-FC-approximation of h. Choose a positive integer n_0 and a sequence $(y_n)_{n=n_0}^{\infty}$ such that $g(\overline{\Omega}_n \times \langle 0, 1\rangle) \subset F_n$ and $\|y_n - y\| < \varepsilon/2$ for $n \geq n_0$. Then

$$\deg(\Phi_n \circ g(\cdot, 0)|\ \overline{\Omega}_n, \Omega_n, \Phi_n(y_n)) = \deg(\Phi_n \circ g(\cdot, 1)|\ \overline{\Omega}_n, \Omega_n, \Phi_n(y_n))$$

for such n. This ends the proof if we notice that $g(\cdot, 0)$ and $g(\cdot, 1)$ are FC-approximations of $h(\cdot, 0)$ and $h(\cdot, 1)$, respectively.

(ii) is an immediate consequence of (i). The proof of (iii) is similar to that of Proposition 5.1, [7]. (iv) is an easy consequence of the definition. ■

PROPOSITION 4.2. *If $f\colon \overline{\Omega} \to F$ is a DC-mapping and $\mathrm{Deg}(f, \Omega, y) \neq 0 \in \mathscr{G}$, then $y \in \overline{f(\Omega)}$.*

Proof. Suppose to the contrary that $y \notin \overline{f(\Omega)}$. For $\varepsilon = d_F(y, f(\Omega)) \leq d_F(y, f(\mathrm{Fr}\,\Omega))$, as in the definition, we obtain an appropriate FC-approximation g as well as a suitable sequence (y_n) such that $y_n \notin g(\overline{\Omega}_n)$ for sufficiently large n. Hence $\deg(\Phi_n \circ (g|\overline{\Omega}_n), \Omega_n, \Phi_n(y_n)) = 0$. ■

Let us consider a double sequence $(s_n^i)_{n,i=1}^{\infty}$ satisfying the following condition:

(A) for any i, $(s_n^i)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} Z$ and, for any n, $(s_n^i)_{i=1}^{\infty} \in \bigoplus_{i=1}^{\infty} Z$. In this case we put $\sum_{i=1}^{\infty} \nu((s_n^i)_{n=1}^{\infty}) = \nu((\sum_{i=1}^{\infty} s_n^i)_{n=1}^{\infty})$.

Proposition 4.3. *Let Ω^i, $i = 1, 2, \ldots$, be a sequence of open disjoint subsets of Ω, and let $f\colon \overline{\Omega} \to F$ be a DC-mapping. If $y \notin \mathrm{cl}\, f(\overline{\Omega} \setminus \bigcup_{i=1}^{\infty} \Omega^i)$, then*

$$\mathrm{Deg}(f, \Omega, y) = \sum_{i=1}^{\infty} \mathrm{Deg}(f, \Omega^i, y) .$$

Proof. Let $\varepsilon = d_F(y, f(\overline{\Omega} \setminus \bigcup_{i=1}^{\infty} \Omega^i))$. Taking an appropriate FC-approximation g of f and a sequence (y_n), we get $y_n \notin g(\overline{\Omega}_n \setminus \bigcup_{i=1}^{\infty} \Omega_n^i)$. The double sequence

$s_n^i = \deg(\Phi_n \circ (g|\overline{\Omega}_n^i), \Omega_n^i, \Phi_n(y_n))$ satisfies condition (A) and, by Proposition 1.4.4, [10], $\deg(\Phi_n \circ (g|\overline{\Omega}_n, \Omega_n, \Phi_n(y_n)) = \sum_{i=1}^{\infty} s_n^i$ for sufficiently large n. ■

Corollary 4.4. *If K is a closed subset of Ω and $y \notin \mathrm{cl}(f(K) \cup f(\mathrm{Fr}\,\Omega))$, then*

$$\mathrm{Deg}(f, \Omega, y) = \mathrm{Deg}(f, \Omega \setminus K, y).$$

Let E, E' be two normed spaces with filtrations $(E_n)_{n=1}^{\infty}$, $(E'_n)_{n=1}^{\infty}$, respectively, and let $\Omega \subset E$, $\Omega' \subset E'$ be open and bounded. Take two DC-mappings $f\colon \overline{\Omega} \to E$ and $f'\colon \overline{\Omega}' \to E'$. It is easy to see that $f \times f'$ is a DC-mapping, where $E \times E'$ is equipped with the filtration $(E_n \times E'_n)_{n=1}$.

Proposition 4.5. *If $y \notin \overline{f(\mathrm{Fr}\,\Omega)}$ and $y' \notin \overline{f'(\mathrm{Fr}\,\Omega')}$ then*

$$(y, y') \notin \mathrm{cl}(f \times f')(\mathrm{Fr}(\Omega \times \Omega'))$$

and

$$\mathrm{Deg}(f \times f', \Omega \times \Omega', (y, y')) = \mathrm{Deg}(f, \Omega, y) \cdot \mathrm{Deg}(f', \Omega', y').$$

Proof. It is a consequence of the simple equality $\mathrm{cl}(f \times f')(\mathrm{Fr}(\Omega \times \Omega')) = \overline{f(\mathrm{Fr}\,\Omega)} \times \overline{f'(\Omega')} \cup \overline{f(\Omega)} \times \overline{f'(\mathrm{Fr}\,\Omega')}$ and Proposition 1.4.6, [10]. ■

We know that each Leray–Schauder operator on $\overline{\Omega}$ of the form $I-f$, where $f\colon \overline{\Omega} \to E$ is compact, is a DC-mapping relative to an arbitrary filtration in E. The next result shows the connection between the notion of the Leray–Schauder degree and ours.

Theorem 4.6. *Let $f\colon \overline{\Omega} \to E$ be a compact mapping and $y \notin \varphi(\mathrm{Fr}\,\Omega)$ where $\varphi = I-f$. Then, for each filtration $(E_n)_{n=1}^{\infty}$ in E, $\mathrm{Deg}(\varphi, \Omega, y)$ can be defined and*

$$\mathrm{Deg}(\varphi, \Omega, y) = \nu((s_n)_{n=1}^{\infty})$$

where $s_n = d(\varphi, \Omega, y)$ for $n = 1, 2, \ldots$; $d(\varphi, \Omega, y)$ stands for the Leray–Schauder degree of φ.

Proof. Firstly, one can easily show that $y \notin \overline{\varphi(\mathrm{Fr}\,\Omega)}$. Hence $\mathrm{Deg}(\varphi, \Omega, y)$ can be defined. For $\varepsilon = d_E(y, \varphi(\mathrm{Fr}\,\Omega)) > 0$, take a finite $\varepsilon/4$-net S for the set $\overline{f(\Omega)}$ belonging to $\bigcup_{n=1}^{\infty} E_n$. Choose n_0 such that $S \subset E_{n_0}$ and $d_E(y, E_{n_0}) < \varepsilon/4$. Let P denote the Schauder projection onto $\mathrm{conv}\,S$. Then Pf is an $\varepsilon/4$-FC-approximation of f. Taking a sequence $(y_n)_{n=n_0}^{\infty}$ such that $y_n \in E_n$ and $\|y_n - y\| < \varepsilon/4$, we have

$$\mathrm{Deg}(\varphi, \Omega, y) = \nu\big((\deg(I-Pf|\overline{\Omega}_n, \Omega_n, y_n))\big). \tag{4.1}$$

On the other hand,

$$d(\varphi, \Omega, y) = d(\varphi, \Omega, y_{n_0}) \tag{4.2}$$

since y, y_{n_0} belong to the same component of $E \setminus \varphi(\mathrm{Fr}\,\Omega)$. Moreover, by definition (see 2.3, [10]),

$$d(\varphi, \Omega, y_{n_0}) = \deg(I-Pf|\overline{\Omega}_{n_0}, \Omega_{n_0}, y_{n_0}). \tag{4.3}$$

As in the standard Leray–Schauder construction, we easily find that, for any $n \geqslant n_0$,

$$\deg(I-Pf|\overline{\Omega}_n, \Omega_n, y_{n_0}) = \deg(I-Pf|\overline{\Omega}_{n_0}, \Omega_{n_0}, y_{n_0}). \tag{4.4}$$

By a straightforward calculation, we show that y_n and y_{n_0} lie in the same component of the set $E_n \setminus (I-Pf)(\mathrm{Fr}\,\Omega_n)$. Hence, for each $n \geqslant n_0$,

$$\deg(I-Pf|\overline{\Omega}_n, \Omega_n, y_n) = \deg(I-Pf|\overline{\Omega}_n, \Omega_n, y_{n_0}). \tag{4.5}$$

To this end, it is sufficient to combine (4.1)–(4.5). ■

As in the case of finite-dimensional and Leray–Schauder operators, one could use the degree introduced above to obtain certain results on the existence of fixed points and solutions of equations involving nonlinear DC-mappings. However, as follows from the properties of our degree, such results would determine only the existence of approximate solutions. In Section 5 we present such theorems.

At present, we generalize the classical theorems on noncontractibility of the unit sphere and on nonretractibility of the unit ball onto its boundary. Let E be a normed space with filtration, and let $B = \{x \in E\colon \|x\| \leqslant 1\}$, $S = \{x \in E\colon \|x\| = 1\}$.

Proposition 4.7. (i) *The sphere S is not DC-contractible, i.e., there exists no DC-mapping $h\colon B \times \langle 0, 1\rangle \to E$ such that, for $x \in B$, $h(x, 0) = x$, $h(x, 1) = e_0 \in S$, and $0 \notin \mathrm{cl}\,h(S \times \langle 0, 1\rangle)$.*

(ii) *There exists no DC-retraction of B onto S.*

Proof. Part (i) follows from Proposition 4.1 (i), (iv) and Proposition 4.2.

(ii) If there existed a DC-retraction $r\colon B \to S$, then, by setting $h(x, t) = r((1-t)x)$ and applying Proposition 4.1 (ii), we would obtain a contradiction with (i). ■

5. Nonlinear alternative and its consequences. Using the notion of degree introduced above, we are able to prove a certain analogue of the nonlinear alternative theorem (comp. [5], p. 61).

Theorem 5.1. *Let Ω be an open bounded neighbourhood of the origin in a normed space E with filtration, and let $f\colon \overline{\Omega} \to E$ be a DC-mapping. Then, at least one of the following conditions is satisfied:*

(i) $\inf_{x \in \Omega} \|x - f(x)\| = 0$,

(ii) $\inf_{x \in \mathrm{Fr}\,\Omega,\, t \in (0,1)} \|x - tf(x)\| = 0$.

Proof. Suppose that (ii) does not hold. Then there exists an $\varepsilon_0 > 0$ such that, for $x \in \mathrm{Fr}\,\Omega$ and $t \in \langle 0, 1\rangle$, $\|x - tf(x)\| \geqslant \varepsilon_0$. It is easily seen that $h(x, t) = x - tf(x)$,

$x \in \overline{\Omega}$, $t \in \langle 0, 1\rangle$, is a DC-mapping and $0 \notin \mathrm{cl}\, h(\mathrm{Fr}\,\Omega \times \langle 0, 1\rangle)$. In view of Proposition 4.1 (i), we have

$$1 = \mathrm{Deg}(I, \Omega, 0) = \mathrm{Deg}(I-f, \Omega, 0),$$

which, together with Proposition 4.2, proves condition (i). ■

COROLLARY 5.2 (comp. [5], p. 61). *If $f\colon E \to E$ is a DC-mapping, then either, for any $\eta > 0$, the set $\varepsilon_\eta(f) = \{x \in E\colon \inf_{t\in(0,1)} \|x - tf(x)\| \leqslant \eta\}$ is unbounded or $\inf_{x\in E} \|x-f(x)\| = 0$.*

Proof. Suppose that $\varepsilon_\eta(f)$ is bounded for some η, i.e., $\varepsilon_\eta(f) \subset B_R = \{x \in E\colon \|x\| < R\}$. The mapping $f\,|\,\overline{B}_R$ satisfies the assumptions of Theorem 5.1 but for this map (ii) does not hold. ■

Now, we prove an ε-fixed point theorem which is an analogue of the well-known results of Rothe and Altman (see [15], [1], [5]).

THEOREM 5.3. *Let $f\colon \overline{\Omega} \to E$ be a DC-mapping, where Ω is an open bounded neighbourhood of the origin in E. If one of the conditions*

(i) (Rothe type condition)

$$\|f(x)\| \leqslant \|x\|, \quad x \in \mathrm{Fr}\,\Omega, \tag{5.1}$$

(ii) (Altman type condition)

$$\|f(x)\|^q \leqslant \|f(x)-x\|^q + \|x\|^q, \quad x \in \mathrm{Fr}\,\Omega \tag{5.2}$$

for some $q > 1$ and

$$\sup_{x\in \mathrm{Fr}\,\Omega} \|f(x)\| < \infty, \tag{5.3}$$

is satisfied, then $\inf_{x\in\Omega} \|x-f(x)\| = 0$.

Proof. Suppose ε_0 is such that f has no ε_0-fixed point, i.e., $\|x-f(x)\| > \varepsilon_0$ for each $x \in \Omega$. By Theorem 5.1, there are sequences $\{x_n\} \subset \mathrm{Fr}\,\Omega$ and $\{t_n\} \subset (0, 1)$ such that $\|u_n\| \to 0$ where $u_n = x_n - t_n f(x_n)$. Then

$$\varepsilon_0 \leqslant \|x_n - f(x_n)\| \leqslant \|x_n - t_n f(x_n)\| + (1-t_n)\|f(x_n)\|$$

whence, for almost all n's, we have $(1-t_n)\|f(x_n)\| \geqslant \varepsilon_0/2$. By (5.3) (which is also a consequence of (5.1)), it thus follows that $\beta = \sup t_n < 1$. Similarly, we infer that $\alpha = \inf t_n > 0$, i.e.,

$$t_n \in \langle \alpha, \beta\rangle, \quad 0 < \alpha < \beta < 1.$$

Now, we easily get a contradiction with (5.1). In fact, under the assumption of (5.1) we have for each n

$$\|x_n\| \leqslant \|u_n\| + t_n\|f(x_n)\| \leqslant \|u_n\| + \beta\|x_n\|,$$

whence $\|u_n\| \geqslant (1-\beta) \inf_{x\in\mathrm{Fr}\,\Omega} \|x\|$, contrary to $\|u_n\| \to 0$.

To get a contradiction with (ii), observe that $|b_n - t_n a_n| \to 0$ and $|c_n - (1-t_n)a_n| \to 0$ where $a_n = \|f(x_n)\|$, $b_n = \|x_n\|$, $c_n = \|f(x_n) - x_n\|$. If Condition (5.2) held, i.e., $a_n^q \leqslant b_n^q + c_n^q$, we would obtain

$$a_n^q[(1-t_n^q) - (1-t_n)^q] \leqslant c_n^q - [(1-t_n)a_n]^q + b_n^q - (t_n a_n)^q.$$

It follows that the right-hand side of this inequality tends to 0 as $n \to \infty$. Since

$$\inf_{t\in\langle\alpha,\beta\rangle} [1 - t^q - (1-t)^q] > 0,$$

we get $a_n \to 0$. But we have already noticed that $\|f(x_n)\| \geqslant \varepsilon_0/2(1-t_n)$ for all but finite n's, which contradicts $a_n \to 0$. ■

PROPOSITION 5.4. *If $(H, (H_n)_{n=1}^\infty)$ is a unitary space with filtration, Ω is an open bounded neighbourhood of the origin $0 \in H$ and $f\colon \overline{\Omega} \to H$ is a DC-mapping such that*

$$\mathrm{Re}(x - f(x)\,|\,x) \geqslant 0 \tag{5.4}$$

for $x \in \mathrm{Fr}\,\Omega$, then $\inf_{x\in\Omega} \|x - f(x)\| = 0$.

Proof. Suppose that the assertion is not true. Using a method similar to that in the proof of Theorem 5.3, we obtain a sequence $\{x_n, t_n\}_{n=n_0}^\infty \subset \mathrm{Fr}\,\Omega \times (0, \beta\rangle$ where $0 < \beta < 1$, for which

$$\|x_n - t_n f(x_n)\| \to 0. \tag{5.5}$$

We have

$$(t_n - 1)\|x_n\|^2 = \mathrm{Re}(t_n x_n - x_n\,|\,x_n) = \mathrm{Re}(t_n x_n - t_n f(x_n)\,|\,x_n) - \mathrm{Re}(x_n - t_n f(x_n)\,|\,x_n),$$

so if (5.4) were true, we would have

$$\mathrm{Re}(x_n - t_n f(x_n)\,|\,x_n) \geqslant m^2(1-\beta)$$

where $m = \inf_{x\in\mathrm{Fr}\,\Omega} \|x\|$, which is inconsistent with (5.5). ■

Remark. Let us observe that (5.2) and (5.4) are equivalent if $q = 2$. Hence, when f is bounded on $\mathrm{Fr}\,\Omega$, the preceding proposition is a straightforward conclusion from Theorem 5.3.

When f is compact, Proposition 5.4 implies Krasnosielski's fixed point theorem. The next statement follows immediately from Proposition 5.4 (comp. [10], 1.6.3).

COROLLARY 5.5. *If $f\colon H \to H$ is a DC-mapping, coercive in the sense of a Hilbert space, i.e.,*

$$\mathrm{Re}(f(x)\,|\,x)\|x\|^{-1} \to \infty$$

as $\|x\| \to \infty$, then $\overline{f(H)} = H$.

6. Some other results. We now present some further results concerning the approximative solvability of equations that involve DC-mappings.

THEOREM 6.1. *Let E be a normed space with filtration, and let $f\colon E \to E$ be a* DC*-mapping, coercive in the sense of a Banach space, i.e., $\|f(x)\| \to \infty$ as $\|x\| \to \infty$. If there exist a $y_0 \in E$ and an open ball B such that* $\operatorname{dist}(f^{-1}(y_0), E \setminus B) > 0$ *and*

$$\operatorname{Deg}(f, B, y_0) \neq 0, \tag{6.1}$$

then $\overline{f(E)} = E$.

Proof. Let $y \in E$, and let $A = \{ty_0 + (1-t)y\colon t \in \langle 0, 1\rangle\}$. Since A is bounded and f is coercive, $f^{-1}(A)$ is bounded as well. Hence, there exists an open ball B' such that $\operatorname{dist}(f^{-1}(A), E \setminus B') > 0$. Without any loss of generality we may assume that $B' \supset B$. Due to Proposition 4.1 (iii), we have

$$\operatorname{Deg}(f, B', y_0) = \operatorname{Deg}(f, B', y).$$

On the other hand, by Corollary 4.4,

$$\operatorname{Deg}(f, B', y_0) = \operatorname{Deg}(f, B, y_0).$$

In view of (6.1) and Proposition 4.2, we get $y \in \overline{f(B')} \subset \overline{f(E)}$. ■

The next theorems are analogous to the classical results connected with the name of K. Borsuk [2]: Antipodensatz, Borsuk's fixed point theorem and the Borsuk–Ulam theorem on antipodes.

Let E, F satisfy the assumptions of Section 3.

THEOREM 6.2. *Let Ω be an open, bounded and symmetric neighbourhood of the origin in E. If $f\colon \overline{\Omega} \to F$ is a* DC*-mapping such that $0 \notin \overline{f(\operatorname{Fr}\Omega)}$ and f is odd on the boundary of Ω, i.e., $f(-x) = -f(x)$ for all $x \in \operatorname{Fr}\Omega$, then*

$$\operatorname{Deg}(f, \Omega, 0) = \nu\big((s_n)_{n=1}^{\infty}\big)$$

where s_n is an odd number for each $n = 1, 2, \ldots$

Proof. Let $\varepsilon = \inf_{x \in \operatorname{Fr}\Omega} \|f(x)\|$, and let $g\colon \overline{\Omega} \to F$ be an arbitrary $\varepsilon/2$-FC-aproximation of f. Consider the following mapping $\bar{g}\colon \overline{\Omega} \to F$:

$$\bar{g}(x) = \big(g(x) - g(-x)\big)/2. \tag{6.2}$$

Obviously, $\bar{g}$ is an FC-mapping and, since

$$\|\bar{g}(x) - f(x)\| \leqslant \|g(x) - f(x)\|/2 + \|g(-x) - f(-x)\|/2 < \varepsilon/2,$$

the homotopy h defined by $h\colon \overline{\Omega} \times \langle 0, 1\rangle \ni (x, t) \mapsto t\bar{g}(x) + (1-t)f(x)$ satisfies the assumptions of Proposition 4.1 (i). It follows that

$$\operatorname{Deg}(f, \Omega, 0) = \operatorname{Deg}(\bar{g}, \Omega, 0).$$

By (6.2), $\bar{g}|\overline{\Omega}_n$ is odd on $\operatorname{Fr}\Omega_n$ for sufficiently large n; therefore, from Borsuk's Antipodensatz, we infer that $\deg(\Phi_n \circ (\bar{g}|\overline{\Omega}_n), \Omega_n, 0)$ is an odd number. ■

COROLLARY 6.3. *Let Ω satisfy the above assumptions, and let $f\colon \overline{\Omega} \to E$ be a* DC*-mapping, odd on the boundary of Ω. Then*

$$\inf_{x \in \Omega} \|x - f(x)\| = 0.$$

Let $(E, (E_n)_{n=1}^{\infty})$ be a normed space with filtration. We consider the filtration $(R^k \oplus E_n)_{n=1}^{\infty}$ in the normed space $R^k \oplus E$. Let Ω be an open, bounded and symmetric neighbourhood of the origin in $R^k \oplus E$, $k \geqslant 1$.

THEOREM 6.4. (i) *If $f\colon \overline{\Omega} \to E$ is a* DC*-mapping odd on $\operatorname{Fr}\Omega$, then $\inf_{x \in \operatorname{Fr}\Omega} \|f(x)\| = 0$.*

(ii) *If $f\colon \overline{\Omega} \to E$ is an arbitrary* DC*-mapping, then $\inf_{x \in \operatorname{Fr}\Omega} \|f(x) - f(-x)\| = 0$.*

Proof. (i) Suppose on the contrary that $\inf_{x \in \operatorname{Fr}\Omega} \|f(x)\| > 0$. Treating f as a DC-mapping into $R^k \oplus E$, we get by Theorem 6.2 that $\operatorname{Deg}(f, \Omega, 0)$ is generated by a sequence of odd numbers. On the other hand, from Proposition 4.1 (iii) and Proposition 4.2 we infer that $\operatorname{Deg}(f, \Omega, 0) = 0 \in \mathcal{G}$.

(ii) is immediate if we apply (i) to the mapping $\overline{\Omega} \ni x \mapsto f(x) - f(-x)$. ■

Finally we should observe that all result concerning approximate solutions are in fact generalizations of the classical theorems on compact operators. If we assume that the mappings are closed (such are the Leray–Schauder maps), we get exact solutions. Another possibility is to assume the demiclosedness of the maps and the weak compactness of their domains.

A mapping $f\colon E \supset A \to F$ is said to be demiclosed [4] if its graph is closed in $E \times F$ while E is endowed with the weak topology and F with the strong one. One can easily prove.

PROPOSITION 6.5. *If A is weakly compact, a mapping $f\colon A \to F$ is demiclosed and $\inf_{x \in A} \|f(x)\| = 0$, then there exists an $x_0 \in A$ such that $f(x_0) = 0$.*

References

[1] M. Altman, *A fixed point theorem for completely continuous operators in Banach spaces*, Bull. Acad. Polon. Sci. 3 (1955), 409–413.

[2] K. Borsuk, *Drei Sätze über die n-dimensionale Euklidische Sphäre*, Fund. Math. 21 (1933), 177–190.

[3] F. E. Browder and W. V. Petryshyn, *The topological degree and Galerkin approximation for noncompact operators in Banach spaces*, Bull. Amer. Math. Soc. 74 (1968), 641–646.

[4] —, — *The solution by iteration of nonlinear functional equation in Banach spaces*, Bull. Amer. Math. Soc. 72 (1966), 571–575.

[5] J. Dugundji and A. Granas, *Fixed Point Theory*, Warszawa 1982.

[6] W. Kryszewski, *O rozwiązalności równań nieliniowych z* DC*-operatorami*, in preparation.

[7] W. Kryszewski and B. Przeradzki, *Multivalued* DJ*-mappings*, submitted for publication.

[8] J. Leray and J. Schauder, *Topologie et équations fonctionnelles*, Ann. Sci. École Norm. Sup. 3 51 (1934), 45–78.

[9] E. Michael, *Continuous selections*, I, Ann. of Math. 63 (1956), 361–382.
[10] L. Nirenberg, *Topics in Nonlinear Functional Analysis*, Lect. Notes, New York Univ. 1974.
[11] B. Nowak, DJ-*odwzorowania i ich homotopie*, Acta Univ. Lodziensis, Łódź 1981.
[12] R. D. Nussbaum, *Degree theory for local condensing maps*, J. Math. Anal. Appl. 37 (1972), 741–766.
[13] W. V. Petryshyn, *On the approximation solvability of equations involving A-proper and pseudo-A-proper mappings*, Bull. Amer. Math. Soc. 81 (1975), 223–312.
[14] B. Przeradzki, *On the homotopical classification of* DJ-*mappings of infinitely dimensional spheres*, Fund. Math. 120 (1984), 145–149.
[15] E. Rothe, *Zur Theorie der topologischen Ordnung und der Vektorfelder in Banachschen Räumen*, Comps. Math. 5 (1937), 177–196.
[16] S. Wereński, *On the fixed point index of noncompact mappings*, Studia Math. 78 (1983), 155–160.

INSTITUTE OF MATHEMATICS
UNIVERSITY OF ŁÓDŹ
Banacha 22, 90-238 Łódź, Poland

Received 5 April 1983;
in revised form 18 April 1984

On some dynamical properties of S-unimodal maps on an interval

by

Tomasz Nowicki (Warszawa)

Abstract. A globally expanding mapping is introduced. It has a uniform hyperbolic structure on the set of periodic points and on the set of preimages of the critical point. For S-unimodal mappings the existence of one such structure is equivalent to the existence of another. When the iterates of the critical point are away from the critical point itself, then the mapping is globally expanding.

0. Introduction. The aim of this paper is to present some results on the dynamics of S-unimodal mappings of an interval. The results are related to the results of Collet and Eckmann [1], Guckenheimer [3] and Misiurewicz [4].

In Section 1 we introduce two notions:

I. *global expanding* — which means that the length of every interval with two consecutive critical points of f^n as endpoints expands exponentially under f^n.

II. *uniform hyperbolic structure* on the set $\mathrm{Per}(f)$ which means that there are two constants $K>0$; $\lambda>1$ such that if $f^s(x) = x$ then $|f^{s\prime}(x)| > K\lambda^s$ and on the set of preimages of the critical point $C_{-\infty}$, i.e., if $f^n(x) = c$ then $|f^{n\prime}(x)| > K\lambda^n$. This notion in another form appears in [1].

In Section 2 we prove that if f is globally expanding, then f has a uniform hyperbolic structure on $\mathrm{Per}(f)$.

In Section 3 we show that a mapping has a uniform hyperbolic structure on $\mathrm{Per}(f)$ if and only if it has a uniform hyperbolic structure on $C_{-\infty}$.

In Section 4 we demonstrate that if f has an uniform hyperbolic structure on $\mathrm{Per}(f)$ then for n large enough f^n has no restrictive central point. Hence f has sensitivity on initial conditions (see [3]).

In Section 5 we prove that if f has a uniform hyperbolic structure on $C_{-\infty}$ and for some $K>0$; $\lambda>1$ and every n we have $|f^{n\prime}(f(c))| > K\lambda^n$ then the length of the interval of monotonicity of f^n diminishes exponentially with n. Hence, if f has no sinks and the images of the critical point are separated from the critical point itself, then f is globally expanding (see [1] and [4]).