Increasing strengthenings of cardinal function inequalities

by

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Abstract. We prove that the following increasing strengthenings of two cardinal function inequalities given in [2] and [1] respectively are valid.

Theorem 1. If $X$ is $T_4$ and $X = \bigcup X_\alpha$ (i.e., $X$ is the union of an increasing chain of its subspaces $X_\alpha$) and $c(X_\alpha) = \aleph_0$ for all $\alpha$ then $|X| \leq 2^\aleph_0$.

Theorem 2. If $X$ is $T_4$ and $X = \bigcup X_\alpha$, where $X_\alpha$ is $T_4$ and $wL(X_\alpha) = \aleph_0$ for all $\alpha$, then $|X| \leq 2^\aleph_0$.

In [3] the first author has initiated the study of strengthening certain cardinal function inequalities in the following manner. A general form of a cardinal function inequality may be given as follows: If $\varphi$ is some given cardinal function and $X$ is a space having some property $P$ then $\varphi(X) \leq \aleph_0$. We call an increasing strengthening of this inequality any statement of the following form: If $X = \bigcup X_\alpha$ is the increasing union of its subspaces $X_\alpha$, where every $X_\alpha$ has property $P$ and $X$ has property $Q$ then $\varphi(X) = \aleph_0$.

A number of such increasing strengthenings of inequalities were proven in [3] as a major problem, however, it remained open whether the inequality $|X| \leq 2^{|\chi(X)|}$, for any $T_4$ space $X$, admits such an increasing strengthening.

Theorem 1 of the present paper gives the affirmative answer to this question. The ideas needed in the proof of Theorem 1, with appropriate modifications, also allowed us to show that the inequality $|X| \leq 2^{\chi(X_\alpha)|X_\alpha|}$ for any $T_4$ space $X$ proved in [1] also admits an increasing strengthening.

Notation and terminology, unless otherwise explained, is identical with that used in [3].

**Theorem 1.** If $X = \bigcup X_\alpha$ is $T_4$ and $c(X_\alpha) = \aleph_0$ holds for each $\alpha$ then $|X| \leq 2^\aleph_0$. 

The proof of Theorem 1 will be based on three lemmas given below.

**Lemma 1.** Let $X$ be an arbitrary space, $Y$ a subspace of $X$ and $x$ a point in $Y$, moreover $\mathcal{B}$ be a complete subalgebra of $RO(X)$ (in symbols $\mathcal{B} \triangleleft RO(X)$), the complete Boolean algebra of all regular open subsets of $X$, such that

$$(*) \quad \text{for every } \mathcal{B} \cup \mathcal{B}, \text{ if } p \in \mathcal{B} \text{ then } p \in \mathcal{B} \cap Y.$$

Then for every open neighbourhood $U$ of $x$ in $X$ there is a member $B(U) \in \mathcal{B}$ such that $p \in B(U)$ and if $B \in \mathcal{B}$ satisfies $U \cap Y \subseteq B$ then $B(U) \subseteq B$ is also valid.

**Proof.** Let $\mathcal{C}$ be the collection of all members $C \in \mathcal{B}$ satisfying

$$C \cap U \cap Y = \emptyset.$$ 

Then $U \cap \cup C \cap U \cap Y = \emptyset$ holds as well, hence $p \notin \cup C \cap U \cap Y$ consequently, by $(*)$, $p \notin \cup C$. We claim that

$$B(U) = X \setminus \cup C \in \mathcal{B}$$

is as required. Now $p \in B(U)$ is obvious.

Next, if $B \in \mathcal{B}$ and $U \cap Y \subseteq B$ then clearly $C = X \setminus B \in \mathcal{C}$, hence $C \subseteq \cup C$, and thus

$$B(U) = X \setminus \cup C \subseteq X \setminus C = B.$$ 

**Remark.** If we have $c(\mathcal{B}) \ll \kappa$, i.e. the cellularity of $\mathcal{B}$ is $\ll \kappa$, which is true e.g. if $c(X) \ll \kappa$, then in Lemma 1 $(*)$ may clearly be replaced by the following weaker condition:

$$(*)_{\kappa} \quad \text{for every } \mathcal{B} \cup \mathcal{B}, \text{ if } p \in \mathcal{B} \text{ then } p \in \mathcal{B} \cap Y.$$ 

Before we formulate our next lemma we need some definitions. First, if a space is the union of its subspaces $X_\alpha$, we say that $X$ has the fine topology with respect to the system $\{X_\alpha\}$ of these subspaces provided that $G \subseteq X$ is open in $X$ if and only if $G \cap X_\alpha$ is open in the subspace $X_\alpha$ for all $\alpha$. Clearly this means that $X$ has the finest topology with respect to which all the $X_\alpha$ have the same induced subspace topology.

We shall need the following simple proposition concerning increasing unions with the fine topology.

**Proposition.** Let $X = \bigcup \{X_\alpha; \alpha \in \lambda\}$ where $\lambda$ is a regular cardinal and $t(p, X_\alpha) < \lambda$ holds for every $\alpha \in \lambda$ and $p \in X_\alpha$ and assume that $X$ has the fine topology with respect to the system $\{X_\alpha; \alpha \in \lambda\}$. Then for every set $A \subseteq X$ we have

$$\mathcal{A} = \bigcup \{A \cap X_\alpha; \alpha \in \lambda\}.$$ 

**Proof.** Clearly it suffices to show that the right-hand side of this equality, let us denote it by $\mathcal{B}$ for short, is closed in $X$. Since $X$ has the fine topology, however, this is equivalent to showing that $B \cap X_\alpha$ is closed in $X_\alpha$ for each $\beta \in \lambda$. But

$$B \cap X_\alpha = \bigcup \{A \cap X_\alpha \cap X_\beta; \alpha \in \lambda\}$$

is an increasing $\lambda$-type union of closed subsets of $X_\alpha$ which is indeed closed in $X_\alpha$ since we have $t(p, X_\alpha) < \lambda$ for all $p \in X_\alpha$.

Now we are ready to formulate the second lemma needed for the proof of Theorem 1.

**Lemma 2.** Let $X = \bigcup \{X_\alpha; \alpha \in \lambda\}$, where $\lambda = (2^\kappa)^+$, $X$ has the fine topology w.r.t. $\{X_\alpha; \alpha \in \lambda\}$ and $x(X_\beta) \ll \kappa$ for each $\alpha \in \lambda$. Then $\mathcal{B} \triangleleft RO(X)$, $c(\mathcal{B}) \ll \kappa$ and $|\mathcal{B}| \ll \lambda$ imply $x(p, \mathcal{B}) \ll \kappa$ for all $p \in X$.

**Proof.** Let us first assume that actually $|\mathcal{B}| \ll \lambda$. Given $p \in X$, for every $\mathcal{B} \cup \mathcal{B}$ there is an ordinal $\alpha \epsilon \lambda$ such that $p \in \mathcal{B} \cup \mathcal{B} \cap X_\alpha$ since $X$ has the fine topology and

$$t(p, X_\alpha) < x(p, X_\alpha) \ll \lambda$$

is valid for all $\alpha \epsilon \lambda$; hence the above proposition can be applied. Since

$$|(\mathcal{B})^{*\kappa}| \ll |\mathcal{B}| = (\kappa)^{\omega} = 2^\kappa = \ll \lambda,$$

we may then find $\alpha_0 \epsilon \lambda$ such that $p \in X_{\alpha_0}$ and $x(p) \ll \alpha_0$ for all $\mathcal{B} \cup \mathcal{B}$. Clearly, then $(\mathcal{B})_{\alpha_0} \ll \kappa$, $x(p) \ll \alpha_0$ also $(\mathcal{B})$, of Lemma 1, will be satisfied for $p, \mathcal{B}$ and $Y = X_{\alpha_0}$.

Now let $\{U_\nu; \nu \epsilon \kappa\}$ be a family of open neighbourhoods of $p$ in $X$ such that $\{U_\nu; \nu \epsilon \kappa\}$ is a neighbourhood base of $p$ in $X_{\alpha_0}$. We may then apply Lemma 1 for $p, \mathcal{B}, Y = X_{\alpha_0}$ and each $U_\nu$ to obtain $B_\nu \epsilon \mathcal{B}$ such that $p \in B_\nu$ and $B_\nu \subseteq B$ whenever $U_\nu \cap X_{\alpha_0} \subseteq B$ hold. However, then $\{B_\nu; \nu \epsilon \kappa\}$ clearly establishes $x(p, \mathcal{B}) \ll \kappa$ since for every $B \cup \mathcal{B}$ with $p \in B$ there is a $\nu \epsilon \kappa$ with $U_\nu \cap X_{\alpha_0} = B$.

Now, assume that $|\mathcal{B}| \ll \lambda$. Applying $c(\mathcal{B}) \ll \kappa$ we may then write

$$\mathcal{B} = \bigcup \{\mathcal{B}_\alpha; \alpha \in \lambda\}$$

where $|\mathcal{B}_\alpha| < \kappa$ and $\mathcal{B}_\alpha \ll \mathcal{B}$ for each $\alpha \in \lambda$. We may also assume that if $\alpha \epsilon \lambda$ and $c(\mathcal{B}_\alpha) > \kappa$ then

$$\mathcal{B}_\alpha = \bigcup \{\mathcal{B}_\beta; \beta \epsilon \alpha\}.$$ 

Let us put $S = \{\alpha \epsilon \lambda; c(\mathcal{B}_\alpha) > \kappa\}$. For every $\alpha \epsilon S$ we may apply the above partial result to $\mathcal{B}_\alpha$ to obtain $\mathcal{B}_\alpha \cup \mathcal{B}_\alpha$ which is a basis of $p$ in $\mathcal{B}_\alpha$. Since $c(\mathcal{B}_\alpha) > \kappa$ and $\mathcal{B}_\alpha \ll \mathcal{B}$ we have some $\phi(\alpha) \epsilon \alpha$ such that

$$\mathcal{B}_\alpha \subseteq \mathcal{B}_{\phi(\alpha)}.$$ 

The function $\phi$ thus defined is regressive on the stationary subset $S$ of $\lambda$, hence by Neumer's theorem there is some $\beta \epsilon \lambda$ and $S_1 \subseteq S$ such that $\phi(\alpha) = \beta$ for all $\alpha \epsilon S_1$. But

$$|(\mathcal{B})^{*\kappa}| \ll (2^\kappa)^+ = 2^\kappa.$$
then implies the existence of some \( \mathcal{U} \in \mathcal{A} \) such that for all \( x \in X \), hence in \( \mathcal{B} \) as well.

The following lemma is actually a variant of the inequality \( |X| \leq \omega^{\omega^{\omega^{\omega^{\omega}}}} \) for \( X \neq \emptyset \).

**Lemma 3.** If \( X \) is a set, \( \mathcal{B} = P(X) \) is a family of subsets of \( X \) that \( T \) separates the points of \( X \), \( \mathcal{B} \) is closed under finite intersections, \( c(\mathcal{B}) \leq \kappa \) and \( c(\mathcal{B}, \kappa) \leq \kappa \) for all \( \kappa \in X \) then \( |X| \leq 2^\kappa \).

**Proof.** A direct proof based on the Erdős–Rado theorem \((\omega^\omega)^+ \rightarrow (\kappa^+)^2\) could be given; however, the lemma also follows from the above cardinal function inequality as applied to the topology on \( X \) generated by \( \mathcal{B} \).

Now we may turn to the proof of Theorem 1. Note that if \( X = \bigcup \{ X_a : a \in \mu \} \in \mathcal{F}_X \) and \( \chi(X_a) : c(X) \leq \kappa \) then \( |X| \leq 2^\kappa \), hence we may assume that \( \mu < \kappa = (2^\kappa)^+ \).

But if \( \mu < \kappa \) then
\[ |X| \leq 2^\mu \cdot |\mu| \leq 2^\kappa, \]

hence it will suffice to show that \( \mu = \kappa \) is impossible.

Assume, reasoning indirectly, that \( \mu = \kappa \). Clearly we may also assume that \( X \) has the fine topology w.r.t. \( \{ X_a : a \in \mu \} \) since this topology on \( X \) is also \( T \). Since \( c(X) \leq \kappa \) holds for all \( x \in X \), we have (e.g. by [13, 6, 1]) \( c(X) \leq \kappa \). Clearly, we have \( |X| = \kappa \); hence by \( X \in \mathcal{F}_X \) and by \( c(\mathcal{R}(X)) \leq c(X) \leq \kappa \) we may find a complete subalgebra \( \mathcal{B}_X < \mathcal{R}(X) \) with \( |\mathcal{B}_X| = \kappa \) that \( T \) separates the points of \( X \). By Lemma 2 then \( \chi(X_a) \leq \kappa \) is valid for all \( a \in X \). But then, by \( c(\mathcal{B}) < c(\mathcal{R}(X)) \leq \kappa \), Lemma 3 may also be applied to \( X \) and \( \mathcal{B} \), consequently we must have \( |X| \leq 2^{\kappa} \), a contradiction. This completes the proof of Theorem 1.

Now we turn to giving an increasing strengthening of the inequality \( |X| \leq 2^{\omega(X#,\omega)} \) proved in [1] for \( X \in \mathcal{F}_X \). Let us note that it is still open whether this inequality is valid for \( X \in \mathcal{F}_X \) as well. In any case our increasing strengthening will only require \( X \) to be \( T_2 \), whereas of course \( X \in \mathcal{F}_X \) will be assumed.

**Theorem 2.** If \( X = \bigcup \{ X_a : a \in \mu \} \in \mathcal{F}_X \) is a subset of \( X \) and \( \omega(X,\omega) \) holds for each \( a \) then \( |X| \leq 2^{\omega(X,\omega)} \).

The proof of Theorem 2 runs analogously to that of Theorem 1 and is based on three analogous lemmas.

**Lemma 4.** Let \( X \) be a space, \( Y \in \mathcal{F}_Y \) a subspace of \( X \) with \( \omega(X,\omega) \leq \kappa \) and \( Y \in \mathcal{S} \) be such that \( \chi(p, Y) < \kappa \) and \( \omega(Y,\omega) = \kappa \). Assume furthermore that \( \mathcal{S} \) is a \( \mu \)-complete subalgebra of \( \mathcal{R}(X) \) in symbols: \( \mathcal{S} = \mathcal{R}(X) \). This means that \( \text{Int} \bigcup \mathcal{S} \in \mathcal{A} \) for all \( \mathcal{S} \in \mathcal{S} \). Let \( (p, X) \in \mathcal{A} \) for all \( \mathcal{S} \in \mathcal{S} \). Assume further that \( \mathcal{S} \) is a \( \mu \)-complete subalgebra of \( \mathcal{R}(X) \) in symbols: \( \mathcal{S} = \mathcal{R}(X) \). This means that \( \text{Int} \bigcup \mathcal{S} \in \mathcal{A} \) for all \( \mathcal{S} \in \mathcal{S} \). Let \( (p, X) \in \mathcal{A} \) for all \( \mathcal{S} \in \mathcal{S} \). Then \( (p, X) \in \mathcal{A} \) for all \( \mathcal{S} \in \mathcal{S} \).

**Proof.** First, since \( (p, X) \in \mathcal{A} \), \( \text{Int} \bigcup \mathcal{S} \in \mathcal{A} \) for all \( \mathcal{S} \in \mathcal{S} \), we can apply the above proposition to conclude that \( X = \bigcup \{ x_a : a \in \kappa \} \) for each set \( X \subseteq \mathcal{X} \). Clearly, this implies then that \( (p, X) \in \mathcal{A} \) for all \( p \in X \).

Let us fix some \( p \in X \). In order to show that \( \chi(p, X) < \kappa \), let us first decompose \( \mathcal{A} \) into an increasing union
\[ \mathcal{A} = \bigcup \{ \mathcal{S}_a : a \in \alpha \}. \]
where \( \mathcal{A}_x \triangleleft \mathcal{A} \) and \( |\mathcal{A}_x| < \lambda \) for each \( x \in \lambda \). We may clearly assume that if \( x \in \lambda \) with \( \gamma(x) > x \) then \( \mathcal{A}_x = \bigcup \{ \mathcal{B}_x : \beta \in x \} \).

In view of our assumptions (which imply \( |X| < 2^\omega < \lambda \) for all \( x \in \lambda \)) we may easily define a map \( \varphi : \lambda \rightarrow \lambda \) such that the following two conditions be valid for all \( x \in \lambda \):

1. if \( \gamma \in [X]^{\aleph \omega} \) and \( \gamma \notin \mathcal{A} \) then there is some \( \mathcal{B} \subseteq \mathcal{A} \) with \( \gamma \subseteq \mathcal{B} \) and \( \mathcal{B} \notin \mathcal{A} \);
2. if \( \gamma \in [X]^{\aleph \omega} \) and \( \gamma \in \bigcup \mathcal{A} \) then \( \mathcal{B} \in \mathcal{A} \).

Let us fix, for \( x \in \lambda \), a family \( \mathcal{G}_x \) of open neighborhoods of \( x \) in \( \mathcal{A}_x \) such that \( |\mathcal{G}_x| < \kappa \) and

\[ \{ U \cap X_x : U \in \mathcal{G}_x \} \]

is a neighborhood basis of \( x \) in \( \mathcal{A}_x \). Then applying Lemma 1' we consider for each \( U \in \mathcal{G}_x \) the set \( \mathcal{B}(U) \subseteq \mathcal{A}_x \) satisfying \( \gamma \in \mathcal{B}(U) \) and \( \mathcal{B}(U) \in \mathcal{A} \) whenever \( \gamma \in \mathcal{A} \) and \( U \cap Y \in \mathcal{B} \). Since \( \gamma(x) > x \) we may then find for every \( x \in \lambda \) an ordinal \( \psi(x) < \kappa \) such that

\[ \mathcal{V}_x = \{ \mathcal{B}(U) : U \in \mathcal{G}_x \} \subseteq \mathcal{A}^{\omega \omega} \]

But then an application of Neumer's theorem and a simple counting argument yields us a set \( S \subseteq \mathcal{V}_x \) of \( \mathcal{V}_x \), ordinal \( \beta \in \lambda \) and a family \( \mathcal{G}_x \in \mathcal{V}_x^\omega \) such that

\[ \mathcal{G}_x = \mathcal{V}_x \text{ for all } \beta \in S \]

We claim that \( \mathcal{G}_x \) is a basis for \( x \) in \( \mathcal{A} \). Assume, indirectly, that \( \gamma \in \mathcal{A} \) but \( \mathcal{C} \cup \mathcal{G}_x \notin \mathcal{A} \) for all \( C \subseteq \mathcal{C} \) such that \( (\mathcal{C} \cup \mathcal{G}_x) \cap X_x \notin \mathcal{B} \) as well. But now \( \mathcal{V}_x \cap \mathcal{G}_x = \mathcal{B}(U) \cap \mathcal{G}_x \) and thus there is some \( U \in \mathcal{G}_x \) with \( U \cap X_x \in \mathcal{B} \) hence \( \mathcal{B}(U) \in \mathcal{A} \) as well, contradicting \( \{ S \cup \mathcal{B}(U) : U \in \mathcal{G}_x \} \) is a \( \mathcal{G}_x \)-basis \( \mathcal{G}_x \in \mathcal{V}_x^\omega \) and \( \mathcal{B}(U) \notin \mathcal{A} \).

Let \( S = \bigcup \{ \mathcal{B}_x : \beta \in \lambda \} \)

Then there is a point \( x \in X \) with \( \chi(x, \mathcal{A}_x) > x \).

Proof. Assume, indirectly, that for each \( x \in X \) there is a \( \mathcal{A}_x \)-basis \( \mathcal{G}_x \in \mathcal{A}_x^\omega \).

For \( x \in \lambda \) we put

\[ \mathcal{G}_x = \{ \gamma \in [X]^{\aleph \omega} : X_x \cap \bigcup \mathcal{A} \neq X \}. \]

By our assumptions we have \( |\mathcal{G}_x| < 2^\omega \). For each \( x \in \lambda \) we may then find an ordinal \( \varphi(x) \in \lambda \) such that

\[ \chi(x, \mathcal{A}_x) \neq \varphi(x) \]

for all \( x \in \mathcal{G}_x \).

Let \( \chi(x) = \beta \) be such that \( \beta \in x \) implies \( \beta \in x \) (there is a closed bounded set of such ordinals \( x \)) and moreover satisfying \( \gamma(x) > x \). Let us pick \( B \subseteq \mathcal{A} \) in such a way that \( X < \gamma \) and \( \gamma \neq \mathcal{A} \). For every \( \gamma \in X \) we may then find a set \( C_\gamma \in \mathcal{G}_\gamma \)

such that \( \gamma \in \bigcup \mathcal{A} \). But

\[ \bigcup \mathcal{A} \subseteq \mathcal{A} \]

hence \( \gamma \supseteq \mathcal{A}_x \) and since \( \gamma(x) > x \) we actually have some \( \beta \in x \) such that \( \gamma \in \mathcal{G}_\gamma \).

But then \( \gamma \in \mathcal{G}_\gamma \) holds, i.e.

\[ \gamma \in \bigcup \mathcal{A} \]

\[ \mathcal{A}_x \subseteq \mathcal{G}_\gamma \]

contradicting \( X \subseteq \bigcup \mathcal{A} \). This completes the proof of Lemma 3.■

The proof of Theorem 2 can now be finished as follows. Since again \( |X| < 2^\omega \) for each \( \gamma \), it suffices to show that our increasing union has length \( < \lambda \).

Assume otherwise, i.e. \( \gamma = \bigcup \{ \gamma_x : x \in \lambda \} \) and \( |X| > \lambda \). Since \( X \) is \( \mathcal{A} \) and for every \( x \in \lambda \) and \( \alpha \in \lambda \)

\[ \chi(p, \mathcal{A}_x) \in \mathcal{A}_x \]

it follows e.g. from [3], 2.5 that \( |X| > 2^\omega \), hence \( X \neq \mathcal{A} \). Thus by the regularity of \( \lambda \) we may clearly find \( \mathcal{B} \subseteq X \) such that \( |\mathcal{B}| \leq \lambda \) and

(i) if \( p \in X \) then \( S \subseteq \mathcal{B} \) and \( \mathcal{B} \notin \mathcal{A} \),

(ii) for every \( \alpha \in \lambda \) there is some \( B \subseteq \mathcal{B} \) with

\[ X_x \subseteq B \neq X \]

Now if we consider the fine topology \( \mathcal{A} \) on \( X \) w.r.t. \( X \) then this topology may not be \( \mathcal{A} \), however the existence of \( \mathcal{A} \subseteq \mathcal{A} \) with \( |\mathcal{A}| \leq \lambda \) and with properties (i) and (ii) will remain valid. For (i) this makes use of the fact that every \( \mathcal{A} \subseteq \mathcal{A} \) is contained in some \( X \), hence by \( |X| < 2^\omega \) we have some \( \beta \in \lambda \) with \( S \subseteq X_x \subseteq X_x \) and thus \( S \subseteq X \). The rest of (i) and (ii) follow easily because for any \( B \in \mathcal{A} \) one clearly has

\[ B \subseteq \bigcup \mathcal{A} \subseteq \mathcal{A} \]

and \( \mathcal{A} \subseteq \mathcal{A} \subseteq \mathcal{A} \).

and \( \mathcal{A} \subseteq \mathcal{A} \subseteq \mathcal{A} \).
Modules over arbitrary domains II

by

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Abstract. Let $R$ be a commutative ring and $S \subseteq R$ a multiplicatively closed subset of $R$. Defining torsion-free modules with respect to $S$, we derive new results of this category extending from $|S| = \aleph_0$. In §§3 we realize any $R$-algebra $A$ with torsion-free, reduced $R$-module structure on modules $G$ as

$$\text{End} G = \mathbb{A} \otimes \text{Ins} G$$

where $\text{Ins} G$ are all endomorphisms of $G$ with $\omega$-complete image in $G$. In §§9 we determine $\text{Ins} G$ more explicitly and derive properties of $G$ from the given algebra $A$.

§ 1. Introduction. We will discuss right $R$-modules $G = G_x$ over nonzero commutative rings $R$. The ring $R$ will have a fixed multiplicatively closed subset $S$ such that $R$ as an $R$-module is $S$-reduced and $S$-torsion-free. These well-known conditions on a module $G$ are $\bigcap G_s = 0$ respectively $(gt = 0 = g = 0)$ for all $g \in G$, $s \in S$.

Many questions on the existence of $R$-modules with prescribed properties can be reduced to representation theorems of $R$-algebras $A$ as endomorphism algebras — in many cases modulo some "small" or "inessential" endomorphisms. Well-known examples for such problems are decomposition-properties related with the Krull–Remak–Schmidt Theorem — respectively related with Kaplansky’s test problems, other derive from questions on prescribed automorphism groups or topologies. The investigation of classical problems in module theory in this sense goes back to a number of fundamental papers by A. L. S. Corner; see [CG] for further references.

In the recent years these investigations have been extended to $R$-modules of arbitrary large size, however under the restriction that $S$ is essentially countable; see [DG 1,2], [GS 1], [S 2,3] and [CG] for a uniform treatment and further extensions, including torsion, mixed and torsion-free $R$-modules.

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2 — Fundamenta Mathematicae CXXVI.