# Paracompactness in the class of closed images of GO-spaces

by

**Witold Bula** (Warszawa)

**Abstract.** We prove that closed images of GO-spaces are paracompact if and only if they contain no closed copies of stationary sets. In addition, we show that every closed continuous map from a GO-space $G$ onto a paracompact space $X$ can be extended to a closed continuous map from some paracompact GO-space containing $G$ onto $X$.

R. Engelking and D. Lutzer [2] proved the following characterization of paracompactness in the class of GO-spaces: a GO-space is not paracompact if and only if it contains a closed subset which is homeomorphic to a stationary subset of a regular uncountable cardinal.

The aim of the present paper is to show that the same property characterizes paracompactness in the (larger) class of images of GO-spaces under closed continuous maps as well. In addition, we prove that every closed continuous map from a GO-space $G$ onto a paracompact space $X$ can be extended to a closed continuous map from some paracompact GO-space containing $G$ onto $X$.

Let us note that our Lemma 2 expresses ideas which are similar to those included either in the conclusion or in the proof of the mentioned result due to R. Engelking and D. Lutzer [2].

All ordinals below are considered as linearly ordered topological spaces. Let us recall that a subset of a regular uncountable cardinal $\kappa$ is said to be *stationary* if it meets all closed unbounded subsets of $\kappa$. The crucial fact about stationary sets is the following

Pressing Down Lemma (G. Fodor; see [6], Theorem 8 p. 347). *Let $S$ be a stationary subset of a regular uncountable cardinal $\kappa$. If $\varphi\colon S \to \kappa$ is a function such that $\varphi(\alpha) < \alpha$ for $\alpha \in S-\{0\}$, then there exists an ordinal $\gamma < \kappa$ such that $\varphi^{-1}(\{\gamma\})$ is stationary.*

By *Lim* we will denote the class of all limit ordinals. By $A^d$ we will denote the set of all cluster points of a set $A$.

Let $G$ be a GO-space. A pair $\langle A, B\rangle$ is called a *pseudo-gap* in $G$ if both $A$ and $B$ are open subsets of $G$ such that $A \cup B = G$, each point of $A$ precedes each point of $B$ and either $A \neq \emptyset$ and $A$ has no last point or $B \neq \emptyset$ and $B$ has no first point.

A pair $\langle a, b\rangle$ of points of $G$ is called a *jump in* $G$ if $a < b$ and $[a, b] = \{a, b\}$; points $a, b$ are called (the *left* and the *right*, respectively) *ends of a jump in* $G$.

Let us recall that every GO-space $G$ has a linearly ordered compactification which preserves the order on $G$ (cf. [1], Problem 3.12.3 (b)). By an easy modification of the usual construction of this, we may obtain a linearly ordered compactification $\lambda G$ of $G$ which preserves the order on $G$ and such that for every pseudo-gap $\langle A, B\rangle$ in $G$ the intersection $\mathrm{cl}_{\lambda G} A \cap \mathrm{cl}_{\lambda G} B$ is empty. Let us note that there is the unique linearly ordered compactification of $G$ which satisfies this condition; namely, $\lambda G$ is maximal among linearly ordered compactifications of $G$ (cf. V. V. Fedorčuk [3] and R. Kaufman [5]). Notice that each point $p \in \lambda G - G$ is an end of a jump in $\lambda G$. Furthermore, a pair $\langle A, B\rangle$ of subsets of $G$ is a pseudo-gap in $G$ if and only if there is a point $p \in \lambda G - G$ such that $A = (\leftarrow, p) \cap G$ and $B = (p, \rightarrow) \cap G$.

A pseudo-gap $\langle A, B\rangle$ in $G$ is called a *left pseudo-Q-gap in* $G$ if there is a regular cardinal $\kappa \geqslant \omega$ and an increasing sequence $P = \{p_\alpha\colon \alpha < \kappa\} \subset A$ such that $P$ has no upper bound in $A$ and for each limit $\gamma$, $\gamma < \kappa$, $\sup_{\lambda G}\{p_\alpha\colon \alpha < \gamma\} \notin G$. A *right pseudo-Q-gap* is defined analogously. A pseudo-gap $\langle A, B\rangle$ is called a *pseudo-Q-gap in* $G$ if either $A$ has the last point or $\langle A, B\rangle$ is the left pseudo-$Q$-gap in $G$ and either $B$ has the first point or $\langle A, B\rangle$ is a right pseudo-$Q$-gap in $G$. A pseudo-gap $\langle A, B\rangle$ in $G$ is called a *left* (*right*) *pseudo-non-Q-gap in* $G$ if $A$ has no last point ($B$ has no first point) and $\langle A, B\rangle$ is not a left (right) pseudo-$Q$-gap in $G$. The concept of (pseudo)-$Q$-gaps for linearly ordered spaces is originally due to L. Gillman and M. Henriksen [4]; for GO-spaces it was introduced by D. Lutzer [7].

A map $\varphi\colon X \overset{\text{into}}{\to} Y$ is called an *embedding* (*of* $X$ *into* $Y$) if it is a homeomorphism when considered as a map from $X$ onto $\varphi(X)$.

Let $G$ and $G'$ be GO-spaces. A function $\varphi\colon G \to G'$ is said to be *order-preserving* (*order-reversing*) if $\varphi(a) \leqslant \varphi(b)$ whenever $a, b \in G$ and $a \leqslant b$ ($a \geqslant b$).

The elementary proof of the following lemma is omitted.

LEMMA 1. *Let $K$ be a compact ordered space, $p$ a point of $K$ and $\kappa$ a regular cardinal. The the weight of the space $(\leftarrow, p] \subset K$ at the point $p$ is equal to $\kappa$ if and only if there exists an order-preserving embedding $\varphi\colon \kappa+1 \to (\leftarrow, p]$ such that $\varphi(\{\kappa\}) = \{p\}$.*

LEMMA 2. *Let $G$ be a GO-space, $\langle A, B\rangle$ a pseudo-gap in $G$ such that $p = \sup_{\lambda G} A \notin G$. Then*

(a) *$\langle A, B\rangle$ is a left pseudo-Q-gap in $G$ if and only if either $\lambda G$ is first countable at $p$ or there is a regular uncountable cardinal $\kappa$ and an order-preserving embedding $\varphi\colon \kappa+1 \to \lambda G - G$ such that $\varphi(\{\kappa\}) = \{p\}$,*

(b) *$\langle A, B\rangle$ is a left pseudo-non-Q-gap in $G$ if and only if there is a stationary subset $S$ of a regular uncountable cardinal and an order-preserving embedding $\varphi\colon S \to A$ such that $\varphi(S)$ is a closed subset of $G$ and $\sup_{\lambda G}\varphi(S) = p$.*



Proof. (a) I. Let $\langle A, B\rangle$ be a left pseudo-$Q$-gap in $G$. Suppose that $\lambda G$ is not first countable at $p$. Let $\{p_\alpha\colon \alpha < \kappa\} \subset A$ be a sequence from the definition of a left pseudo-$Q$-gap. Notice that $\kappa > \omega$. Put $q_\alpha = \sup_{\lambda G}\{p_\xi\colon \xi < \alpha\}$ for limit $\alpha < \kappa$. Notice that $q_\alpha \in \lambda G - G$. Let $\psi\colon \kappa \to \kappa \cap Lim$ be an increasing enumeration of limit ordinals. Put $\varphi(\alpha) = q_{\psi(\alpha)}$ for $\alpha < \kappa$, and $\varphi(\{\kappa\}) = \{p\}$. Notice that $\varphi$ satisfies the conclusion of our lemma.

II. If $\lambda G$ is first countable at $p$, then there is an increasing sequence $P = \{p_n\colon n < \omega\} \subset \lambda G$ which converges to $p$. Since $G$ is dense in $\lambda G$, we may assume that $P \subset G$. Thus, $\langle A, B\rangle$ is a left pseudo-$Q$-gap in $G$.

Now, suppose that there exists a regular uncountable cardinal $\kappa$ and an order-preserving embedding $\varphi\colon \kappa+1 \to \lambda G - G$ such that $\varphi\{\kappa\}) = \{p\}$. Notice that for every $\alpha < \kappa$ the set $(\varphi(\alpha), \varphi(\alpha+2)) \cap G$ is nonempty. Choose a point $r_\alpha$ from this for every limit $\alpha$, $\alpha < \kappa$. Let $\psi\colon \kappa+1 \to (\kappa+1) \cap Lim$ be an increasing enumeration of limit ordinals. Put $p_\alpha = r_{\psi(\alpha)}$ for $\alpha < \kappa$. Notice that $p_\alpha \in G$ and, for limit $\beta < \kappa$, $\sup_{\lambda G}\{p_\alpha\colon \alpha < \beta\} = \varphi(\psi(\beta)) \in \lambda G - G$. Thus, $\langle A, B\rangle$ is a left pseudo-$Q$-gap in $G$.

(b) I. Let $\langle A, B\rangle$ be a left pseudo-non-$Q$-gap in $G$. Then $\lambda G$ is not first countable at $p$, and so, by Lemma 1, there exists a regular uncountable cardinal $\kappa$ and an order-preserving embedding $\tilde{\varphi}\colon \kappa+1 \to \lambda G$ such that $\tilde{\varphi}(\{\kappa\}) = \{p\}$. By (a), the set $\kappa \cap \tilde{\varphi}^{-1}(\lambda G - G)$ does not contain any closed unbounded subset of $\kappa$, and so the set $\tilde{\varphi}^{-1}(G)$ is stationary. Put $S = \tilde{\varphi}^{-1}(G)$ and $\varphi = \tilde{\varphi}|S$. Observe that $\varphi(S) = G \cap \tilde{\varphi}(\kappa+1)$, and so $\varphi(S)$ is closed relatively to $G$. Obviously, $\varphi(S) \subset A$ and $\sup_{\lambda G}\varphi(S) = p$.

II. Let $S$ be a stationary subset of a regular uncountable cardinal $\kappa$ and let $\varphi\colon S \to A$ be an order-preserving embedding such that $\varphi(S)$ is closed in $G$ and $\sup_{\lambda G}\varphi(S) = p$. Let $\mu$ be a regular cardinal and let $\{p_\alpha\colon \alpha < \mu\} \subset A$ be an increasing sequence such that $\sup_{\lambda G}\{p_\alpha\colon \alpha < \mu\} = p$. Notice that $\mu = \kappa$, since both $\mu$ and $\kappa$ are regular. Both sets $\varphi(S)$ and $\{p_\alpha\colon \alpha < \kappa\}$ have no upper bound in $A$, and so we can define by induction increasing functions $\psi\colon \kappa \to S$ and $\chi\colon \kappa \to \kappa$ such that $\varphi(\psi(\alpha)) < p_{\chi(\alpha)} < \varphi(\psi(\alpha+1))$ for $\alpha < \kappa$. The set $\psi(\kappa)$ is unbounded in $\kappa$, and so there exists an ordinal $\beta \in S \cap [\psi(S)]^d$. Denote $\Lambda = \{\alpha < \kappa\colon \psi(\alpha) < \beta\}$. Observe that $\sup_\kappa \psi(\Lambda) = \beta \in S$, and so $\sup_G \varphi(\psi(\Lambda)) = \varphi(\beta) \in A$. Thus, $\sup\{p_{\chi(\alpha)}\colon \alpha \in \Lambda\} = \sup_G \varphi(\psi(\Lambda)) = \varphi(\beta) \in A$. Hence, $\langle A, B\rangle$ is a left pseudo-non-$Q$-gap in $G$.

Let $G$ be a GO-space. The following notation will be used below.

$G^- = \{\sup A \in \lambda G\colon \langle A, B\rangle$ is a left pseudo-non-$Q$-gap in $G\}$,

$G^+ = \{\inf B \in \lambda G\colon \langle A, B\rangle$ is a right pseudo-non-$Q$-gap in $G\}$, and

$G^* = G \cup G^- \cup G^+$.

LEMMA 3. *The space $G^*$ is paracompact for every* GO-*space $G$.*

Proof. Let $\langle A, B\rangle$ be a pseudo-gap in $G$. Assume that $A \neq \emptyset$ and $p = \sup_{\lambda G} A \notin G$. If $\lambda G$ is first countable at $p$, then $\langle A, B\rangle$ is a left pseudo-$Q$-gap in $G^*$. Assume that $\lambda G$ is not first countable at $p$. Notice that $\langle A \cap G, B \cap G\rangle$ is a pseudo-gap in $G$, and, since $p \notin G^*$, it is a left pseudo-$Q$-gap in $G$. Thus, by

Lemma 2 (a), there exists a regular uncountable cardinal $\kappa$ and an order-preserving embedding $\varphi\colon \kappa+1 \to \lambda G - G$ such that $\varphi(\{\kappa\}) = \{p\}$.

Fix a limit ordinal $\gamma < \kappa$. Let $\psi_\gamma\colon \mathrm{cf}\gamma \to \gamma$ be an increasing continuous function. Define $\hat{\psi}_\gamma\colon (\mathrm{cf}\gamma)+1 \to \gamma+1$ by assuming $\hat{\psi}_\gamma|\mathrm{cf}\gamma = \psi_\gamma$ and $\hat{\psi}_\gamma(\{\mathrm{cf}\gamma\}) = \{\gamma\}$. Notice that the map $\varphi \circ \hat{\psi}_\gamma\colon (\mathrm{cf}\gamma)+1 \to \lambda G - G$ is an order-preserving embedding, and so the pair $\langle(\leftarrow, \varphi(\gamma)) \cap G,\ (\varphi(\gamma), \rightarrow) \cap G\rangle$ is a left pseudo-$Q$-gap in $G$. Thus, $\varphi(\gamma) \notin G^*$.

Hence, $\varphi((\kappa+1) \cap Lim) \subset \lambda G - G$, and so $\langle A, B\rangle$ is a left pseudo-$Q$-gap in $G^*$. Thus, each gap in $G^*$ is a pseudo-$Q$-gap, and so $G^*$ is paracompact in view of D. Lutzer Theorem [5].

Remark. The paracompactness of $G^*$ has an interesting interpretation in the space $G$. For formulating this, let us define a special type of coverings of $G$. An open cover $\mathcal{U}$ of a GO-space $G$ will be called a *$Q$-cover of $G$* if for each left (right) pseudo-non-$Q$-gap $\langle A, B\rangle$ in $G$ there is a set $U \in \mathcal{U}$ ($V \in \mathcal{U}$) and a point $p \in A$ ($q \in B$) such that $(p, \rightarrow) \cap A \subset U$ ($(\leftarrow, q) \cap B \subset V$). Now, Lemma 3 can be equivalently expressed as follows.

Lemma 3′. *Every $Q$-cover of a* GO-*space has a locally finite open refinement.*

Lemma 4. *Let $S$ be a stationary subset of a regular uncountable cardinal $\kappa$ and let $f$ be a closed map from $S$ onto a space $X$. If each fibre of $f$ is a nonstationary subset of $\kappa$, then $X$ contains a closed copy of a stationary set.*

Proof. The family $\{f^{-1}(x)\colon x \in X\}$ is a partition of $S$ onto nonstationary sets, and so, by Pressing Down Lemma, the set $\{\min f^{-1}(x)\colon x \in X\}$ contains a stationary subset $S'$ which is closed relatively to $S$. Thus, the map $f|F\colon S' \to f(S')$ is closed and one-to-one, and so it is a homeomorphism.

Lemma 5. *Let $\varphi$ be an order-preserving embedding of a stationary subset $S$ of a regular uncountable cardinal into a* GO-*space $G$. If $\varphi(S)$ is a closed unbounded subset of $G$, then for every open subset $U$ of $G$ containing $\varphi(S)$ there exists a point $p \in G$ such that $(p, \rightarrow) \subset U$.*

Proof. Let $C(\alpha)$, where $\alpha \in S$, be a convex component of $U$ which contains $\varphi(\alpha)$. Put $f(\alpha) = \min(\varphi^{-1}(C(\alpha)))$. Notice that $f(\gamma) < \gamma$ for every $\gamma \in S \cap S^d$, and so, by Pressing Down Lemma, there exists an element $\gamma_0$ such that $f^{-1}(\{\gamma_0\})$ is stationary. Observe that the point $p = \varphi(\gamma_0)$ satisfies the conclusion of our lemma.

Let $G$ be a GO-space and $U$ an open subset of $G$. We will use the following notation.

$$\tilde{U} = U \cup \{p \in G^-\colon \text{there is a point } r < p \text{ such that } (r, p) \cap G \subset U\} \cup$$
$$\cup \{q \in G^+\colon \text{there is a point } s > q \text{ such that } (q, s) \cap G \subset U\}.$$

Lemma 6. *Let $G$ be a* GO-*space and $U$ an open subset of $G$. If $L$ is an open convex subset of $\lambda G$ such that $L \cap G \subset U$, then $L \cap G^* \subset \tilde{U}$.*

Proof. Fix a point $p \in L \cap G^-$. The point $p$ is not the right end of a jump in $\lambda G$, and so there is a point $r < p$ such that $(r, p] \subset L$. Thus, $(r, p) \cap G \subset L \cap$ $\cap G \subset U$, and so $p \in \tilde{U}$. Hence, $L \cap G^- \subset \tilde{U}$; analogously, $L \cap G^+ \subset \tilde{U}$, and so $L \cap G^* \subset \tilde{U}$.





Lemma 7. *Let $G$ be a* GO-*space. Then for every open subset $U$ of $G$ the set $\tilde{U}$ is open in $G^*$.*

Proof. Let $U$ be an open subset of $G$ and $p$ a point of $\tilde{U}$.

If $p \in U$, then there exist points $a, b \in \lambda G$ such that $p \in \mathrm{int}_{\lambda G}[a, b] \cap G \subset U$. By Lemma 6, we have $p \in \mathrm{int}_{\lambda G}[a, b] \cap G^* \subset \tilde{U}$, and so $p \in \mathrm{int}_{G^*}\tilde{U}$.

If $p \in \tilde{U} \cap G^-$, then there is a point $r < p$ such that $(r, p) \cap G \subset U$. By Lemma 6, $(r, p) \cap G^* \subset \tilde{U}$. Since $p$ is the left end of a jump in $\lambda G$, the set $(r, p]$ is open in $\lambda G$. Thus, $p \in (r, p] \cap G^* \subset \mathrm{int}_{G^*}\tilde{U}$.

The case of $p \in \tilde{U} \cap G^+$ is analogous.

Lemma 8. *Let $G$ be a* GO-*space, $F$ a closed subset of $G^*$ and $U$ an open subset of $G^*$ containing $F$. Then there exists an open set $W \subset G^*$ such that $F \subset W \subset U$ and for every convex component $C$ of $W$, if $\sup_{\lambda G} C \in G^-$, then $\sup_{\lambda G} C \in W$, and if $\inf_{\lambda G} C \in G^+$, then $\inf_{\lambda G} C \in W$.*

Proof. Let $D$ be a convex component of $U$. Notice that the set $F \cap U$ is closed in $G^*$.

Assume that $x = \inf_{\lambda G} D \in G^+ - D$. Then $x \notin F$ and, since $x$ is not the left end of a jump in $\lambda G$, there is a point $p \in D \cap G$ such that $(x, p] \cap F = \emptyset$. Put $D' = D \cap$ $\cap (p, \rightarrow)$. In the case when $x \notin G^+ - D$, put $D' = D$.

Analogously, if $y = \sup_{\lambda G} D \in G^- - D$, then there is a point $q \in D \cap G$ such that $[q, y) \cap F = \emptyset$. In this case, put $C(D) = D' \cap (\leftarrow, q)$; in the opposite case, put $C(D) = D'$. Notice, that the set $W = \bigcup \{C(D)\colon D$ is a convex component of $U\}$ satisfies the conclusion of our lemma.

Lemma 9. *Let $G$ be a* GO-*space and $V$ an open subset of $G^*$ such that for every convex component $C$ of $V$, if $\sup_{\lambda G} C \in G^-$, then $\sup_{\lambda G} C \in V$ and if $\inf_{\lambda G} C \in G^+$, then $\inf_{\lambda G} C \in V$. Then $\widetilde{V \cap G} = V$.*

Proof. Fix a point $p \in \widetilde{V \cap G} \cap G^-$.

Since $p \in \widetilde{V \cap G} - G$, there is a point $q < p$ such that $(q, p) \cap G \subset V$. The point $p \in G^-$, and so, by Lemma 2 (b), there is a stationary subset S of a regular uncountable cardinal and an order-preserving embedding $\varphi\colon S \to (q, p) \cap G$ such that $\varphi(S)$ is closed in $G$ and $\sup \varphi(S) = p$.

Suppose that $p \notin V$. Put $F = ([q, p) \cap G^*) - V$. Observe that $F$ is closed in $G^* - \{p\}$. Let $C$ be a convex component (in $G^*$) of the set $(q, p) \cap V$. Since $p \notin V$, we have $\sup C < p$. Thus, $\sup F = p$.

Define a function $\psi\colon F \to \varphi(S)$ by assuming $\psi(x) = \min\{y \in \varphi(S)\colon x < y\}$. The set $\varphi^{-1}(\psi(F))$ is unbounded in $S$, and so there is an ordinal $\alpha \in S \cap [\varphi^{-1}(\psi(F))]^d$. Notice that $\varphi(\alpha) \in F^d \cap \varphi(S)$, and so $F \cap V \supset F \cap \varphi(S) \neq \emptyset$; a contradiction.

Hence, $\widetilde{V \cap G} \subset V$. The reverse inclusion is obvious.

The following lemma results immediately from Lemmas 8 and 9.

LEMMA 10. *Let $G$ be a* GO*-space, $F$ a closed subset of $G^*$ and $U$ an open subset of $G^*$ containing $F$. Then there exists an open set $V \subset G^*$ such that $F \subset V \subset U$ and $\widetilde{V \cap G} = V$.*

THEOREM. *Let $f$ be a closed continuous map from a* GO*-space $G$ onto a space $X$. If $X$ does not contain any closed subset homeomorphic to a stationary subset of a regular uncountable cardinal, then there exists a closed continuous map $\tilde{f}$ from $G^*$ onto $X$ such that $\tilde{f}|G = f$. Consequently, the space $X$ is paracompact.*

Proof. Put $\tilde{f}(p) = f(p)$, for $p \in G$.

Fix a point $q \in G^-$. By Lemma 2 (b), there is a stationary subset $\hat{S}$ of a regular uncountable cardinal and an order-preserving embedding $\varphi\colon \hat{S} \to G$ such that $\varphi(\hat{S})$ is closed in $G$ and $\sup_{G^*}\varphi(\hat{S}) = q$. The space $X$ does not contain closed copies of stationary sets, and so, by Lemma 4, there is a (unique) point $x \in X$ such that $\varphi^{-1}(\varphi(\hat{S}) \cap f^{-1}(x))$ is a stationary subset of $\hat{S}$. Denote $S(q) = \varphi(\hat{S}) \cap f^{-1}(x)$. Notice that $\sup_{G^*} S(q) = q$. Put $\tilde{f}(q) = x$.

Analogously, if $q \in G^+$, then there exists a closed subset $S(q)$ of $G$ which is homeomorphic to a stationary set (by an order-reversing homeomorphism) and such that $\inf_{G^*} S(q) = q$, and a point $x \in X$ such that $f(S(q)) = \{x\}$. Define $\tilde{f}(q) = x$.

Let us observe that $\tilde{f}^{-1}(U) = \widetilde{f^{-1}(U)}$ for each open subset $U$ of $X$.

In fact, $\tilde{f}^{-1}(U) \cap G = f^{-1}(U) = \widetilde{f^{-1}(U)} \cap G$. If $q \in \tilde{f}^{-1}(U) \cap G^-$, then $S(q) \subset f^{-1}(U)$. Thus, by Lemma 5, there is a point $r < q$ such that $(r, q) \cap G \subset f^{-1}(U)$. But this implies that $q \in \widetilde{f^{-1}(U)}$. Conversely, if $q \in \widetilde{f^{-1}(U)} \cap G^-$, then there is a point $r < q$ such that $(r, q) \cap G \subset f^{-1}(U)$. Notice that the set $S' = S(q) \cap (r, q)$ is nonempty, and so $f(\{q\}) = f(S(q)) = f(S') \subset U$. Thus, $q \in \tilde{f}^{-1}(U)$, and so $\widetilde{f^{-1}(U)} \subset \tilde{f}^{-1}(U)$.

Hence, in view of Lemma 7, the map $\tilde{f}$ is continuous. We will show that it is closed as well.

Fix a point $x \in X$ and an open set $U \subset G^*$ which contains $\tilde{f}^{-1}(x)$. By Lemma 10, there is an open set $V \subset G^*$ such that $\tilde{f}^{-1}(x) \subset V \subset U$ and $\widetilde{V \cap G} = V$. The map $f\colon G \to X$ is closed, and so there is an open set $W \subset X$ such that $f^{-1}(x) \subset f^{-1}(W) \subset V \cap G$. Thus, $\tilde{f}^{-1}(x) \subset \tilde{f}^{-1}(W) = \widetilde{f^{-1}(W)} \subset \widetilde{V \cap G} = V \subset U$, and so $\tilde{f}$ is a closed map from $G^*$ onto $X$.

Finally let us note that, by Lemma 3, the space $G^*$ is paracompact, an so $X$ is paracompact as well, in view of E. Michael's Theorem [8] (see also [1], Theorem 5.1.33).

COROLLARY 1. *If a paracompact space is an image of a* GO*-space under a closed continuous map, then it is an image of some paracompact* GO*-space under a closed continuous map as well.*

In view of E. Michael's Theorem [9] (see also [1], Problem 5.5.11 (e)), every closed continuous map from a paracompact space is compact-covering. Thus, we have the following.



COROLLARY 2. *If a compact space is an image of a* GO*-space under a closed continuous map, then it is a continuous image of some compact ordered space as well.*

In the following corollary we formulate the announced characterization of paracompactness.

COROLLARY 3. *Let a space $X$ be an image of a* GO*-space under a closed continuous map. The $X$ is not paracompact if and only if it contains a closed subset which is homeomorphic to a stationary subset of a regular uncountable cardinal.*

COROLLARY 4. *Let a space $X$ be an image of a* GO*-space under a closed continuous map. Then $X$ is not hereditarily paracompact if and only if it contains a subset which is homeomorphic to a stationary subset of a regular uncountable cardinal.*

**References**

[1] R. Engelking, *General Topology*, Warszawa 1977.
[2] R. Engelking and D. Lutzer, *Paracompactness in ordered spaces*, Fund. Math. 44 (1977), 49–58.
[3] V. V. Fedorčuk, (В. В. Федорчук), *Некоторые вопросы теории упорядоченных пространств*, Сибирский Мат. Журн. 10 (1969), 172–187. (Some questions in the theory of ordered spaces; English transl.: Siberian Math. J. 10 (1969), 124–132).
[4] L. Gillman and M. Henriksen, *Concerning rings of continuous functions*, Trans. Amer. Math. Soc. 77 (1954), 340–362.
[5] R. Kaufman, *Ordered sets and compact spaces*, Colloq. Math. 17 (1967), 35–39.
[6] K. Kuratowski and A. Mostowski, *Set Theory*, Warszawa 1976.
[7] D. Lutzer, *Ordinals and paracompactness in ordered spaces*; in: TOPO 72, Proc. of the Second Pittsburgy Topology Conference, Lecture Notes in Math. 378, 258–266.
[8] E. Michael, *Another note on paracompact spaces*, Proc. Amer. Math. Soc. 8 (1957), 822–828.
[9] E. Michael, *A note on closed maps and compact sets*, Israel J. Math. 2 (1964), 173–176.

INSTYTUT MATEMATYCZNY PAN
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
Akademicka 2
15-267 Białystok

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