On a Corson compact space of Todorčević

by

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Abstract. Examples of (1) a metalindelöf space which is not weakly submetacompact, and (2) a compact Radon space which is not hereditarily weakly submetacompact are provided, by showing that the square of a certain Corson compact non-Eberlein compact space constructed by S. Todorčević is a hereditarily metalindelöf Radon space which is not hereditarily weakly submetacompact. The relationship between Todorčević's example, which is constructed from a certain tree, and a construction of compact spaces from trees due to P. Nyikos is also discussed, and we give a simple characterization of when these spaces are Eberlein compact.

1. Introduction. In this note we show that a certain Corson compact, non-Eberlein compact space X constructed by S. Todorčević ([T$_1$], [T$_2$]) has the property that $X^2 \setminus \Delta$, where Δ is the diagonal, is an example of a metalindelöf space which is not weakly submetacompact (see Section 2 for the definitions). Such an example was previously known only under the continuum hypothesis [GG]. We also show that if the continuum is not a real-valued measurable cardinal, then X^2 is a Radon space. R. J. Gardner [Ga$_1$] has shown that every compact hereditarily weakly submetacompact space is a Radon space, as long as it does not contain discrete subsets of measurable cardinality, and has asked if the converse holds, i.e., whether every compact Radon space is hereditarily weakly submetacompact. This shows that the answer is "no". Gardner [Ga$_2$] had previously constructed a counterexample assuming the continuum hypothesis.

Todorčević's example is of the following type: Given a tree T, a certain compact space $X(T)$ is constructed which is Corson compact if and only if all chains of T are countable. P. Nyikos [N] has a construction of a compact space $Y(T)$ from a tree T which is very similar to Todorčević's construction. In Section 3 we discuss the relationship between these two constructions, and show that Todorčević's space $X(T)$ will be Eberlein compact if and only if T is special, and Nyikos's space $Y(T)$ will be Eberlein compact if and only if T is R-embeddable.

All our space are presumed to be at least Hausdorff. For basic set-theoretic notation and definitions, see Kunen [K$_1$].

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* Partially supported by NSF grant MCS-8301932.

2. A metalindelöf non-weakly-submetacompact space. Recall that a compact space X is Corson compact if and only if X embeds in a Σ-product

$$\Sigma(\Gamma) = \{x \in R^{\Gamma}: |\{\alpha \in \Gamma: x(\alpha) \neq 0\}| \leqslant \omega\}$$

of real lines, and is Eberlein compact if it embeds so that, for each $x \in X$ and $\varepsilon > 0$,

$$|\{\alpha \in \Gamma: ||x(\alpha)|| > \varepsilon\}| < \omega$$

(viewing X as a subspace of $\Sigma(\Gamma)$). The following topological characterization of these spaces is well-known: A compact space X is Corson (Eberlein) compact if and only if X has a point-countable (σ-point-finite) T_0-separating cover by open F_σ's. (A cover $\mathcal{U}$ is *T_0-separating* if for each pair of distinct points, some $U \in \mathcal{U}$ contains exactly one of them.) The author recently obtained the following characterization [Gr]: A compact space X is Corson (Eberlein) compact if and only if $X^2 \setminus \Delta$ is metalindelöf (σ-metacompact). Recall that a space Y is *metalindelöf* (*σ-metacompact*) if and only if every open cover of Y has a point-countable (σ-point-finite) open refinement.

Now, a space Y is *weakly submetacompact* (or *weakly θ-refinable*) if every open cover of Y has a refinement $\mathcal{U} = \bigcup_{n \in \omega} \mathcal{U}_n$ such that, for each $x \in X$ there exists $n_x \in \omega$ such that x is in at least one but only finitely many members of $\mathcal{U}_{n_x}$ (i.e., $1 \leqslant \operatorname{ord}(x, \mathcal{U}_{n_x}) < \omega$). Note that we do not require each $\mathcal{U}_n$ to cover Y — had we done this, we would have defined *submetacompact* (or *θ-refinable*) spaces. There are a number of known examples of metalindelöf spaces which are not submetacompact. A suitable subspace of Bing's Example G will work (see Burke [Bu]). But metalindelöf spaces which are not weakly submetacompact seem to be more difficult to construct. Perhaps this is in part because any countable union of weakly submetacompact spaces is weakly submetacompact — this eliminates many common "counterexample machines" and, for example, eliminates subspaces of Bing's G as possible places to look for an example. In fact, the only known example seems to be one constructed by R. J. Gardner and the author [GG] assuming the continuum hypothesis.

Now note that if a compact space X is Corson compact but not Eberlein compact, then $X^2 \setminus \Delta$ is metalindelöf but not σ-metacompact and that σ-metacompactness is very similar in spirit, though somewhat stronger than, weak submetacompactness. This observation led us to consider various known examples of Corson compact, non-Eberlein compact spaces, and we show here that a certain example due to Todorčević can be used to obtain a first countable, locally compact, metalindelöf space which is not weakly submetacompact. The space is constructed as follows. Let S be any stationary, co-stationary subset of ω_1, and let

$$T = \{C \subset S: C \text{ is closed in } \omega_1\}.$$

For $t_1, t_2 \in T$, let $t_1 \leqslant t_2$ if and only if t_1 is an initial segment of t_2. Then T with this ordering is a tree with no uncountable chains (since S does not contain a club). Let

$$P(T) = \{p \subset T: t \in p \wedge s < t \Rightarrow s \in p, \text{ and } p \text{ is totally ordered}\}$$

be the set of all paths of T viewed as a subspace of ${}^{T}2$, where a path is identified with its characteristic function. Then $P(T)$ is easily seen to be a closed, hence compact, subset of ${}^{T}2$, and since every path is countable, $P(T)$ must be Corson compact. Todorčević [T$_1$] shows that $P(T)$ has no dense metrizable subset, hence, by a result of [AL], is not Eberlein compact. This $P(T)$ is not first countable, but if we modify T (following Todorčević) by "sticking a Cantor tree" between every node and its immediate successors, the resulting $P(T)$ is now first countable (what is needed for first countability is for every node to have only countably many immediate successors), and still has the other properties.

Instead of showing that $P(T)$ is our desired space, it will be slightly more convenient to show instead that a similar space constructed from T using an idea of Nyikos [N] has all the desired properties. Let $\tilde{T}$ be the tree obtained from T by adding a node at the end of each maximal chain. Let $S(T)$ be the set of all successor nodes of T, i.e., all nodes of T which have an immediate predecessor. For each $t \in S(T)$, let $V_t = \{t' \in \tilde{T}: t' \geqslant t\}$. Then it is not difficult to show that the V_t's and their complements form a subbase for a supercompact topology on $\tilde{T}$ (noting that every cover of $\tilde{T}$ by V_t's and their complements has a two element subcover).

Let X be $\tilde{T}$ with the above topology; we will show that $X^2 \setminus \Delta$ is metalindelöf but not weakly submetacompact. ($X^2 \setminus \Delta$ is of course locally compact, and will be first countable if T is also modified as suggested earlier.)

CLAIM 1. *$X^2 \setminus \Delta$ is metalindelöf.*

For each $t \in S(T)$, let $V_t^c = X \setminus V_t$. Note that

$$\mathcal{V} = \{V_t \times V_t^c: t \in S(T)\} \cup \{V_t^c \times V_t: t \in S(T)\}$$

is a point-countable cover of $X^2 \setminus \Delta$ by compact open sets. Claim 1 easily follows.

CLAIM 2. *$X^2 \setminus \Delta$ is not weakly submetacompact.*

Let $S \subset \omega_1$ be stationary. Then it can be shown using a pressing down argument that $S^2 \setminus \Delta$ is not weakly submetacompact. Todorčević [T$_2$] shows that an analogue of the usual pressing down lemma on ω_1 holds for T. The author originally obtained a proof of Claim 2 by mimicking in $X^2 \setminus \Delta$ the proof that $S^2 \setminus \Delta$ is not weakly submetacompact. However, this proof is rather tedious and involved. So we will give a much shorter and in our opinion clearer proof using a rather simple forcing argument (¹).

In this paragraph, we state the forcing facts needed to understand the rest of the proof. We use the terminology of Kunen [K_1]. Let M be a countable transitive model of ZFC, and consider S, T, X, etc. to have been defined within M. A set

(¹) The author would like to thank Alan Dow and Juris Steprans for suggestions which led to this proof.

$D \subset T$ is *dense* in T if $V_t \cap D \neq \emptyset$ for each $t \in T$, and $G \subset T$ is T-*generic over* M if $G \cap D \neq \emptyset$ for each dense $D \subset T$, $D \in M$. The *generic extension* $M[G]$ of M is the smallest transitive model of ZFC extending M and containing G as an element; M and $M[G]$ have the same ordinals. In general forcing, cardinals of M may become smaller cardinals or just ordinals in $M[G]$ (e.g., the first uncountable cardinal ω_1 in M may be a countable ordinal in $M[G]$, and ω_2 in M may be the first uncountable cardinal in $M[G]$), but it is known that in this case, the cardinal ω_1 in M is also the first uncountable ordinal in $M[G]$. This follows from the facts that (1) T is Baire (in the topology generated by the V_t's), and that (2) for Baire partial orders ${}^{\omega}M \cap M = {}^{\omega}M \cap M[G]$, i.e., if a countable sequence of members of M is in $M[G]$, then it must be in M. (So, if ω_1 became a countable ordinal in $M[G]$, there would be in $M[G]$ a countable sequence of ordinals whose limit is ω_1. But this sequence would also be in M, a contradiction.) (Jensen was the first to use this T in forcing; see also [BHK]. See [T_1], Lemma 9.12, for a proof that T is Baire. Fact (2) above is an exercise in [K_1].)

Consider a fixed T-generic set G. Since the union of all levels of T beyond any given level is dense in T, it is clear that a generic G must be an uncountable chain through T, and that $\bigcup G$ is a club subset of ω_1 contained in S (of course, $G \notin M$).

We now describe the idea of the proof. Let τ be the topology on X in M; we show that (X, τ) is not weakly submetacompact in M. Let $\mathscr{V}$ be the cover of $X^2 \setminus \Delta$ in Claim 1, and suppose $\mathscr{V}$ has a weak submetacompact refinement $\mathscr{U}$ in M. In $M[G]$, the same V_t's and V_t^c's form a subbase for a topology τ' on X, with $\tau \subseteq \tau'$. Hence $\mathscr{U}$ is also a weak submetacompact refinement of $\mathscr{V}$ in $M[G]$. We will arrive at a contradiction by showing that in $M[G]$, $X^2 \setminus \Delta$ contains a copy of $\omega_1^2 \setminus \Delta$ such that the trace of $\mathscr{V}$ on this copy has no weak submetacompact refinement.

In $M[G]$, X contains as a closed subset the uncountable chain

$$C = \{t \in T\colon \exists s \in G (t \leqslant s)\}.$$

It is easy to check that the relative topology on C is precisely the order topology, hence C is naturally homeomorphic to ω_1, and that the trace of on $C^2 \setminus \Delta$ corresponds to the cover

$$\mathscr{W} = \{[0, \alpha) \times [\alpha, \omega_1)\colon \alpha \in \omega_1 \setminus \mathrm{LIM}\} \cup \{[\alpha, \omega_1) \times [0, \alpha)\colon \alpha \in \omega_1 \setminus \mathrm{LIM}\}$$

of $\omega_1^2 \setminus \Delta$, where LIM denotes the set of limit ordinals. It remains to show by a standard pressing down argument that $\mathscr{W}$ has no weak submetacompact refinement.

Suppose $\mathcal{O} = \bigcup_{n \in \omega} \mathcal{O}_n$ is a weak submetacompact refinement of $\mathscr{W}$. Let $\alpha \in \mathrm{LIM}$, and let

$$X_{\alpha,n} = \{\beta > \alpha\colon 1 \leqslant \mathrm{ord}(\langle \alpha, \beta \rangle, \mathcal{O}_n) < \omega\}.$$

For some $n(\alpha) \in \omega$, $X_{\alpha,n(\alpha)}$ is stationary. Since $\mathcal{O}_{n(\alpha)}$ is a point-finite open cover of a stationary subset of ω_1 (more precisely, of $\{\alpha\} \times \omega_1$), it must, by an easy pressing down argument, have a member $O_\alpha \supset \{\alpha\} \times [\beta(\alpha), \omega_1)$ for some $\alpha < \beta(\alpha) < \omega_1$. By another pressing down argument, we can assume there exists $\gamma(\alpha) < \alpha$ such that $O_\alpha \supset [\gamma(\alpha), \alpha] \times [\beta(\alpha), \omega_1)$. And since $O_\alpha \subset W$ for some $W \in \mathscr{W}$, there exists $\delta(\alpha) > \alpha$ with $O_\alpha \subset [0, \delta(\alpha)] \times [\delta(\alpha), \omega_1)$.

Now for some $k \in \omega$, $S_k = \{\alpha\colon n(\alpha) = k\}$ is stationary. Thus there exists $\beta \in \omega_1$ and an uncountable $S' \subset S_k$ such that $\beta(\alpha) = \beta$ for all $\alpha \in S'$. Choose an increasing sequence of ordinals $\alpha_n \in S'$ with $\alpha_{n+1} > \delta(\alpha_n)$ for each $n \in \omega$. Let $\lambda \in X_{\alpha_0,k}$ with $\gamma > \sup\{\gamma(\alpha_n)\colon n \in \omega\}$. Then $\langle \alpha_0, \gamma \rangle \in \bigcap_{n \in \omega} O_{\alpha_n}$, contradicting $\mathrm{ord}(\langle \alpha_0, \gamma \rangle, \mathcal{O}_k) < \omega$. That completes the proof.

3. Radon spaces. A *Radon measure* on a space X is a finite Borel measure μ which satisfies

$$\mu(B) = \sup\{\mu(K)\colon K \subset B, K \text{ compact}\}$$

for every Borel set B. We say that X is a *Radon space* if each Borel measure on X is a Radon measure.

As mentioned in the introduction, Gardner [Ga] showed that every compact hereditarily weakly submetacompact space is a Radon space, as long as it does not contain a discrete subset of (real-valued) measurable cardinality. In the previous section, we showed that the square X^2 of Todorčević's space is not hereditarily weakly submetacompact. In this section, we will show that X^2 is nevertheless a Radon space, as long as the continuum $\underline{c}$ is not real-valued measurable (for convenience, this will be assumed throughout this section), thereby proving that the converse of Gardner's theorem does not hold.

First we will note that X itself is hereditarily paracompact, hence Radon. While a direct proof of this is not difficult, we will save space by noting that this follows from the facts

(1) X is monotonically normal, hence hereditarily collectionwise normal [N], and

(2) every collectionwise normal metalindelöf locally compact space is paracompact [B].

Now we show further that every Borel measure μ on X has "small" support. Let X_α be the subspace of X consisting of all $t \in \tilde{T}$ on or below the αth level. We claim that $\mu(X \setminus X_\alpha) = 0$ for some $\alpha < \omega_1$. Suppose $\mu(X \setminus X_\alpha) > 0$ for all $\alpha < \omega_1$. Then since μ is Radon, μ cannot be locally 0 on $X \setminus X_\alpha$, so $\mu(V_{t(\alpha)}) > 0$ for some $t(\alpha) \in X \setminus X_\alpha$. Let $n \in \omega$ be such that

$$T_n = \left\{t(\alpha)\colon \mu(V_{t(\alpha)}) > \frac{1}{2^n}\right\}$$

is uncountable. Now $V_t \cap V_{t'}' \neq \emptyset$ if and only if t and t' are related in $\tilde{T}$, so if k is such that $k \cdot 1/2^n \geqslant \mu(X)$, then every $k+1$ elements of T_n are related. We put pairs $\{s, t\}$ of elements of T_n into two pots: Pot I if s and t are related, Pot II if not. Erdos's theorem $\omega_1 \to (\omega_1, \omega)^2$ (see [K_2]) says that either there is an uncountable set $A \subset T_n$ all of whose pairs belong to Pot I, or a countable set $B \subset T_n$ all pairs of which belong to Pot II. Since the latter is impossible, the former holds. But this means T contains an uncountable chain, so we have a contradiction. Hence $\mu(X\setminus X_\alpha) = 0$ for some $\alpha < \omega_1$, as claimed.

Now suppose ν is a Borel measure on X^2. Let ν_i, $i = 1, 2$, be the Borel measure on X defined by

$$\nu_i(A) = \nu(\pi_i^{-1}(A)),$$

where π_i is the projection on the ith factor. By the above paragraph there exists $\alpha < \omega_1$ such that $\nu_i(X\setminus X_\alpha) = 0$, $i = 1, 2$. Then $\nu(X^2\setminus X_\alpha^2) = 0$, so ν is fully supported on X_α^2. Note that each level of $\tilde{T}$ is metrizable as a subspace of X (it is easily seen to have a σ-discrete base). Hence X_α^2 is the union of countably many metrizable spaces, hence is hereditarily weakly submetacompact, hence is a Radon space. It follows that ν is a Radon measure, which completes the proof that X^2 is Radon.

4. Compact spaces from trees. In this section we discuss the relationship between Todorčević's and Nyikos's construction of compact T_2-spaces from trees, and characterize, in terms of simple properties of the tree, when the resulting spaces are Eberlein compact.

Given a tree T, Todorčević considers the set $P(T)$ of all paths of T viewed as with a subspace of ${}^T2$, with a path identified with its characteristic function. $P(T)$ is closed, hence compact, in ${}^T2$. Nyikos's idea is to first extend T to $\tilde{T}$ by adding a node at the end of each path (including the empty path) in T which does not already have a unique supremum in T. If $S(\tilde{T})$ is the set of successors of $\tilde{T}$, and $V_t = \{t' \in T: t \leqslant t'\}$, then, as mentioned earlier, the V_t's and their complements are a subbase for a compact Hausdorff topology on $\tilde{T}$.

The above constructions are very closely related. Note that $P(T)$ is itself a tree ordered by the "initial segment" relation, and that every path in $P(T)$ has a unique supremum, hence $\widetilde{P(T)} = P(T)$. Also, a path $p \in P(T)$ is a successor node in $P(T)$ if and only if it has the form $(\cdot, t_p] = \{s \in T: s \leqslant t_p\}$ for some $t_p \in T$. Thus V_p is precisely the set of all paths containing t_p, and this is the same as

$$\{f \in ({}^T2) \cap P(T): f(t_p) = 1\}.$$

Thus we see that Nyikos's topology on $P(T)$ is the same as its subspace topology in ${}^T2$.

Also note that the function $h: \tilde{T} \to P(S(\tilde{T}))$ defined by $h(t) = (\cdot, t] \cap S(T)$ is an isomorphism of $\tilde{T}$ onto the path tree of the subtree $S(\tilde{T})$ of T. By the above remarks, $h(V_t) = V_{h(t)}$ for $t \in S(\tilde{T})$, hence h is also a homeomorphism of topological spaces. So the precise relationship between the two constructions can be stated as follows: *Applying Nyikos's construction to T yields the same space as applying Todorčević's construction to $S(\tilde{T})$, while applying Todorčević's construction to T yields the same space as applying Nyikos's construction to $P(T)$.*

Now we obtain a simple characterization of those trees T for which $P(T)$ or $\tilde{T}$ is Eberlein compact. Recall that T is *special* if it is a countable union of antichains,

and that T is *R-embeddable* if $S(T)$ is a countable union of antichains. (The usual definition of R-embeddable is that there exists $f: T \to R$ such that $s < t \Rightarrow f(s) < f(t)$; that our definition is equivalent to this one is a result of Galvin (see Baumgartner [Ba]).

THEOREM. *Let T be a tree. Then*

(a) *$P(T)$ is Eberlein compact $\Leftrightarrow T$ is special*

(b) *$\tilde{T}$ is Eberlein compact $\Leftrightarrow T$ is R-embeddable.*

Proof. Part (a) follows easily from (b) and the above remarks — for assuming (b), $P(T)$ is Eberlein compact if and only if $P(T)$ is R-embeddable if and only if $S(P(T))$ is special. But the map $k: S(P(T)) \to T$ defined by $k((\cdot, t]) = t$ is an isomorphism, so (a) follows.

To prove (b), assume first that T is R-embeddable. Let $S(T) = \bigcup_{n\in\omega} T_n$, where each T_n is an antichain, and let $\mathcal{U}_n = \{V_t: t \in T_n\}$. Then $\mathcal{U}_n$ is a disjoint collection of clopen sets, and $\mathcal{U} = \bigcup_{n\in\omega} \mathcal{U}_n$ is T_0-separating. Thus $\tilde{T}$ is Eberlein compact.

Finally, assume that $\mathcal{U} = \bigcup_{n\in\omega} \mathcal{U}_n$ is a point-finite T_0-separating collection of open F_σ's in the space $\tilde{T}$. Since each $U \in \mathcal{U}$ is an F_σ and $\tilde{T}$ is 0-dimensional, it is easy to see that we may assume each member of $\mathcal{U}$ is a clopen set. We may also assume $\mathcal{U}_0 \subset \mathcal{U}_1 \subset \ldots$.

Now let

$$\mathcal{V}_n = \{U \times (\tilde{T}\setminus U): U \in \mathcal{U}_n\} \cup \{(\tilde{T}\setminus U) \times U: U \in \mathcal{U}_n\},$$

and let $\mathcal{V} = \bigcup_{n\in\omega} \mathcal{V}_n$. Then $\mathcal{V}$ is a cover of $\tilde{T}^2\setminus\Delta$, and each $\mathcal{V}_n$ is point-finite. For $t \in S(T)$, choose from $\mathcal{V}$ a finite minimal cover $\mathcal{W}_t$ of $V_t \times V_t^c$, and put $t \in T_{n,m}$ if and only if $|\mathcal{W}_t| = m$ and n is the least such that $\mathcal{W}_t \subset \mathcal{V}_n$.

We shall complete the proof by showing that every increasing chain in $T_{n,m}$ is finite, hence $T_{n,m}$, and so also $S(T)$, is a countable union of antichains. To this end, suppose $t_0 < t_1 < \ldots$ is an increasing sequence in $T_{n,m}$. Since $|\mathcal{W}_{t_i}| = m$, we may assume that the $\mathcal{W}_{t_i}$'s form a Δ-system with root $\mathcal{R}$. Let $t = \sup_{n\in\omega} \{t_n\}$. Since $\langle t, t\rangle$ is not in the clopen set $\bigcup \mathcal{R}$, there exists $k \in \omega$ such that $\langle t, t_k\rangle \notin \bigcup \mathcal{R}$. Since $\langle t, t_k\rangle \in V_{t_n} \times V_{t_n}^c$ for $n > k$, there exist $W_n \in \mathcal{W}_{t_n}\setminus\mathcal{R}$ with $\langle t, t_k\rangle \in W_n$. But the W_n's are distinct elements of $\mathcal{V}_n$, which contradicts the point-finiteness of $\mathcal{V}_n$.

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Received 10 December 1984

Tiling with smooth and rotund tiles

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Abstract. Without assuming local finiteness, this paper studies tilings of topological vector spaces by convex sets that are bounded or finitely bounded. The paper was motivated by a wish to ascertain, in the infinite as well as the finite-dimensional case, to what extent the tiles can be smooth or rotund. Various limitations are established. For example, no space of dimension ≥ 2 admits a countable rotund tiling, and tilings that are uniformly smooth or uniformly rotund are excluded under certain hypotheses. On the other hand, some nonseparable locally convex spaces admit tilings in which each tile is both smooth and rotund. Several unsolved problems remain.

Introduction. A collection $\mathscr{C}$ of subsets of a topological space S is a *covering* if $S = \bigcup \mathscr{C}$. It is a *packing* if $|\mathscr{C}| > 1$, each member of $\mathscr{C}$ is the closure of its nonempty interior, and the interiors are disjoint. A *tiling* is a collection that is both a covering and a packing, and the members of a tiling are *tiles*.

A subset of a topological vector space is here called a *bc-set* (resp. *fc-set*) if it is closed, convex and bounded (resp. *finitely bounded* (has bounded intersection with each finite-dimensional subspace)). Along with certain other adjectives (e.g. closed, convex, smooth, rotund), the prefixes *bc* and *fc* are applied to a collection $\mathscr{C}$ if they apply to each member of $\mathscr{C}$. However, some adjectives refer to $\mathscr{C}$ as a collection or to the interactions among members of $\mathscr{C}$, and we rely on context for the necessary distinctions. For example, $\mathscr{C}$ is *countable* if $|\mathscr{C}| \leq \aleph_0$, *disjoint* if no two members intersect, and *locally finite* if each point of the space has a neighborhood that intersects only finitely many members of $\mathscr{C}$.

In a locally finite bc-tiling of $\boldsymbol{R}^d$, each tile is a d-polytope [3] [17], and at least for $d \leq 3$ an arbitrary d-polytope P may serve as a prototile in the sense that $\boldsymbol{R}^d$ admits a locally finite tiling in which each tile is combinatorially equivalent to P [4] [13]. However, without the assumption of local finiteness, little is known even in the plane, and that assumption is inappropriate for the study of bc-tilings of in-

* Research of the first author was partially supported by the U. S. National Science Foundation. Research of the second and third authors was supported by Consiglio Nazionale delle Ricerche and Ministero della Pubblica Istruzione of Italy, while they were visiting the University of Washington, Seattle, U. S. A.