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Increasing strengthenings of cardinal function inequalities

by

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Abstract. We prove that the following *increasing strengthenings* of two cardinal function inequalities given in [2] and [1] respectively are valid.

THEOREM 1. *If X is T_2 and $X = \bigcup\limits_{\alpha}^{\uparrow} X_\alpha$ (i.e. X is the union of an increasing chain of its subspaces X_α) and $c(X_\alpha)\cdot\chi(X_\alpha) \leqslant \varkappa$ for all α then $|X| \leqslant 2^{\varkappa}$.*

THEOREM 2. *If X is T_3 and $X = \bigcup\limits_{\alpha}^{\uparrow} X_\alpha$, where X_α is T_4 and $wL(X_\alpha)\cdot\chi(X_\alpha) \leqslant \varkappa$ for all α then $|X| \leqslant 2^{\varkappa}$.*

In [3] the first author has initiated the study of strengthening certain cardinal function inequalities in the following manner. A general form of a cardinal function inequality may be given as follows: If φ is some given cardinal function and X is a space having some property P then $\varphi(X) \leqslant \varkappa$. We call an *increasing strengthening* of this inequality any statement of the following form: If $X = \bigcup\limits_{\alpha}^{\uparrow} X_\alpha$ is the increasing union of its subspaces X_α, where every X_α has property P and X has property Q then $\varphi(X) \leqslant \varkappa$.

A number of such increasing strengthenings of inequalities were proven in [3], as a major problem, however, it remained open whether the inequality $|X| \leqslant 2^{c(X)\chi(X)}$, for any T_2 space X, admits such an increasing strengthening.

Theorem 1 of the present paper gives the affirmative answer to this question. The ideas needed in the proof of Theorem 1, with appropriate modifications, also allowed us to show that the inequality $|X| \leqslant 2^{wL(X)\cdot\chi(X)}$ for any T_4 space X proved in [1] also admits an increasing strengthening.

Notation and terminology, unless otherwise explained, is identical with that used in [3].

THEOREM 1. *If $X = \bigcup\limits_{\alpha}^{\uparrow} X_\alpha$ is T_2 and*

$$c(X_\alpha)\cdot\chi(X_\alpha) \leqslant \varkappa$$

holds for each α then

$$|X| \leqslant 2^{\varkappa}.$$

The proof of Theorem 1 will be based on three lemmas given below.

LEMMA 1. *Let X be an arbitrary space, Y a subspace of X and p a point in Y, moreover $\mathscr{B}$ be a complete subalgebra of $RO(X)$ (in symbols $\mathscr{B} \prec RO(X)$), the complete Boolean algebra of all regular open subsets of X, such that*

$(*)$ *for every $\mathscr{C} \subset \mathscr{B}$ if $p \in \overline{\bigcup \mathscr{C}}$ then $p \in \overline{\bigcup \mathscr{C} \cap Y}$.*

Then for every open neighbourhood U of p in X there is a member $B(U) \in \mathscr{B}$ such that $p \in B(U)$ and if $B \in \mathscr{B}$ satisfies $U \cap Y \subset B$ then $B(U) \subset B$ is also valid.

Proof. Let $\mathscr{C}$ be the collection of all members $C \in \mathscr{B}$ satisfying

$$C \cap U \cap Y = \emptyset .$$

Then $\bigcup \mathscr{C} \cap U \cap Y = \emptyset$ holds as well, hence $p \notin \overline{\bigcup \mathscr{C} \cap Y}$ consequently, by $(*)$, $p \notin \overline{\bigcup \mathscr{C}}$. We claim that

$$B(U) = X \setminus \overline{\bigcup \mathscr{C}} \in \mathscr{B}$$

is as required. Now $p \in B(U)$ is obvious.

Next, if $B \in \mathscr{B}$ and $U \cap Y \subset B$ then clearly $C = X \setminus \bar{B} \in \mathscr{C}$, hence $\bar{C} \subset \overline{\bigcup \mathscr{C}}$, and thus

$$B(U) = X \setminus \overline{\bigcup \mathscr{C}} \subset X \setminus \bar{C} = B . \ \blacksquare$$

Remark. If we have $c(\mathscr{B}) \leqslant \varkappa$, i.e. the cellularity of $\mathscr{B}$ is $\leqslant \varkappa$, which is true e.g. if $c(X) \leqslant \varkappa$, then in Lemma 1 $(*)$ may clearly be replaced by the following weaker condition:

$(*)_\varkappa$ for every $\mathscr{C} \in [\mathscr{B}]^{\leqslant \varkappa}$, if $p \in \overline{\bigcup \mathscr{C}}$ then $p \in \overline{\bigcup \mathscr{C} \cap Y}$.

Before we formulate our next lemma we need some definitions. First, if a space is the union of its subspaces X_α, we say that X has the fine topology with respect to the system $\{X_\alpha\}$ of these subspaces provided that $G \subset X$ is open in X if and only if $G \cap X_\alpha$ is open in the subspace X_α for all α. Clearly this means that X has the finest topology with respect to which all the X_α have the same induced subspace topology.

We shall need the following simple proposition concerning increasing unions with the fine topology.

PROPOSITION. *Let $X = \mathop{\dot{\bigcup}} \{X_\alpha \colon \alpha \in \lambda\}$ where λ is a regular cardinal and $t(p, X_\alpha) < \lambda$ holds for every $\alpha \in \lambda$ and $p \in X_\alpha$ and assume that X has the fine topology with respect to the system $\{X_\alpha \colon \alpha \in \lambda\}$. Then for every set $A \subset X$ we have*

$$\bar{A} = \mathop{\dot{\bigcup}} \{\overline{A \cap X_\alpha} \colon \alpha \in \lambda\} .$$

Proof. Clearly it suffices to show that the right-hand side of this equality, let us denote it by B for short, is closed in X. Since X has the fine topology, however, this is equivalent to showing that $B \cap X_\beta$ is closed in X_β for each $\beta \in \lambda$. But

$$B \cap X_\beta = \mathop{\dot{\bigcup}} \{\overline{A \cap X_\alpha} \cap X_\beta \colon \alpha \in \lambda\}$$

is an increasing λ-type union of closed subsets of X_β which is indeed closed in X_β since we have $t(p, X_\beta) < \lambda$ for all $p \in X_\beta$.

Now we are ready to formulate the second lemma needed for the proof of Theorem 1.

LEMMA 2. *Let $X = \mathop{\dot{\bigcup}} \{X_\alpha \colon \alpha \in \lambda\}$, where $\lambda = (2^\varkappa)^+$, X has the fine topology w.r.t. $\{X_\alpha \colon \alpha \in \lambda\}$ and $\chi(X_\alpha) \leqslant \varkappa$ for each $\alpha \in \lambda$. Then $\mathscr{B} \prec RO(X)$, $c(\mathscr{B}) \leqslant \varkappa$ and $|\mathscr{B}| \leqslant \lambda$ imply $\chi(p, \mathscr{B}) \leqslant \varkappa$ for all $p \in X$.*

Proof. Let us first assume that actually $|\mathscr{B}| < \lambda$. Given $p \in X$, for every $\mathscr{C} \in [\mathscr{B}]^{\leqslant \varkappa}$ there is an ordinal $\alpha_{\mathscr{C}} \in \lambda$ such that $p \in \overline{\bigcup \mathscr{C}}$ implies $p \in \overline{\bigcup \mathscr{C} \cap X_{\alpha_{\mathscr{C}}}}$ since X has the fine topology and

$$t(p, X_\alpha) \leqslant \chi(p, X_\alpha) \leqslant \varkappa < \lambda$$

is valid for all $\alpha \in \lambda$; hence the above proposition can be applied. Since

$$|[\mathscr{B}]^{\leqslant \varkappa}| \leqslant |\mathscr{B}|^\varkappa \leqslant (2^\varkappa)^\varkappa = 2^\varkappa < \lambda ,$$

we may then find $\alpha_0 \in \lambda$ such that $p \in X_{\alpha_0}$ and $\alpha_{\mathscr{C}} \leqslant \alpha_0$ for all $\mathscr{C} \in [\mathscr{B}]^{\leqslant \varkappa}$. Clearly, then $(*)_\varkappa$, hence by $c(\mathscr{B}) \leqslant \varkappa$ also $(*)$ of Lemma 1, will be satisfied for p, $\mathscr{B}$ and $Y = X_{\alpha_0}$.

Now let $\{U_\nu \colon \nu \in \varkappa\}$ be a family of open neighbourhoods of p in X such that $\{U_\nu \cap X_{\alpha_0} \colon \nu \in \varkappa\}$ is a neighbourhood base of p in X_{α_0}. We may then apply Lemma 1 for p, $\mathscr{B}$, $Y = X_{\alpha_0}$ and each U_ν to obtain $B_\nu \in \mathscr{B}$ such that $p \in B_\nu$ and $B_\nu \subset B$ whenever $U_\nu \cap X_{\alpha_0} \subset B \in \mathscr{B}$. However, then $\{B_\nu \colon \nu \in \varkappa\}$ clearly establishes $\chi(p, \mathscr{B}) \leqslant \varkappa$ since for every $B \in \mathscr{B}$ with $p \in B$ there is a $\nu \in \varkappa$ with $U_\nu \cap X_{\alpha_0} \subset B$.

Now, assume that $|\mathscr{B}| = \lambda$. Applying $c(\mathscr{B}) \leqslant \varkappa$ we may then write

$$\mathscr{B} = \mathop{\dot{\bigcup}} \{\mathscr{B}_\alpha \colon \alpha \in \lambda\}$$

where $|\mathscr{B}_\alpha| < \lambda$ and $\mathscr{B}_\alpha \prec \mathscr{B}$ for each $\alpha \in \lambda$. We may also assume that if $\alpha \in \lambda$ and $cf(\alpha) > \varkappa$ then

$$\mathscr{B}_\alpha = \mathop{\dot{\bigcup}} \mathscr{B}_\beta \colon \beta \in \alpha\} .$$

Let us put $S = \{\alpha \in \lambda \colon cf(\alpha) > \varkappa\}$. For every $\alpha \in S$ we may apply the above partial result to $\mathscr{B}_\alpha$ to obtain $\mathscr{C}_\alpha \in [\mathscr{B}_\alpha]^{\leqslant \varkappa}$ which is a basis of p in $\mathscr{B}_\alpha$. Since $cf(\alpha) > \varkappa$ and $\mathscr{B}_\alpha = \bigcup_{\beta \in \alpha} \mathscr{B}_\beta$ we have some $\varphi(\alpha) \in \alpha$ such that

$$\mathscr{C}_\alpha \subset \mathscr{B}_{\varphi(\alpha)} .$$

The function φ thus defined is regressive on the stationary subset S of λ, hence by Neumer's theorem there is some $\beta \in \lambda$ and $S_1 \in [S]^\lambda$ such that $\varphi(\alpha) = \beta$ for all $\alpha \in S_1$. But

$$|[\mathscr{B}_\beta]^{\leqslant \varkappa}| \leqslant (2^\varkappa)^\varkappa = 2^\varkappa$$

then implies the existence of some $\mathscr{C} \in [\mathscr{B}_\beta]^{\leqslant \varkappa}$ and $S_2 \in [S_1]^\lambda$ such that $\mathscr{C}_\alpha = \mathscr{C}$ for all $\alpha \in S_2$. Then $\mathscr{C}$ is a basis of p in $\mathscr{B}_\alpha$ for cofinally many $\alpha \in \lambda$ hence in $\mathscr{B}$ as well. ■

The following lemma is actually a variant of the inequality $|X| \leqslant 2^{c(X)\cdot\chi(X)}$ for $X \in \mathscr{T}_2$.

LEMMA 3. *If X is a set, $\mathscr{B} \subset P(X)$ is a family of subsets of X that T_2-separates the points of X, $\mathscr{B}$ is closed under finite intersections, $c(\mathscr{B}) \leqslant \varkappa$ and $\chi(p, \mathscr{B}) \leqslant \varkappa$ for all $p \in X$ then $|X| \leqslant 2^\varkappa$.*

Proof. A direct proof based on the Erdös–Rado theorem $(2^\varkappa)^+ \to (\varkappa^+)^2_\varkappa$ could be given, however the lemma also follows from the above cardinal function inequality as applied to the topology on X generated by $\mathscr{B}$. ■

Now we may turn to the proof of Theorem 1. Note that if $X = \dot{\bigcup} \{X_\alpha: \alpha \in \mu\} \in \mathscr{T}_2$ and $\chi(X_\alpha)\cdot c(X_\alpha) \leqslant \varkappa$ then $|X_\alpha| \leqslant 2^\varkappa$, hence we may assume that $\mu \leqslant \lambda = (2^\varkappa)^+$. But if $\mu < \lambda$ then

$$|X| \leqslant 2^\varkappa\cdot|\mu| \leqslant 2^\varkappa,$$

hence it will suffice to show that $\mu = \lambda$ is impossible.

Assume, reasoning indirectly, that $\mu = \lambda$. Clearly we may also assume that X has the fine topology w.r.t. $\{X_\alpha: \alpha \in \lambda\}$ since this topology on X is also T_2. Since $c(X_\alpha) \leqslant \varkappa$ holds for all $\alpha \in \lambda$, we have (e.g. by [3], 6.1) $c(X) \leqslant \varkappa$. Clearly, we have $|X| = \lambda$; hence by $X \in \mathscr{T}_2$ and by $c(RO(X)) \leqslant c(X) \leqslant \varkappa$ we may find a complete subalgebra $\mathscr{B} \prec RO(X)$ with $|\mathscr{B}| \leqslant \lambda$ that T_2-separates the points of X. By Lemma 2 then $\chi(p, \mathscr{B}) \leqslant \varkappa$ is valid for all $p \in X$. But then, by $c(\mathscr{B}) \leqslant c(RO(X)) \leqslant \varkappa$, Lemma 3 may also be applied to X and $\mathscr{B}$, consequently we must have $|X| \leqslant 2^\varkappa < \lambda$, a contradiction. This completes the proof of Theorem 1. ■

Now we turn to giving an increasing strengthening of the inequality $|X| \leqslant 2^{wL(X)\cdot\chi(X)}$ proved in [1] for $X \in \mathscr{T}_4$. Let us note that it is still open whether this inequality is valid for $X \in \mathscr{T}_3$ as well. In any case our increasing strengthening will only require X to be T_3, while of course $X_\alpha \in \mathscr{T}_4$ will be assumed.

THEOREM 2. *If $X = \dot{\bigcup} X_\alpha$ is T_3 where X_α is T_4 and $wL(X_\alpha)\cdot\chi(X_\alpha) \leqslant \varkappa$ holds for each α then $|X| \leqslant 2^\varkappa$.*

The proof of Theorem 2 runs analogously to that of Theorem 1 and is based on three analogous lemmas.

LEMMA 1′. *Let X be a space, Y a T_4 subspace of X with $wL(Y) \leqslant \varkappa$ and $p \in Y$ be such that $\chi(p, Y) \leqslant \varkappa$ and $t(p, X) \leqslant \varkappa$. Assume furthermore that $\mathscr{B}$ is a $\varkappa$-complete subalgebra of $RO(X)$, in symbols: $\mathscr{B} \prec_\varkappa RO(X)$. (This means that $\operatorname{Int}\overline{\bigcup\mathscr{C}} \in \mathscr{B}$ for all $\mathscr{C} \in [\mathscr{B}]^{\leqslant\varkappa}$). If p, $\mathscr{B}$ and Y satisfy condition $(*)_\varkappa$ formulated in the remark made after Lemma 1 as well as condition $(**)_\varkappa$ to be formulated below, then for every open neighbourhood U of p there is a member $B(U) \in \mathscr{B}$ such that $p \in B(U)$ and for every $B \in \mathscr{B}$ if $U \cap Y \subset B$ then $B(U) \cap Y \subset B$.*

$(**)_\varkappa$ For every $S \in [Y]^{\leqslant\varkappa}$ if $p \notin \bar{S}$ then there is a $B \in \mathscr{B}$ such that $\bar{S} \subset B$ and $p \notin \bar{B}$.

Proof. Let us start by fixing a family $\mathscr{V}$ of open neighbourhoods of p in X such that $|\mathscr{V}| \leqslant \varkappa$ and $\{V \cap Y: V \in \mathscr{V}\}$ is a neighbourhood basis of p in Y.

For any neighbourhood V of p in X let us put

$$\mathscr{C}(V) = \{B \in \mathscr{B}: B \cap V \cap Y = \emptyset\}.$$

We claim that $p \notin \overline{\bigcup\mathscr{C}(V)}$. Indeed, if $p \in \overline{\bigcup\mathscr{C}(V)}$ then by $t(p, X) \leqslant \varkappa$ there is some $\mathscr{C}_1 \in [\mathscr{C}(V)]^{\leqslant\varkappa}$ with $p \in \overline{\bigcup\mathscr{C}_1}$ as well, hence $(*)_\varkappa$ implies $p \in \overline{\bigcup\mathscr{C}_1 \cap Y}$ which is clearly impossible since $\bigcup\mathscr{C}_1 \cap V \cap Y = \emptyset$.

Now, we claim that given U there is a neighbourhood $V \in \mathscr{V}$ of p such that

$$F_U = Y \cap \overline{\bigcup\mathscr{C}(U)} \subset \bigcup\mathscr{C}(V).$$

Again, we reason indirectly, i.e. assume that for every $V \in \mathscr{V}$ there is a point

$$q_V \in F_U \setminus \bigcup\mathscr{C}(V).$$

Then $S = \{q_V: V \in \mathscr{V}\} \in [Y]^{\leqslant\varkappa}$ and $S \subset \overline{\bigcup\mathscr{C}(U)}$ implies $p \notin \bar{S}$, hence by $(**)_\varkappa$ there is some $B \in \mathscr{B}$ with $\bar{S} \subset B$ and $p \notin \bar{B}$. Let $V \in \mathscr{V}$ be such that $V \cap Y \subset X \setminus \bar{B}$. Then $B \in \mathscr{C}(V)$ and $q_V \in S \subset B$, contradicting that $q_V \notin \bigcup\mathscr{C}(V) \supset B$.

Thus we may indeed fix $V \in \mathscr{V}$ such that $\mathscr{C}(V)$ covers F_U. But F_U is closed in Y, hence $Y \in \mathscr{T}_4$ and $wL(Y) \leqslant \varkappa$ imply (cf. [3], 2.35) that there is some $\mathscr{C}_1(U) \in [\mathscr{C}(V)]^{\leqslant\varkappa}$ such that $F_U \subset \overline{\bigcup\mathscr{C}_1(U)}$. We claim that

$$B(U) = X \setminus \overline{\bigcup\mathscr{C}_1(U)} \in \mathscr{B}$$

is as required. That $B(U) \in \mathscr{B}$ follows from the $\varkappa$-completeness of $\mathscr{B}$. Next, $p \in B(U)$ holds because $\mathscr{C}_1(U) \subset \mathscr{C}(V)$ and $p \notin \overline{\bigcup\mathscr{C}(V)}$. Finally, if $B \in \mathscr{B}$ and $U \cap Y \subset B$ then $X \setminus \bar{B} \in \mathscr{C}(U)$, consequently

$$B(U) \cap Y = Y \setminus \overline{\bigcup\mathscr{C}_1(U)} \subset Y \setminus F_U = Y \setminus \overline{\bigcup\mathscr{C}(U)} \subset X \setminus \overline{X \setminus \bar{B}} = B. ■$$

In our next lemma we shall again use the notation $\lambda = (2^\varkappa)^+$.

LEMMA 2′. *Let $X = \dot{\bigcup}\{X_\alpha: \alpha \in \lambda\}$ where X has the fine topology w.r.t. $\{X_\alpha: \alpha \in \lambda\}$, $X_\alpha \in \mathscr{T}_4$ and $wL(X_\alpha)\cdot\chi(X_\alpha) \leqslant \varkappa$ for all $\alpha \in \lambda$, furthermore $\mathscr{B} \prec_\varkappa RO(X)$ is such that $|\mathscr{B}| \leqslant \lambda$ and for every $p \in X$ and $S \in [X]^{\leqslant\varkappa}$ if $p \notin \bar{S}$ then there is some $B \in \mathscr{B}$ with $\bar{S} \subset B$ and $p \notin \bar{B}$. Then for every $p \in X$ we have $\chi(p, \mathscr{B}) \leqslant \varkappa$.*

Proof. First, since $t(p, X_\alpha) \leqslant \chi(p, X_\alpha) \leqslant \varkappa < \lambda$ holds for all $\alpha \in \lambda$ and $p \in X_\alpha$ we can apply the above proposition to conclude that $\bar{A} = \bigcup\{\overline{A \cap X_\alpha}: \alpha \in \lambda\}$ for each set $A \subset X$. Clearly, this implies then that $t(p, X) \leqslant \varkappa$ for all $p \in X$.

Let us fix some $p \in X$. In order to show that $\chi(p, \mathscr{B}) \leqslant \varkappa$ let us first decompose $\mathscr{B}$ into an increasing union

$$\mathscr{B} = \dot{\bigcup}\{\mathscr{B}_\alpha: \alpha \in \lambda\},$$

where $\mathscr{B}_\alpha \prec_\varkappa \mathscr{B}$ and $|\mathscr{B}_\alpha| < \lambda$ for each $\alpha \in \lambda$. We may clearly assume that if $\alpha \in \lambda$ with $cf(\alpha) > \varkappa$ then $\mathscr{B}_\alpha = \overset{\uparrow}{\bigcup} \{\mathscr{B}_\beta\colon \beta \in \alpha\}$.

In view of our assumptions (which imply $|X_\alpha| \leqslant 2^\varkappa < \lambda$ for all $\alpha \in \lambda$) we may easily define a map $\varphi\colon \lambda \to \lambda$ such that the following two conditions be valid for all $\alpha \in \lambda$:

(1) if $S \in [X_\alpha]^{\leqslant\varkappa}$ and $p \notin \bar{S}$ then there is some $B \in \mathscr{B}_{\varphi(\alpha)}$ with $\bar{S} \subset B$ and $p \notin \bar{B}$;

(2) if $\mathscr{C} \in [\mathscr{B}_\alpha]^{\leqslant\varkappa}$ and $p \in \overline{\bigcup \mathscr{C}}$ then $p \in \overline{\bigcup \mathscr{C} \cap X_{\varphi(\alpha)}}$.

Let us put

$$C = \{\alpha \in \lambda\colon \forall \beta (\beta \in \alpha \to \varphi(\beta) \in \alpha)\},$$

then C is closed unbounded in λ. Thus if $S = \{\alpha \in \lambda\colon p \in X_\alpha \text{ and } cf(\alpha) > \varkappa\}$ then $C \cap S$ is stationary in λ. It is easy to check that if $\alpha \in C \cap S$ then the conditions of Lemma 1′ are satisfied for $X, p, Y = X_\alpha$ and $\mathscr{B} = \mathscr{B}_\alpha$.

Let us fix, for $\alpha \in C \cap S$, a family $\mathscr{U}_\alpha$ of open neighbourhoods of p in X such that $|\mathscr{U}_\alpha| \leqslant \kappa$ and

$$\{U \cap X_\alpha\colon U \in \mathscr{U}_\alpha\}$$

is a neighbourhood basis of p in X_α. Then applying Lemma 1′ we consider for each $U \in \mathscr{U}_\alpha$ the set $B(U) \in \mathscr{B}_\alpha$ satisfying $p \in B(U)$ and $B(U) \cap Y \subset B$ whenever $B \in \mathscr{B}_\alpha$ and $U \cap Y \subset B$. Since $cf(\alpha) > \varkappa$ we may then find for every $\alpha \in C \cap S$ an ordinal $\psi(\alpha) < \alpha$ such that

$$\mathscr{C}_\alpha = \{B(U)\colon U \in \mathscr{U}_\alpha\} \subset \mathscr{B}_{\psi(\alpha)}.$$

But then an application of Neumer's theorem and a simple counting argument yields us a set $S_1 \in [C \cap S]^\lambda$, an ordinal $\beta \in \lambda$ and a family $\mathscr{C} \in [\mathscr{B}_\beta]^{\leqslant\varkappa}$ such that $\mathscr{C}_\alpha = \mathscr{C}$ for all $\alpha \in S_1$.

We claim that $\mathscr{C}$ is a basis for p in $\mathscr{B}$. Assume, indirectly, that $p \in B \in \mathscr{B}$ but $C \setminus B \neq \emptyset$ for all $C \in \mathscr{C}$, then there is some $\alpha \in S_1$ such that $(C \setminus B) \cap X_\alpha \neq \emptyset$ for all $C \in \mathscr{C}$ as well. But now $\mathscr{C} = \mathscr{C}_\alpha = \{B(U)\colon U \in \mathscr{U}_\alpha\}$ and thus there is some $U \in \mathscr{U}_\alpha$ with $U \cap X_\alpha \subset B$ hence $B(U) \cap X_\alpha \subset B$ as well, contradicting that $(B(U) \setminus B) \cap X_\alpha \neq \emptyset$. ■

Lemma 3′. *Let $X = \overset{\uparrow}{\bigcup} \{X_\alpha\colon \alpha \in \lambda\}$ (where $\lambda = (2^\varkappa)^+$), $|X_\alpha| \leqslant 2^\varkappa$ and $wL(X_\alpha) \leqslant \varkappa$ for all $\alpha \in \lambda$, moreover $\mathscr{B} \subset RO(X)$ be such that for every $\alpha \in \lambda$ there is some $B \in \mathscr{B}$ with*

$$X_\alpha \subset B \subset \bar{B} \neq X.$$

Then there is a point $p \in X$ with $\chi(p, \mathscr{B}) > \varkappa$.

Proof. Assume, indirectly, that for each $p \in X$ there is a $\mathscr{B}$-basis $\mathscr{C}_p \in [\mathscr{B}]^{\leqslant\varkappa}$. For $\alpha \in \lambda$ we put

$$\mathscr{C}_\alpha = \bigcup \{\mathscr{C}_p\colon p \in X_\alpha\},$$

furthermore

$$\mathscr{W}_\alpha = \{\mathscr{V} \in [\mathscr{C}_\alpha]^{\leqslant\varkappa}\colon X_\alpha \subset \overline{\bigcup \mathscr{V}} \neq X\}.$$

By our assumptions we have $|\mathscr{W}_\alpha| \leqslant 2^\varkappa$. For each $\alpha \in \lambda$ we may then find an ordinal $\varphi(\alpha) \in \lambda$ such that

$$X_{\varphi(\alpha)} \setminus \overline{\bigcup \mathscr{V}} \neq \emptyset$$

for all $\mathscr{V} \in \mathscr{W}_\alpha$.

Let $\alpha \in \lambda$ be such that $\beta \in \alpha$ implies $\varphi(\beta) \in \alpha$ (there is a closed unbounded set of such ordinals α) and moreover satisfying $cf(\alpha) > \varkappa$. Let us pick $B \in \mathscr{B}$ in such a way that $X_\alpha \subset B$ and $\bar{B} \neq X$. For every $p \in X_\alpha$ we may then find a set $C_p \in \mathscr{C}_p$ with $p \in C_p \subset B$, and applying $wL(X_\alpha) \leqslant \varkappa$ to the open cover $\{C_p\colon p \in X_\alpha\}$ of X_α we can choose

$$\mathscr{V} \in [\{C_p\colon p \in X_\alpha\}]^{\leqslant\varkappa}$$

such that $X_\alpha \subset \overline{\bigcup \mathscr{V}}$. But

$$\overline{\bigcup \mathscr{V}} \subset \bar{B} \neq X,$$

hence $\mathscr{V} \in \mathscr{W}_\alpha$, and since $cf(\alpha) > \varkappa$ we actually have some $\beta \in \alpha$ such that $\mathscr{V} \in \mathscr{W}_\beta$. But then $\varphi(\beta) < \alpha$ holds, i.e.

$$X_\alpha \setminus \overline{\bigcup \mathscr{V}} \neq \emptyset,$$

contradicting $X_\alpha \subset \overline{\bigcup \mathscr{V}}$. This completes the proof of Lemma 3′. ■

The proof of Theorem 2 can now be finished as follows. Since again $|X_\alpha| \leqslant 2^\varkappa$ for each α, it suffices to show that our increasing union has length $< \lambda$.

Assume otherwise, i.e. $X = \overset{\uparrow}{\bigcup} \{X_\alpha\colon \alpha \in \lambda\}$ and $|X| = \lambda$. Since X is T_3 and for every $p \in X$ and $\alpha \in \lambda$

$$\chi(p, X_\alpha \cup \{p\}) \leqslant \varkappa,$$

it follows e.g. from [3], 2.5 that $|\overline{X_\alpha}| \leqslant 2^\varkappa$, hence $\bar{X}_\alpha \neq X$. Thus by the regularity of X we may clearly find $\mathscr{B} \prec_\varkappa RO(X)$ such that $|\mathscr{B}| \leqslant \lambda$ and

(i) if $p \in X$, $S \in [X]^{\leqslant\varkappa}$ and $p \notin \bar{S}$ then there is some $B \in \mathscr{B}$ with $\bar{S} \subset B$ and $p \notin \bar{B}$;

(ii) for every $\alpha \in \lambda$ there is some $B \in \mathscr{B}$ with

$$X_\alpha \subset B \quad \text{and} \quad \bar{B} \neq X.$$

Now if we consider the fine topology ϱ on X w.r.t. $\{X_\alpha\colon \alpha \notin \lambda\}$ then this topology may not be T_3, however the existence of $\mathscr{B} \prec_\varkappa RO(X, \varrho)$ with $|\mathscr{B}| \leqslant \lambda$ and with properties (i) and (ii) will remain valid. For (i) this makes use of the fact that every $S \in [X]^{\leqslant\varkappa}$ is contained in some X_α, hence by $|\bar{X}_\alpha| \leqslant 2^\varkappa$ we have some $\beta \in \lambda$ with $S \subset \bar{X}_\alpha \subset X_\beta$ and thus $\bar{S}^\varrho = \bar{S}$. The rest of (i) and (ii) follow easily because for any $B \in RO(X)$ one clearly has

$$B \subset \operatorname{Int}_\varrho \bar{B}^\varrho \subset \bar{B}^\varrho \subset \bar{B},$$

and $\operatorname{Int}_\varrho \bar{B}^\varrho \in RO(X, \varrho)$.

Since all we need of the regularity of X is just the existence of such a $\varkappa$-complete subalgebra $\mathscr{B}$ of $RO(X)$, we assume in what follows that X has the fine topology w.r.t. $\{X_\alpha: \alpha \in \lambda\}$.

But then, in view of (i), Lemma 2′ applies and yields us $\chi(p, \mathscr{B}) \leqslant \varkappa$ for all $p \in X$. On the other hand since (ii) is satisfied Lemma 3′ can also be applied and this gives us $\chi(p, \mathscr{B}) > \varkappa$ for some $p \in X$. This contradiction then finishes the proof. ■

COROLLARY. *If X is T_4 and $X = \bigcup\limits_\alpha^{\uparrow} X_\alpha$ with $wL(X_\alpha)\cdot\chi(X_\alpha) \leqslant \varkappa$ for all α then $|X| \leqslant 2^\varkappa$.*

Proof. Assume, indirectly, that $X = \bigcup^{\uparrow}\{X_\alpha: \alpha \in \lambda\}$ and $|X| = \lambda = (2^\varkappa)^+$. Similarly as in the above proof we can see that $|\overline{X}_\alpha| \leqslant 2^\varkappa$ for each α, consequently $wL(\overline{X}_\alpha)\cdot\chi(\overline{X}_\alpha) \leqslant \varkappa$ is also valid because $\overline{X}_\alpha \subset X_\beta$ holds for some $\beta \in \lambda$. But $\overline{X}_\alpha$ is also T_4 and thus by $X = \bigcup^{\uparrow}\{\overline{X}_\alpha: \alpha \in \lambda\}$ we get a contradiction with Theorem 2.

Note that this corollary does not follow immediately from Theorem 2 because a subspace of a T_4 space is not necessarily T_4.

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Modules over arbitrary domains II

by

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Abstract. Let R be a commutative ring and $S \subseteq R$ a multiplicatively closed subset of R. Defining torsion-free modules with respect to S, we derive new results of this category extending from $|S| = \aleph_0$. In §8 we realize any R-algebra A with torsion-free, reduced R-module structure on modules G as

$$\operatorname{End} G = A \oplus \operatorname{Ines} G$$

where $\operatorname{Ines} G$ are all endomorphisms on G with ω-complete image in G. In §9 we determine $\operatorname{Ines} G$ more explicitely and derive properties of G from the given algebra A.

§ 1. Introduction. We will discuss right R-modules $G = G_R$ over nonzero commutative rings R. The ring R will have a fixed multiplicatively closed subset S such that R as an R-module is S-reduced and S-torsion-free. These well-known conditions on a module G are $\bigcap\limits_{s \in S} Gs = 0$ respectively ($gs = 0 \Rightarrow g = 0$) for all $g \in G$, $s \in S$.

Many questions on the existence of R-modules with prescribed properties can be reduced to representation theorems of R-algebras A as endomorphism algebras — in many cases modulo some "small" or "inessential" endomorphisms. Well-known examples for such problems are decomposition-properties related with the Krull–Remak–Schmidt Theorem — respectively related with Kaplansky's test problems, other derive from questions on prescribed automorphism groups or topologies. The investigation of classical problems in module theory in this sense goes back to a number of fundamental papers by A. L. S. Corner; see [CG] for further references.

In the recent years these investigations have been extended to R-modules of arbitrary large size, however under the restriction that S is essentially countable; see [DG 1,2], [GS 1], [S 2,3] and [CG] for a uniform treatment and further extensions, including torsion, mixed and torsion-free R-modules.

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