Normality and hereditary countable paracompactness of Pixley–Roy hyperspaces

by

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Abstract. In this paper, for the Pixley–Roy hyperspace $\mathscr{F}[M]$ of a metric space M, it will be shown that $\mathscr{F}[M]$ is normal if and only if $\mathscr{F}[M]$ is hereditarily countably paracompact.

Introduction. Throughout this paper, all spaces are assumed to be T_1-spaces and M always denotes a metric space. Pixley–Roy hyperspace of the real line was defined by C. Pixley and P. Roy in [5] and later generalized by E. K. van Douwen in [1]. The *Pixley–Roy hyperspace* $\mathscr{F}[X]$ of a space X has as its underlying set the collection of all nonempty finite subsets of X. If $F \in \mathscr{F}[X]$, then the basic open neighborhoods of F are of the form $[F, U] = \{G \in \mathscr{F}[X]\colon F \subset G \subset U\}$, where U is an open subset of X containing F. Therefore $[F, U] \cap [H, V] \neq \emptyset$ if and only if $F \subset V$ and $H \subset U$. It was pointed out in [1] that every Pixley–Roy hyperspace is a zero-dimensional hereditarily metacompact space and $\mathscr{F}[X]$ is a Moore space if and only if X is first countable.

M is said to be a *q-set* if every subset of M is an F_σ-set in M and a *strong q-set* if every finite power of M is a *q-set*. If, in addition, M is separable, then M is said to be a *Q-set* and a *strong Q-set* respectively. It is well known that the existence of an uncountable Q-set is undecidable in ZFC and is equivalent to the existence of a separable normal nonmetrizable Moore space. T. Przymusiński [7] showed that the existence of an uncountable Q-set is equivalent to the existence of an uncountable strong Q-set.

Studying countable paracompactness in separable Moore spaces led to the notion of Δ-sets. M is said to be a *Δ-set* if M is separable and for any decreasing sequence $\{A_n\colon n \in N\}$ of subsets of M having $\bigcap \{A_n\colon n \in N\} = \emptyset$, there is a sequence $\{U_n\colon n \in N\}$ of open subsets of M having $A_n \subset U_n$ for each $n \in N$ and $\bigcap \{U_n\colon n \in N\} = \emptyset$ and a *strong Δ-set* if every finite power of M is a Δ-set. It is clear that any Q-set (strong Q-set) is a Δ-set (strong Δ-set). The argument given by T. Przymusiński in [6] shows that any Δ-set must have cardinality $< \mathfrak{c}$, where $\mathfrak{c}$ is the cardinality of continuum. Thus any Δ-set is strongly zero-dimensional. E. K. van Douwen, T. Przymusiński and G. M. Reed showed that the existence of a separable countably paracompact nonmetrizable Moore space is equivalent to the existence of an uncountable Δ-set (see [6]).

For normality and countable paracompactness of Pixley–Roy hyperspaces of metric spaces, M. E. Rudin [10], T. Przymusiński [8] and D. J. Lutzer [4] obtained the following elegant results (see also T. Przymusiński and F. D. Tall [9]): (1) if M is separable then $\mathscr{F}[M]$ is normal if and only if M is a strong Q-set; (2) if M is strongly zero-dimensional then $\mathscr{F}[M]$ is normal if and only if M is a strong q-set and (3) if M is a strong Δ-set then $\mathscr{F}[M]$ is countably paracompact.

Our purpose of this paper is to show the equivalence of normality and hereditary countable paracompactness of Pixley–Roy hyperspaces of metric spaces. To do so we introduce a notion of an almost strong q-set, which is intermediate between the notions of a q-set and a strong q-set. Our result may be of interest in connection with the following D. J. Lutzer's problem in [4]: Is every strong Δ-set a strong Q-set?

Let N denote the set of natural numbers. For Q-sets and Δ-sets, see W. G. Fleissner [2].

§ 1. Preliminaries. Let $n \in N$ and let τ be a permutation of $\{1, \ldots, n\}$. For a point $x = (x_1, \ldots, x_n) \in X^n$, let $\tau(x) = (x_{\tau(1)}, \ldots, x_{\tau(n)})$. A subset A of X^n, $n \in N$, is *symmetric* if for any permutation τ of $\{1, \ldots, n\}$, $\tau(A) = A$.

DEFINITION. M is said to be an *almost strong q-set* if for each $n \in N$, every symmetric subset of M^n is an F_σ-set in M^n.

Clearly every strong q-set is an almost strong q-set and every almost strong q-set is a q-set. Some results concerning almost strong q-sets are given.

LEMMA 1.1. *If M is strongly zero-dimensional, then M is a strong q-set if and only if M is an almost strong q-set.*

Proof. It suffices to prove the "if" part. Let M be a strongly zero-dimensional almost strong q-set. Then M is linearly orderable (see H. Herrlich [3]). Let $\leqslant$ be a linear order on M generating the topology of M. Take $n \in N$ and assume that we have already proved that M^n is a q-set. Let

$$Z = \{(x_1, \ldots, x_{n+1}) \in M^{n+1}\colon x_1 < \ldots < x_{n+1}\}.$$

Then M^{n+1} is a finite union of F_σ-sets which are either homeomorphic to Z or to some M^k $(k \leqslant n)$. In order to prove that M^{n+1} is a q-set, it is enough to prove that Z is a q-set. Let A be an arbitrary subset of Z and let $\tilde{A} = \bigcup \{\tau(A)\colon \tau$ is a permutation of $\{1, \ldots, n+1\}\}$. Then $\tilde{A}$ is a symmetric subset of M^{n+1}. Since M is an almost strong q-set, $\tilde{A}$ is an F_σ-set in M^{n+1}. Thus $A = \tilde{A} \cap Z$ is an F_σ-set in Z. Hence Z is a q-set. It follows that M is a strong q-set.

PROPOSITION 1.2. *For every non σ-discrete almost strong q-set M, there are a non σ-discrete strong q-set M' and a one-to-one continuous mapping from M' onto M.*

Proof. By Lemma 1.1, this is essentially proved by T. Przymusiński (see [8], Lemma 5.10).

The following lemma is the key to our theorem. For a point $x = (x_1, \ldots, x_n) \in X^n$ and $n \in N$, let $F_x = \{x_1, \ldots, x_n\}$. Let $|A|$ denote the cardinality of a set A.

LEMMA 1.3. *Let X and Y be subsets of a metric space M such that $M = X \cup Y$ and $X \cap Y = \emptyset$. If X is an almost strong q-set and Y is a closed subset of M with $|Y| \leqq \aleph_0$, then M is an almost strong q-set.*

Proof. Let A be a subset of M. Then $A \cap X$ is an F_σ-set in X. Since X is an F_σ-set in M, $A \cap X$ is an F_σ-set in M. Since $|Y| \leqslant \aleph_0$, it follows that $A = (A \cap X) \cup \cup (A \cap Y)$ is an F_σ-set in M. Hence M is a q-set. Take $n \in N$ and assume that we have already proved that every symmetric subset of M^n is an F_σ-set in M^n. Let

$$Z = \{(z_1, \ldots, z_{n+1}) \in M^{n+1}\colon z_i \neq z_j \text{ for } i, j \leqslant n+1 \text{ and } i \neq j\}.$$

Then Z is an open subset of M^{n+1}. We shall show that every symmetric subset of Z is an F_σ-set in Z. Let A be a symmetric subset of Z. Since X is an almost strong q-set and Y is a closed subset of M with $|Y| \leqslant \aleph_0$, we may assume that for each point $z = (z_1, \ldots, z_{n+1}) \in A$, $1 \leqslant |F_z \cap Y| \leqq n$. Let $S = \{s = (y_1, \ldots, y_k)\colon s$ is an ordered pair of distinct elements of Y and $1 \leqslant k \leqslant n\}$. Then we have $|S| \leqslant \aleph_0$. Fix $s = (y_1, \ldots, y_k) \in S$. Define A_s as follows: $z = (z_1, \ldots, z_{n+1}) \in A_s$ if and only if $z \in A$, $F_z \cap Y = F_s$ and $z_{i_j} = y_j$ for some $\{i_1, \ldots, i_k\} \subset \{1, \ldots, n+1\}$ such that if $j < j'$ and $j, j' \leqq k$ then $i_j < i_{j'}$. Let $z = (z_1, \ldots, z_{n+1}) \in A_s$ and let $\{i_1^z, \ldots, i_k^z\}$ be a subset of $\{1, \ldots, n+1\}$ such that $z_{i_j^z} = y_j$ for $j \leqslant k$. Let $\{m_1^z, \ldots, m_{n-k+1}^z\} = \{1, \ldots, n+1\} - \{i_1^z, \ldots, i_k^z\}$ such that if p, $t \leqq n-k+1$ and $p < t$ then $m_p^z < m_t^z$. Define $x_z = (z_{m_1^z}, \ldots, z_{m_{n-k+1}^z})$. Then x_z is a point of M^{n-k+1}. Let $B_s = \{x_z\colon z \in A_s\}$. Then B_s is a symmetric subset of M^{n-k+1}. Thus there is a sequence $\{E_{s,p}\colon p \in N\}$ of closed subsets of M^{n-k+1} such that $B_s = \bigcup \{E_{s,p}\colon p \in N\}$. Without loss of generality, we can assume that each $E_{s,p}$ is symmetric. For each $p \in N$, let $H_{s,p} = \{z \in A_s\colon F_z - F_s = F_w$ for some $w \in E_{s,p}\}$. It is easy to check that each $H_{s,p}$ is a closed subset of M^{n+1}. Since $A = \bigcup \{H_{s,p}\colon s \in S$ and $p \in N\}$, A is an F_σ-set in Z. Since M^{n+1} is a finite union of Z and F_σ-sets which are homeomorphic to some M^k $(k \leqslant n)$, it follows that every symmetric subset of M^{n+1} is an F_σ-set in M^{n+1}. Thus M is an almost strong q-set.

LEMMA 1.4. *Let $f\colon M \to M'$ be a perfect mapping from a metric space M onto a metric space M'. If M is an almost strong q-set, then M' is also an almost strong q-set.*

Proof. For each $n \in N$, let $f^n\colon M^n \to M'^n$ be a perfect mapping from M^n onto M'^n induced by f. Let A be a symmetric subset of M'^n and $n \in N$. Then $(f^n)^{-1}(A)$ is a symmetric subset of M^n. Since M is an almost strong q-set, $(f^n)^{-1}(A)$ is an F_σ-set in M^n. Thus A is an F_σ-set in M'^n. Hence M' is an almost strong q-set.

§ 2. Normality and hereditary countable paracompactness. For each $n \in N$, let $\mathscr{F}_n[X] = \{F \in \mathscr{F}[X]\colon |F| \leqslant n\}$. Notice that every $\mathscr{F}_n[X]$ is a closed subspace of $\mathscr{F}[X]$ and in particular, $\mathscr{F}_1[X]$ is a discrete closed subspace of $\mathscr{F}[X]$.

Let d be a compatible metric on M. For each $F \in \mathscr{F}[M]$, let $B(F, 1/n) = \bigcup \{B(x, 1/n)\colon x \in F\}$, where $B(x, 1/n) = \{y \in M\colon d(x, y) < 1/n\}$.

We give the main theorem in this paper.

THEOREM 2.1. *The following are equivalent.*

(a) *$\mathscr{F}[M]$ is normal,*

(b) *$\mathscr{F}[M]$ is hereditarily countably paracompact,*

(c) *M is an almost strong q-set.*

Proof. (a) → (b). Since $\mathscr{F}[M]$ is a perfectly normal space, this implication is obvious.

(b) → (c). We may assume that M is not discrete. Let x be a nonisolated point of M and let $\{x_n\colon n \in N\}$ be a sequence of distinct points of $M - \{x\}$ converging to x. Let $Z = \{x_n\colon n \in N\} \cup \{x\}$ and let $Y = M - Z$. Then Z is a compact subset of M. In order to prove this implication, from Lemma 1.3, it suffices to prove that Y is an almost strong q-set. To see this, let Y' be a space considering the following new topology on Y: for each $y \in Y'$, the basic open neighborhoods of y are of the form $(B(y, 1/n) \cup B(x, 1/n)) \cap Y'$ and $n \in N$. Then Y' is first countable and consequently, $\mathscr{F}[Y']$ is a Moore space. It is clear that if O is an open neighborhood of $F \cup \{x, x_n\}$ in M, where $F \in \mathscr{F}[Y']$ and $n \in N$ then $O \cap Y'$ is an open neighborhood of F in Y'. We need the following claim.

CLAIM. *$\mathscr{F}_n[Y']$ is normal for each $n \in N$.*

Proof of Claim. We shall show that every $\mathscr{F}_n[Y']$ is perfectly normal. If $n = 1$, then $\mathscr{F}_1[Y']$ is a discrete space. Thus $\mathscr{F}_1[Y']$ is normal. Let $n \geqslant 2$ and assume that $\mathscr{H}$ is a closed subset of $\mathscr{F}_n[Y']$. It is enough to prove that there is a sequence $\{\mathscr{U}_m\colon m \in N\}$ of open subsets of $\mathscr{F}_n[Y']$ satisfying $\mathscr{H} = \bigcap \{\mathscr{U}_m\colon m \in N\} = \bigcap \{\mathrm{cl}_{\mathscr{F}_n[Y']}\mathscr{U}_m\colon m \in N\}$ (see P. Zenor [11]). For each $m \in N$, let

$$\mathscr{H}_m = \{F \cup \{x, x_m\}\colon F \in \mathscr{H}\}.$$

Define $\mathscr{Y}_m$, $m \in N$, as follows: $F \in \mathscr{Y}_m$ if and only if $F \in \mathscr{H}_m$ or $F = G \cup \{x, x_m\}$, where $G \in \mathscr{F}_n[Y'] - \mathscr{H}$ and there is an open subset O in M containing F such that $[G, O \cap Y'] \cap \mathscr{H} = \emptyset$. Let $\mathscr{Y} = \{F \cup \{x\}\colon F \in \mathscr{F}_n[Y'] - \mathscr{H}\}$ and let

$$\mathscr{E} = \bigcup \{\mathscr{Y}_m\colon m \in N\} \cup \mathscr{Y}.$$

We consider $\mathscr{E}$ with the subspace topology of $\mathscr{F}[M]$. Then each $\mathscr{Y}_m$ is an open-and-closed subset of $\mathscr{E}$. Let $\mathscr{I}_m = \bigcup \{\mathscr{H}_s\colon s \geqslant m\}$ for each $m \in N$. Then $\{\mathscr{I}_m\colon m \in N\}$ is a decreasing sequence of closed subsets of $\mathscr{E}$ having $\bigcap \{\mathscr{I}_m\colon m \in N\} = \emptyset$. To see this, assume that $F \notin \mathscr{I}_m$ and $m \in N$. In case of $F \in \mathscr{Y}_s$ and $s < m$. Then $\mathscr{Y}_s$ is an open neighborhood of F in $\mathscr{E}$ such that $\mathscr{Y}_s \cap \mathscr{I}_m = \emptyset$. In case of $F \in \mathscr{Y}_s$ and $s \geqslant m$. Then $F = G \cup \{x, x_s\}$ for some $G \in \mathscr{F}_n[Y']$. Then we have $G \notin \mathscr{H}$. Thus there is an open neighborhood O of F in M such that $[G, O \cap Y'] \cap \mathscr{H} = \emptyset$. Assume that $K \in [F, O] \cap \mathscr{I}_m \neq \emptyset$. Then $K = J \cup \{x, x_p\}$ for some $J \in \mathscr{H}$ and $p \geqslant m$. From the construction of $\mathscr{E}$, we have $p = s$. Thus we have $K = J \cup \{x, x_s\}$ and consequently, $J \in [G, O \cap Y'] \cap \mathscr{H}$, which is a contradiction. Thus it follows that $[F, O] \cap \mathscr{I}_m = \emptyset$. In case of $F \in \mathscr{Y}$. Then $F = G \cup \{x\}$ for some $G \in \mathscr{F}_n[Y'] - \mathscr{H}$. Then there is a basic open neighborhood O' of G in Y' such that $[G, O'] \cap \mathscr{H} = \emptyset$. From the definition of the topology of Y', there is an open set O in M containing F such that $O \cap Y' = O'$. Assume that $K \in [F, O] \cap \mathscr{I}_m$. Then $K = J \cup \{x, x_s\}$ for some $J \in \mathscr{H}$ and $s \geqslant m$. Then it analogously follows that $J \in [G, O'] \cap \mathscr{H}$. This is a contradiction. Thus, in each case, there is an open subset $\mathscr{O}$ in $\mathscr{E}$ containing F such that $\mathscr{O} \cap \mathscr{I}_m = \emptyset$. Hence each $\mathscr{I}_m$ is a closed subset of $\mathscr{E}$. It is clear that $\bigcap \{\mathscr{I}_m\colon m \in N\} = \emptyset$. Since $\mathscr{F}[M]$ is hereditarily countably paracompact, $\mathscr{E}$ is countably paracompact. So there is a decreasing sequence $\{\mathscr{V}_m\colon m \in N\}$ of open subsets of $\mathscr{E}$ having $\mathscr{I}_m \subset \mathscr{V}_m$ for each $m \in N$ and $\bigcap \{\mathrm{cl}_{\mathscr{E}}\mathscr{V}_m\colon m \in N\} = \emptyset$. Without loss of generality, we can assume that each $\mathscr{V}_m$ is contained in $\bigcup \{\mathscr{Y}_s\colon s \geqslant m\} \cup \mathscr{Y}$. For each $F \in \mathscr{V}_m$ and $m \in N$, take an open subset $O_{F,m}$ in M containing F such that $[F, O_{F,m}] \cap \mathscr{E} \subset \mathscr{V}_m$. For each $m \in N$, define

$$\mathscr{U}_m = \bigcup \{[F - \{x, x_s\}, O_{F,m} \cap Y'] \cap \mathscr{F}_n[Y']\colon F \in \mathscr{Y}_s \cap \mathscr{V}_m \text{ and } s \geqslant m\} \cup \big(\bigcup \{[F - \{x\}, O_{F,m} \cap Y'] \cap \mathscr{F}_n[Y']\colon F \in \mathscr{Y} \cap \mathscr{V}_m\}\big).$$

Then each $\mathscr{U}_m$ is an open subset of $\mathscr{F}_n[Y']$ and it is obvious that $\mathscr{H} \subset \mathscr{U}_m$ for each $m \in N$. Suppose that $F \in \mathscr{F}_n[Y'] - \mathscr{H}$. Then we have $F \cup \{x\} \in \mathscr{Y}$. Then there are open subsets O_1 and O_2 in M containing $F \cup \{x\}$ and a natural number m_1 such that $[F, O_1 \cap Y'] \cap \mathscr{H} = \emptyset$ and $[F \cup \{x\}, O_2] \cap \mathscr{V}_{m_1} = \emptyset$. Let $O = O_1 \cap O_2$. Since O is an open neighborhood of x in M, there is a natural number m_2 such that if $s \geqslant m_2$ then $x_s \in O$. Let $m = \max\{m_1, m_2\}$. Since $\{\mathscr{V}_s\colon s \in N\}$ is a decreasing sequence, we have $[F, O \cap Y'] \cap \mathscr{H} = \emptyset$ and $[F \cup \{x\}, O] \cap \mathscr{V}_m = \emptyset$. We shall show that $[F, O \cap Y'] \cap \mathscr{U}_m = \emptyset$. To see this, assume that $G \in [F, O \cap Y'] \cap \mathscr{U}_m$. Then the following two cases are considered.

Case 1. There is an $I = J \cup \{x, x_s\} \in \mathscr{V}_m$ $(s \geqslant m)$ such that $G \in [J, O_{I,m} \cap Y'] \cap \mathscr{F}_n[Y']$. If $G \in \mathscr{H}$, then we have $G \cup \{x, x_s\} \in \mathscr{V}_m$. If $G \notin \mathscr{H}$, then we have $G \cup \{x, x_s\} \in \mathscr{Y}_s \subset \mathscr{E}$, because O is an open neighborhood of $G \cup \{x, x_s\}$ such that $[G, O \cap Y'] \cap \mathscr{H} \subset [F, O \cap Y'] \cap \mathscr{H} = \emptyset$. From the way of taking $O_{I,m}$, we have $G \cup \{x, x_s\} \subset O_{I,m}$. Then $G \cup \{x, x_s\} \in [I, O_{I,m}] \cap \mathscr{E} \subset \mathscr{V}_m$. Thus, in each case, we have $G \cup \{x, x_s\} \in \mathscr{V}_m$. But $G \cup \{x, x_s\} \in [F \cup \{x\}, O] \cap \mathscr{V}_m$, which is a contradiction.

Case 2. Not Case 1. Then there is an $I = J \cup \{x\} \in \mathscr{V}_m$ such that $G \in [J, O_{I,m} \cap Y'] \cap \mathscr{F}_n[Y']$. If $G \in \mathscr{H}$, then $G \in [F, O \cap Y'] \cap \mathscr{H}$, which is a contradiction. Thus we have $G \notin \mathscr{H}$. Then $G \cup \{x\} \in \mathscr{Y} \subset \mathscr{E}$ and consequently, $G \cup \{x\} \in [I, O_{I,m}] \cap \mathscr{E} \subset \mathscr{V}_m$. But $G \cup \{x\} \in [F \cup \{x\}, O] \cap \mathscr{V}_m$. This is a contradiction.

Thus it follows that $[F, O \cap Y'] \cap \mathscr{U}_m = \emptyset$. Hence we have

$$\mathscr{H} = \bigcap \{\mathrm{cl}_{\mathscr{F}_n[Y']}H_m\colon m \in N\}.$$

Therefore $\mathscr{F}_n[Y']$ is perfectly normal for each $n \in N$.

Now we show that Y is an almost strong q-set. Let A be a subset of Y. Let $\mathscr{A} = \{\{y\}: y \in A\}$ and $\mathscr{B} = \{\{y\}: y \in Y-A\}$. Then $\mathscr{A}$ and $\mathscr{B}$ are disjoint closed subsets of $\mathscr{F}_2[Y']$. Since $\mathscr{F}_2[Y']$ is normal, there are disjoint open subsets $\mathscr{U}$ and $\mathscr{V}$ of $\mathscr{F}_2[Y']$ such that $\mathscr{A} \subset \mathscr{U}$ and $\mathscr{B} \subset \mathscr{V}$. For each $n \in N$, let

$$A_n = \{y \in A: [\{y\}, (B(y, 1/n) \cup B(x, 1/n)) \cap Y'] \cap \mathscr{F}_2[Y'] \subset \mathscr{U}\}.$$

Since $A = \bigcup \{A_n: n \in N\}$, in order to prove that A is an F_σ-set in Y, it suffices to prove that $\mathrm{cl}_Y A_n \subset A$ for each $n \in N$. Suppose that $y \in (Y-A) \cap \mathrm{cl}_Y A_n$ for some $n \in N$. Then $\{y\} \in \mathscr{B} \subset \mathscr{V}$. Then there is a natural number m $(m \geqslant n)$ such that $[\{y\}, (B(y, 1/m) \cup B(x, 1/m)) \cap Y'] \cap \mathscr{F}_2[Y'] \subset \mathscr{V}$. Since $y \in \mathrm{cl}_Y A_n$, there is a $y' \in A_n$ such that $y' \in B(y, 1/m) \cap Y$. Then

$$\{y, y'\} \in [\{y'\}, (B(y', 1/n) \cup B(x, 1/n)) \cap Y'] \cap [\{y\}, (B(y, 1/m) \cup B(x, 1/m)) \cap Y'] \cap \mathscr{F}_2[Y'] \subset \mathscr{U} \cap \mathscr{V}.$$

This is a contradiction. Thus it follows that $\mathrm{cl}_Y A_n \subset A$ for each $n \in N$. Hence A is an F_σ-set in Y. Thus Y is a q-set. Take $n \in N$ and assume that we have already proved that every symmetric subset of Y^n is an F_σ-set in Y^n. Let

$$O = \{(y_1, \ldots, y_{n+1}) \in Y^{n+1}: y_i \neq y_j \text{ for } i, j \leqslant n+1 \text{ and } i \neq j\}.$$

Since Y^{n+1} is a finite union of O and F_σ-sets which are homeomorphic to some Y^k $(k \leqslant n)$, it is enough to prove that every symmetric subset of O is an F_σ-set in O. Let A be a symmetric subset of O. Let $\mathscr{A} = \{F_y: y \in A\}$ and $\mathscr{B} = \{F_y: y \in O-A\}$. Then $\mathscr{A}$ and $\mathscr{B}$ are disjoint closed subsets of $\mathscr{F}_{2n+2}[Y'] - \mathscr{F}_n[Y']$. By the Claim, $\mathscr{F}_{2n+2}[Y']$ is perfectly normal and consequently, $\mathscr{F}_{2n+2}[Y']$ is hereditarily normal. Thus $\mathscr{F}_{2n+2}[Y'] - \mathscr{F}_n[Y']$ is normal. Hence there are disjoint open subsets $\mathscr{U}$ and $\mathscr{V}$ of $\mathscr{F}_{2n+2}[Y'] - \mathscr{F}_n[Y']$ such that $\mathscr{A} \subset \mathscr{U}$ and $\mathscr{B} \subset \mathscr{V}$. For each $m \in N$, let $A_m = \{y \in A: [F_y, B(F_y, 1/m) \cup B(x, 1/m)) \cap Y'] \cap (\mathscr{F}_{2n+2}[Y'] - \mathscr{F}_n[Y']) \subset \mathscr{U}\}$. Clearly $A = \bigcup \{A_m: m \in N\}$. We shall show that $\mathrm{cl}_O A_m \subset A$ for each $m \in N$. Assume that $y = (y_1, \ldots, y_{n+1}) \in (O-A) \cap \mathrm{cl}_O A_m$ for some $m \in N$. Then we have $F_y \in \mathscr{B} \subset \mathscr{V}$. Then there is a $k \in N$ $(k \geqslant m)$ such that $\{B(y_i, 1/k): i = 1, \ldots, n+1\}$ is pairwise disjoint in M and

$$[F_y, (B(F_y, 1/k) \cup B(x, 1/k)) \cap Y'] \cap (\mathscr{F}_{2n+2}[Y'] - \mathscr{F}_n[Y']) \subset \mathscr{V}.$$

Since $y \in \mathrm{cl}_O A_m$, there is a $y' = (y'_1, \ldots, y'_{n+1}) \in \prod_{i=1}^{n+1} B(y_i, 1/k) \cap A_m$. Thus we have $y'_i \in B(y_i, 1/k)$ for each $i = 1, \ldots, n+1$. Hence we have $F_{y'} \subset B(F_y, 1/k)$. Since $k \geqslant m$, we have $F_y \subset B(F_{y'}, 1/m)$. Hence

$$F_y \cup F_{y'} \in [F_{y'}, (B(F_{y'}, 1/m) \cup B(x, 1/m)) \cap Y'] \cap [F_y, (B(F_y, 1/k) \cup B(x, 1/k)) \cap Y'] \cap (\mathscr{F}_{2n+2}[Y'] - \mathscr{F}_n[Y']) \subset \mathscr{U} \cap \mathscr{V},$$

which is a contradiction. Thus it follows that $\mathrm{cl}_O A_m \subset A$ for each $m \in N$. Hence A is an F_σ-set in O. Therefore it follows that Y is an almost strong q-set.

(c) $\to$ (a). The idea of the proof of this implication is due to T. Przymusiński and F. D. Tall [9]. We shall show that $\mathscr{F}[M]$ is a perfectly normal space. Let $\mathscr{U}$ be an open subset of $\mathscr{F}[M]$. For each $F \in \mathscr{U}$, there is a natural number $\mu(F)$ such that $[F, B(F, 1/\mu(F))] \subset \mathscr{U}$ and for each $F \in \mathscr{U}$ with $|F| \geqslant 2$, define $\varrho(F) = \min \{d(x, y): x, y \in F \text{ and } x \neq y\}$. For each n $(\geqslant 2)$, $m \in N$, let

$$\mathscr{A}_{n,m} = \{F \in \mathscr{U}: |F| = n,\ \mu(F) \leqslant m \text{ and } \varrho(F) \geqslant 1/m\}$$

and for each $m \in N$, let

$$\mathscr{A}_{1,m} = \{F \in \mathscr{U}: |F| = 1 \text{ and } \mu(F) \leqslant m\}.$$

Then $\mathscr{U} = \bigcup \{\mathscr{A}_{n,m}: n, m \in N\}$. For each $n, m \in N$, let $A_{n,m} = \{z \in M^n: F_z \in \mathscr{A}_{n,m}\}$. Since each $A_{n,m}$ is a symmetric subset of M^n, there is a sequence $\{E_{n,m,k}: k \in N\}$ of symmetric closed subsets of M^n such that $A_{n,m} = \bigcup \{E_{n,m,k}: k \in N\}$ for $n, m \in N$. For each $n, m, k \in N$, let $\mathscr{V}_{n,m,k} = \bigcup \{[F_z, B(F_z, 1/2m)]: z \in E_{n,m,k}\}$. Clearly $\mathscr{U} = \bigcup \{\mathscr{V}_{n,m,k}: n, m, k \in N\}$. In order to prove that $\mathscr{F}[M]$ is perfectly normal, it suffices to prove that $\mathrm{cl}_{\mathscr{F}[M]} \mathscr{V}_{n,m,k} \subset \mathscr{U}$ for each $n, m, k \in N$. Fix n, m, k. Assume that $F = \{z_1, \ldots, z_r\} \in \mathrm{cl}_{\mathscr{F}[M]} \mathscr{V}_{n,m,k}$. Then for each $s \in N$, $[F, B(F, 1/s)] \cap \mathscr{V}_{n,m,k} \neq \emptyset$. So there is an $F^s \in [F, B(F, 1/s)] \cap [F_{x_s}, B(F_{x_s}, 1/2m)]$ for some $x_s = (x_{s,1}, \ldots, x_{s,n}) \in E_{n,m,k}$ and $s \in N$. Hence we have

(i) $F \cup F_{x_s} \subset F^s \subset B(F, 1/s) \cap B(F_{x_s}, 1/2m)$ for each $s \in N$.

By using the same technique in [9], we have natural numbers $i_1, \ldots, i_n$ and an infinite subset P of N such that

(ii) $d(x_{s,j}, z_{i_j}) < 1/s$ for each $s \in P$ and for each j, $1 \leqslant j \leqslant n$, and

(iii) $\min\{d(z_{i_j}, z_{i_{j'}}): j, j' \leqslant n \text{ and } j \neq j'\} \geqslant 1/m$.

Let $G = \{z_{i_1}, \ldots, z_{i_n}\}$. By (iii), G is a finite subset of distinct elements which is contained in F. Let B be a subset of M^n such that for each $x \in B$, $F_x = G$. Then $B \subset E_{n,m,k}$. For if not, there is an $\varepsilon > 0$ such that $\{B(z_{i_j}, \varepsilon): j = 1, \ldots, n\}$ is pairwise disjoint in M and for each $x = (x_1, \ldots, x_n) \in B$, $\prod_{j=1}^{n} B(x_j, \varepsilon) \cap E_{n,m,k} = \emptyset$. Take $s \in P$ such that $1/s < \varepsilon$. By (ii), we have $\prod_{j=1}^{n} B(z_{i_j}, \varepsilon) \cap E_{n,m,k} \ni x_s = (x_{s,1}, \ldots, x_{s,n})$, which is a contradiction. Thus it follows that $B \subset E_{n,m,k}$. Hence we have $G \in \mathscr{A}_{n,m}$ and consequently, we have $[G, B(G, 1/m)] \subset \mathscr{U}$. Take any $y \in F$ and $s \in P$ such that $1/s < 1/2m$. By (i), $y \in B(F_{x_s}, 1/2m)$. So there is a j $(j \leqslant n)$ such that $d(y, x_{s,j}) < 1/2m$. By (ii), $d(x_{s,j}, z_{i_j}) < 1/s$, so we have $d(y, z_{i_j}) \leqslant d(y, x_{s,j}) + d(x_{s,j}, z_{i_j}) < 1/m$. Since y is an arbitrary element of F, it follows that $F \subset B(G, 1/m)$. Since $G \subset F$, we have $F \in [G, B(G, 1/m)] \subset \mathscr{U}$. The proof is complete.

THEOREM 2.2. *Let $f\colon M \to M'$ be a perfect mapping from a metric space M onto a metric space M'. If $\mathscr{F}[M]$ is normal, then $\mathscr{F}[M']$ is also normal.*

Proof. This follows from Lemma 1.4 and Theorem 2.1 immediately.

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Increasing strengthenings of cardinal function inequalities

by

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Abstract. We prove that the following *increasing strengthenings* of two cardinal function inequalities given in [2] and [1] respectively are valid.

THEOREM 1. *If X is T_2 and $X = \overset{\uparrow}{\bigcup_\alpha} X_\alpha$ (i.e. X is the union of an increasing chain of its subspaces X_α) and $c(X_\alpha)\cdot\chi(X_\alpha) \leqslant \varkappa$ for all α then $|X| \leqslant 2^\varkappa$.*

THEOREM 2. *If X is T_3 and $X = \overset{\uparrow}{\bigcup_\alpha} X_\alpha$, where X_α is T_4 and $wL(X_\alpha)\cdot\chi(X_\alpha) \leqslant \varkappa$ for all α then $|X| \leqslant 2^\varkappa$.*

In [3] the first author has initiated the study of strengthening certain cardinal function inequalities in the following manner. A general form of a cardinal function inequality may be given as follows: If φ is some given cardinal function and X is a space having some property P then $\varphi(X) \leqslant \varkappa$. We call an *increasing strengthening* of this inequality any statement of the following form: If $X = \overset{\uparrow}{\bigcup_\alpha} X_\alpha$ is the increasing union of its subspaces X_α, where every X_α has property P and X has property Q then $\varphi(X) \leqslant \varkappa$.

A number of such increasing strengthenings of inequalities were proven in [3], as a major problem, however, it remained open whether the inequality $|X| \leqslant 2^{c(X)\chi(X)}$, for any T_2 space X, admits such an increasing strengthening.

Theorem 1 of the present paper gives the affirmative answer to this question. The ideas needed in the proof of Theorem 1, with appropriate modifications, also allowed us to show that the inequality $|X| \leqslant 2^{wL(X)\cdot\chi(X)}$ for any T_4 space X proved in [1] also admits an increasing strengthening.

Notation and terminology, unless otherwise explained, is identical with that used in [3].

THEOREM 1. *If $X = \overset{\uparrow}{\bigcup_\alpha} X_\alpha$ is T_2 and*

$$c(X_\alpha)\cdot\chi(X_\alpha) \leqslant \varkappa$$

holds for each α then

$$|X| \leqslant 2^\varkappa.$$