Normality and hereditary countable paracompactness of Pixley–Roy hyperspaces

by

Hidenori Tanaka (Ibaraki)

Abstract. In this paper, for the Pixley–Roy hyperspace \( \mathcal{F}(M) \) of a metric space \( M \), it will be shown that \( \mathcal{F}(M) \) is normal if and only if \( \mathcal{F}(M) \) is hereditarily countably paracompact.

Introduction. Throughout this paper, all spaces are assumed to be \( T_1 \)-spaces and \( M \) always denotes a metric space. Pixley–Roy hyperspace of the real line was defined by C. Pixley and P. Roy in [5] and later generalized by B. K. van Douwen in [1]. The Pixley–Roy hyperspace \( \mathcal{F}(X) \) of a space \( X \) has as its underlying set the collection of all nonempty finite subsets of \( X \). If \( F \in \mathcal{F}(X) \), then the basic open neighborhoods of \( F \) are of the form \( \{G \in \mathcal{F}(X) : F \subset G \subset U\} \), where \( U \) is an open subset of \( X \) containing \( F \). Therefore \( \{F \in \mathcal{F}(X) : F \subset U\} \) is closed if and only if \( F \subset V \) and \( H \subset U \). It was pointed out in [1] that every Pixley–Roy hyperspace is a zero-dimensional hereditarily metacompact space and \( \mathcal{F}(X) \) is a Moore space if and only if \( X \) is first countable.

\( M \) is said to be a \( q \)-set if every subset of \( M \) is an \( F_{\sigma} \)-set in \( M \) and a strong \( q \)-set if every finite power of \( M \) is a \( q \)-set. If, in addition, \( M \) is separable, then \( M \) is said to be a \( Q \)-set and a strong \( Q \)-set respectively. It is well known that the existence of an uncountable \( Q \)-set is indecomposable in ZFC and is equivalent to the existence of a separable normal nonmetrizable Moore space. T. Przymusinski [7] showed that the existence of an uncountable \( Q \)-set is equivalent to the existence of an uncountable strong \( Q \)-set.

Studying countable paracompactness in separable Moore spaces led to the notion of \( \mathcal{A} \)-sets. \( M \) is said to be a \( \mathcal{A} \)-set if \( M \) is separable and for any decreasing sequence \( \{A_n : n \in \mathbb{N}\} \) of subsets of \( M \) having \( \bigcap \{A_n : n \in \mathbb{N}\} = \emptyset \), there is a sequence \( \{U_n : n \in \mathbb{N}\} \) of open subsets of \( M \) having \( A_n \subset U_n \) for each \( n \in \mathbb{N} \) and \( \bigcap \{U_n : n \in \mathbb{N}\} = \emptyset \) and a strong \( \mathcal{A} \)-set if every finite power of \( M \) is a \( \mathcal{A} \)-set. It is clear that any \( Q \)-set (strong \( Q \)-set) is a \( \mathcal{A} \)-set (strong \( \mathcal{A} \)-set). The argument given by T. Przymusinski in [6] shows that any \( \mathcal{A} \)-set must have cardinality less than \( c \), where \( c \) is the cardinality of continuum. Thus any \( \mathcal{A} \)-set is strongly zero-dimensional.

E. K. van Douwen, T. Przymusinski and G. M. Reed showed that the existence...
of a separable countably paracompact nonmetrizable Moore space is equivalent to the existence of an uncountable $\mathcal{A}$-set (see [6]).

For normality and countable paracompactness of Pixley-Roy hyperspaces of metric spaces, M. E. Rudin [10], T. Przymusiński [8] and D. J. Lutzer [4] obtained the following elegant results (see also T. Przymusiński and F. D. Tall [9]): (1) if $M$ is separable then $\mathcal{F}[M]$ is normal if and only if $M$ is a strong $\mathcal{A}$-set; (2) if $M$ is strongly zero-dimensional then $\mathcal{F}[M]$ is normal if and only if $M$ is a strong $\mathcal{A}$-set; and (3) if $M$ is a strong $\mathcal{A}$-set then $\mathcal{F}[M]$ is countably paracompact.

Our purpose of this paper is to show the equivalence of normality and hereditary countable paracompactness of Pixley-Roy hyperspaces of metric spaces. To do so we introduce a notion of an almost strong $\mathcal{A}$-set, which is intermediate between the notions of a $\mathcal{A}$-set and a strong $\mathcal{A}$-set. Our result may be of interest in connection with the following D. J. Lutzer’s problem in [4]: Is every strong $\mathcal{A}$-set a strong $\mathcal{A}$-set?

Let $N$ denote the set of natural numbers. For $\mathcal{A}$-sets and $\mathcal{A}$-sets, see W. G. Fleissner [2].

§ 1. Preliminaries. Let $n \in N$ and let $\pi$ be a permutation of $(1, ..., n)$. For a point $x = (x_1, ..., x_n) \in X^n$, let $(x) = (x_{\pi(1)}, ..., x_{\pi(n)})$. A subset $A$ of $X^n$, $n \in N$, is symmetric if for any permutation $(1, ..., n)$, $(x) \in A$. Let $M$ be said to be an almost strong $\mathcal{A}$-set if for any $n \in N$, every symmetric subset of $M^n$ is an $F_{\sigma}$-set in $\mathcal{A}$.

Clearly every strong $\mathcal{A}$-set is an almost strong $\mathcal{A}$-set and every almost strong $\mathcal{A}$-set is a $\mathcal{A}$-set. Some results concerning almost strong $\mathcal{A}$-sets are given.

Lemma 1.1. If $M$ is strongly zero-dimensional, then $M$ is a strong $\mathcal{A}$-set if and only if $M$ is an almost strong $\mathcal{A}$-set.

Proof. It suffices to prove the “if” part. Let $M$ be a strongly zero-dimensional almost strong $\mathcal{A}$-set. Then $M$ is linearly orderable (see H. Herrlich [3]). Let $\pi$ be a linear order on $M$ generating the topology of $M$. Take $n \in N$ and assume that we have already proved that $M^n$ is a $\mathcal{A}$-set. Then $Z = \{(x_1, ..., x_n) : x_1 < x_2 < ... < x_{n+1}\}$.

Then $M^{n+1}$ is a finite union of $F_{\sigma}$-sets which are either homeomorphic to $Z$ or to some $M^n(k \leq n)$. In order to prove that $M^{n+1}$ is a $\mathcal{A}$-set, it is enough to prove that $Z$ is a $\mathcal{A}$-set. Let $\mathbb{A}$ be an arbitrary subset of $Z$ and let $\mathbb{A} = \bigcup \{\mathcal{A}(\pi) : \pi \text{ is a permutation of } \{1, ..., n+1\}\}$. Then $\mathbb{A}$ is a symmetric subset of $M^{n+1}$. Since $M$ is an almost strong $\mathcal{A}$-set, $\mathbb{A}$ is an $F_{\sigma}$-set in $M^{n+1}$. Thus $\mathbb{A} = \mathbb{A} \cap Z$ is an $F_{\sigma}$-set in $Z$. Hence $Z$ is a $\mathcal{A}$-set. It follows that $M$ is a strong $\mathcal{A}$-set.

Proposition 1.2. For every non-\(\sigma\)-discrete almost strong $\mathcal{A}$-set $M$, there are non-\(\sigma\)-discrete strong $\mathcal{A}$-set $M'$ and a one-to-one continuous mapping from $M'$ onto $M$.

Proof. By Lemma 1.1, it is essentially proved by T. Przymusiński (see [3], Lemma 5.10).

The following lemma is the key to our theorem. For a point $x = (x_1, ..., x_n) \in X^n$ and $n \in N$, let $F_x = \{x_1, ..., x_n\}$. Let $|M|$ denote the cardinality of a set $A$.

Lemma 1.3. Let $X$ and $Y$ be subsets of a metric space $M$ such that $M = X \cup Y$ and $X \cap Y = \emptyset$. If $X$ is an almost strong $\mathcal{A}$-set and $Y$ is a closed subset of $M$ with $|Y| \leq n_0$, then $M$ is an almost strong $\mathcal{A}$-set.

Proof. Let $A$ be a subset of $M$. Then $A \cap X$ is an $F_{\sigma}$-set in $X$. Since $X$ is an $F_{\sigma}$-set in $M$, $A \cap X$ is an $F_{\sigma}$-set in $M$. Since $|Y| \leq n_0$, it follows that $A = (A \cap X) \cup (A \cap Y)$ is an $F_{\sigma}$-set in $M$. Hence $M$ is a $\mathcal{A}$-set. Take $n \in N$ and assume that we have already proved that every symmetric subset of $M^n$ is an $F_{\sigma}$-set in $M^n$. Let $Z = \{(x_1, ..., x_{n+1}) : x_i \neq x_j \text{ for } i, j \leq n+1 \text{ and } i \neq j\}$.

Then $Z$ is an open subset of $M^{n+1}$. We shall show that every symmetric subset of $Z$ is an $F_{\sigma}$-set in $Z$. Let $A$ be a symmetric subset of $Z$. Since $X$ is an almost strong $\mathcal{A}$-set and $Y$ is a closed subset of $M$ with $|Y| \leq n_0$, we may assume that for each point $z = (z_1, ..., z_{n+1}) \in A$, $1 \leq |F_z \cap Y| \leq n$. Let $S = \{z = (y_1, ..., y_n) : A \text{ is a

ordered pair of distinct elements of } Y \text{ and } 1 \leq k \leq n\}$. Then we have $|S| \leq n_0$. Fix $z = (y_1, ..., y_n) \in S$. Define $A_z$ as follows: $z = (z_1, ..., z_{n+1}) \in A_z$ if and only if $z \in A$, $F_z \cap Y = F_z$ and $y_i = y_j$ for some $(y_1, ..., y_n) = (y_1, ..., y_n)$ such that $i < j$ and $i, j \leq k$ or $i < j < k$. Let $x = (z_1, ..., z_{n+1}) \in A_z$ and let $\{z_1, ..., z_{n+1}\}$ be a symmetric subset of $(1, ..., n+1)$ such that $z_i = y_j$ for $j \leq k$. Let $m_1, ..., m_{n+1}$ be the following sequence: $m_{n+k} = (1, ..., n+1) - \{z_1, ..., z_{n+1}\}$ such that if $p, q \leq n+k-1$ and $p \neq q$, then $m_p \neq m_q$. Define $x_n = (a_{n+1}, ..., a_{n+k})$. Then $x_n$ is a point of $M^{n-k+1}$. Let $B_n = \{x_n : x \in A_z\}$. Then $B_n$ is a symmetric subset of $M^{n-k+1}$. Thus there is a sequence $\{B_n : p \in N\}$ of closed subsets of $M^{n-k+1}$ such that $B_n = \bigcup \{B_n : p \in N\}$. Without loss of generality, we can assume that each $B_n$ is symmetric. For each $p \in N$, let $H_n = \{z \in A : F_z = F_z \text{ for some } w \in E_n\}$. It is easy to check that each $H_n$ is closed in $M^{n-k+1}$ since $A = \bigcup \{B_n : p \in N\}$ is an $F_{\sigma}$-set in $M^{n-k+1}$. Thus $M^{n-k+1}$ is a finite union of $Z$ and $F_{\sigma}$-sets which are homeomorphic to some $M^{n-k+1}$, it follows that every symmetric subset of $M^{n+1}$ is an $F_{\sigma}$-set in $M^{n+1}$. Thus $M$ is an almost strong $\mathcal{A}$-set.

Lemma 1.4. Let $f : M \to M'$ be a perfect mapping from a metric space $M$ onto a metric space $M'$. If $M$ is an almost strong $\mathcal{A}$-set, then $M'$ is also an almost strong $\mathcal{A}$-set.

Proof. For each $n \in N$, let $f^n : M^n \to M''$ be a perfect mapping from $M^n$ onto $M''$ induced by $f$. Let $A$ be a symmetric subset of $M^n$ and $n \in N$. Then $(f^n)^{-1}(A)$ is a symmetric subset of $M^n$. Since $M$ is an almost strong $\mathcal{A}$-set, $(f^n)^{-1}(A)$ is an $F_{\sigma}$-set in $M^n$. Thus $A$ is an $F_{\sigma}$-set in $M^n$. Hence $M'$ is an almost strong $\mathcal{A}$-set.

§ 2. Normality and hereditary countable paracompactness. For each $n \in N$, let $\mathcal{F}[X] = \{F \subseteq \mathcal{F}[X] : |F| \leq n\}$. Notice every $\mathcal{F}[X]$ is a closed subspace of $\mathcal{F}[X]$ and in particular, $\mathcal{F}[X]$ is a discrete closed subspace of $\mathcal{F}[X]$. 


Let $d$ be a compatible metric on $M$. For each $F \in \mathcal{F}[M]$, let $B(F, 1/n) = \bigcup \{B(x, 1/n) : x \in F\}$, where $B(x, 1/n) = \{y \in M : d(x, y) < 1/n\}$.

We give the main theorem in this paper.

**Theorem 2.1.** The following are equivalent.

(a) $\mathcal{F}[M]$ is normal.
(b) $\mathcal{F}[M]$ is hereditarily countably paracompact.
(c) $M$ is an almost strong $g$-set.

**Proof.** (a) $\Rightarrow$ (b). Since $\mathcal{F}[M]$ is a perfectly normal space, this implication is obvious.

(b) $\Rightarrow$ (c). We may assume that $M$ is not discrete. Let $x$ be a nonisolated point of $M$ and let $\{x_n : n \in \mathbb{N}\}$ be a sequence of distinct points of $M - \{x\}$ converging to $x$. Let $Z = \{x_n : n \in \mathbb{N}\} \cup \{x\}$ and let $Y = M - Z$. Then $Z$ is a compact subset of $M$. In order to prove this implication, from Lemma 1.3, it suffices to prove that $Y$ is an almost strong $g$-set. To see this, let $Y'$ be a space considering the following new topology on $Y$: for each $y \in Y'$, the basic open neighborhoods of $y$ are of the form $B(y, 1/n) \cup B(z, 1/n) \cap Y'$ and $n \in \mathbb{N}$. Then $Y'$ is first countable and consequently, $\mathcal{F}[Y']$ is a Moore space. It is clear that if $O$ is an open neighborhood of $F \cup \{x, x_n\}$ in $M$, where $F \in \mathcal{F}[Y']$ and $n \in \mathbb{N}$ then $O \cap Y'$ is an open neighborhood of $F$ in $Y'$, we need the following claim.

**Claim.** $\mathcal{F}[Y']$ is normal for each $n \in \mathbb{N}$.

**Proof of Claim.** We shall show that every $\mathcal{F}[Y']$ is perfectly normal. If $n = 1$, then $\mathcal{F}[Y']$ is a discrete space. Thus $\mathcal{F}[Y']$ is normal. Let $n \geq 2$ and assume that $\mathcal{F}$ is a closed subset of $\mathcal{F}[Y']$. It is enough to prove that there is a sequence $\{\mathcal{F}_m : m \in \mathbb{N}\}$ of open subsets of $\mathcal{F}[Y']$ satisfying $\mathcal{F} = \bigcap \{\mathcal{F}_m : m \in \mathbb{N}\}$ (see P. Zarnor). (11) [For each $m \in \mathbb{N}$, let

$$\mathcal{F}_m = \{F \cup \{x, x_n\} : F \in \mathcal{F}\}.$$  

Define $\mathcal{F}_m = \{F \cup \{x, x_n\} : F \in \mathcal{F}\}$, if and only if $F \in \mathcal{F}_m$ or $F = G \cup \{x, x_n\}$, where $G \in \mathcal{F}[Y'] - \mathcal{F}$ and there is an open subset $O$ in $M$ containing $F$ such that $[G, O \cap Y'] \cap \mathcal{F} = \emptyset$. Let $\mathcal{F} = \{F \cup \{x, x_n\} : F \in \mathcal{F}\}$ and let

$$\mathcal{F} = \bigcup \{\mathcal{F}_m : m \in \mathbb{N}\}.$$  

We consider $\mathcal{F}$ with the subspace topology of $\mathcal{F}[M]$. Then each $\mathcal{F}_m$ is an open-and-closed subset of $\mathcal{F}$. Let $\mathcal{F}_m = \{F \cup \{x, x_n\} : m \in \mathbb{N}\}$ be the decreasing sequence of closed subsets of $\mathcal{F}$ having $\bigcap \{\mathcal{F}_m : m \in \mathbb{N}\} = \emptyset$. To see this, assume that $F \not\in \mathcal{F}_m$ and $m \in \mathbb{N}$. In case of $F \in \mathcal{F}$ and $m \in \mathbb{N}$, then $\mathcal{F}_m$ is an open neighborhood of $F$ in $\mathcal{F}$ such that $\mathcal{F}_m \cap \mathcal{F} = \emptyset$. In case of $F \in \mathcal{F}$ and $m \not\in \mathbb{N}$, then $F = G \cup \{x, x_n\}$ for some $G \in \mathcal{F}[Y']$. Then we have $G \not\in \mathcal{F}$. Thus there is an open neighborhood $O$ of $F$ in $M$ such that $[G, O \cap Y'] \cap \mathcal{F} = \emptyset$. Assume that $K \in [F, O] \cap \mathcal{F} = \emptyset$. Then $K = J \cup \{x, x_n\}$ for some $J \in \mathcal{F}$ and $p \not\in m$. From the construction of $\mathcal{F}$, we have $p = r$. Thus we have $K = J \cup \{x, x_n\}$ and consequently, $J \not\in [G, O \cap Y'] \cap \mathcal{F} = \emptyset$. This is a contradiction. Thus it follows that $[F, O \cap Y'] \cap \mathcal{F} = \emptyset$.

Therefore $\mathcal{F}[Y']$ is perfectly normal for each $n \in \mathbb{N}$.
Now we show that $Y$ is an almost strong $q$-set. Let $A$ be a subset of $Y$. Let \( \mathcal{A} = \{ (x, y) : x \in A \} \) and \( \mathcal{B} = \{ (y, x) : y \in Y \setminus A \} \). Then \( \mathcal{A} \) and \( \mathcal{B} \) are disjoint closed subsets of \( \mathcal{S}_Y[Y] \). Since \( \mathcal{S}_Y[Y] \) is normal, there are disjoint open subsets \( U \) and \( \nu' \) of \( \mathcal{S}_Y[Y] \) such that \( U \in \mathcal{A} \) and \( \nu' \in \mathcal{B} \). For each \( n \in \mathbb{N} \), let

\[
A_n = \{ (y, x) : (y, x) \in \mathcal{B} \} \cap Y \cap \mathcal{S}_Y[Y] \subset U \cap \nu'.
\]

Since \( A = \bigcup \{ A_n : n \in \mathbb{N} \} \), in order to prove that \( A \) is an \( F_\nu \)-set in \( Y \), it suffices to prove that \( c_1A_n \in A \) for each \( n \in \mathbb{N} \). Suppose that \( (y, x) \in Y \setminus A \). Then there is a natural number \( m \) such that \( (y, x) \in B(y, 1/m) \cap \mathcal{S}_Y[Y] \). Since \( y \in A_m \), there is an \( (y, x) \in A_n \) such that \( y \in B(y, 1/m) \cap Y \). Then

\[
(y, y') \in \mathcal{B}(y', 1/m) \cap B(x, 1/m) \cap Y \cap \mathcal{S}_Y[Y] \setminus \mathcal{S}_Y[Y] = \emptyset.
\]

This is a contradiction. Thus it follows that \( c_1A_n \in A \) for each \( n \in \mathbb{N} \). Hence \( A \) is an \( F_\nu \)-set in \( Y \). Thus \( Y \) is a \( q \)-set. Take \( n \in \mathbb{N} \) and assume that we have already proved that every symmetric subset of \( Y \) is an \( F_\nu \)-set in \( Y \). Let

\[
O = \{(x_1, ..., x_n) : i \neq j \text{ for } i, j \leq n+1 \text{ and } i \neq j \}
\]

Since \( Y^{n+1} \) is a finite union of \( O \) and \( F_\nu \)-sets which are homeomorphic to some \( Y^k \) (\( k \leq n \)), it is enough to prove that every symmetric subset of \( O \) is an \( F_\nu \)-set in \( O \). Let \( A \) be a symmetric subset of \( O \). Let \( \mathcal{A} = \{ (y, x) : y \in A \} \) and \( \mathcal{B} = \{ (x, y) : x \in A \} \). Then \( \mathcal{A} \) and \( \mathcal{B} \) are disjoint closed subsets of \( \mathcal{S}_{n+1}[Y] \). By the Claim, \( \mathcal{S}_{n+1}[Y] \) is perfectly normal and consequently, \( \mathcal{S}_{n+1}[Y] \) is hereditarily normal. Thus \( \mathcal{S}_{n+1}[Y] \setminus \mathcal{S}_Y[Y] \) is normal. Hence there are disjoint open subsets \( U \) and \( \nu' \) of \( \mathcal{S}_{n+1}[Y] \setminus \mathcal{S}_Y[Y] \) such that \( U \in \mathcal{A} \) and \( \nu' \in \mathcal{B} \). For each \( m \in \mathbb{N} \), let \( A_m = \{ (y, x) : (y, x) \in \mathcal{B} \} \cap \mathcal{S}_Y[Y] \setminus \mathcal{S}_Y[Y] \subset U \cap \nu'. \) Clearly \( A = \bigcup \{ A_n : m \in \mathbb{N} \} \). We shall show that \( c_1A_m \in A \) for each \( m \in \mathbb{N} \). Assume that \( y = (y_1, ..., y_n) \in O \setminus A \cap c_1A_m \) for some \( m \in \mathbb{N} \). Then we have \( F_y \in \mathcal{B} \setminus \nu' \). Then there is a \( k \in \mathbb{N} \) such that \( (y_1, 1/k) \mid 1 = 1, ..., n+1 \) is pairwise disjoint in \( M \) and

\[
F_y \cap F_0 \cap B(y, 1/k) \cap B(x, 1/k) \cap \mathcal{B}(x, 1/m) < \mathcal{S}_{n+1}[Y] \setminus \mathcal{S}_Y[Y] \subset U \cap \nu'.
\]

Since \( y \in c_1A_m \), there is a \( y' = (y_1', ..., y_n') \in y \cap B(y, 1/k) \cap A_n \). Thus we have \( y' \in B(y, 1/k) \) for each \( i = 1, ..., n+1 \). Hence we have \( F_y \subset B(F_0, 1/k) \). Since \( k \geq m \), we have \( F_0 \subset B(F_0, 1/m) \). Hence

\[
F_y \cap F_0 \cap B(F_0, 1/m) \cap B(x, 1/m) \subset \mathcal{B}(x, 1/m) \cap \mathcal{S}_Y[Y] \setminus \mathcal{S}_Y[Y] \subset U \cap \nu',
\]

which is a contradiction. Thus it follows that \( c_1A_m \in A \) for each \( m \in \mathbb{N} \). Hence \( A \) is an \( F_\nu \)-set in \( O \). Therefore it follows that \( Y \) is an almost strong \( q \)-set.
Increasing strengthenings of cardinal function inequalities

by

I. Juhasz and Z. Szentmiklochy (Budapest)

Abstract. We prove that the following increasing strengthenings of two cardinal function inequalities given in [2] and [1] respectively are valid.

THEOREM 1. If $X = T_4$ and $X = \bigcup X_n$ (i.e. $X$ is the union of an increasing chain of its subspaces $X_n$) and $c(X_n) \cdot c(X) \leq \kappa$ for all $n$ then $|X| \leq 2^\kappa$.

THEOREM 2. If $X = T_4$ and $X = \bigcup X_n$, where $X_n = T_4$ and $nL(X_n) \cdot c(X) \leq \kappa$ for all $n$ then $|X| \leq 2^\kappa$.

In [3] the first author has initiated the study of strengthening certain cardinal function inequalities in the following manner. A general form of a cardinal function inequality may be given as follows: If $\phi$ is some given cardinal function and $X$ is a space having some property $P$ then $\phi(X) \leq \kappa$. We call an increasing strengthening of this inequality any statement of the following form: If $X = \bigcup X_n$ is the increasing union of its subspaces $X_n$, where every $X_n$ has property $P$ and $X$ has property $Q$ then $\phi(X) \leq \kappa$.

A number of such increasing strengthenings of inequalities were proven in [3], as a major problem, however, it remained open whether the inequality $|X| \leq 2^{|X_n|}$, for any $T_2$ space $X$, admits such an increasing strengthening.

Theorem 1 of the present paper gives the affirmative answer to this question. The ideas needed in the proof of Theorem 1, with appropriate modifications, also allowed us to show that the inequality $|X| \leq 2^{nL(X) \cdot c(X)}$ for any $T_4$ space $X$ proved in [1] also admits an increasing strengthening.

Notation and terminology, unless otherwise explained, is identical with that used in [3].

THEOREM 1. If $X = \bigcup X_n$ is $T_2$ and $c(X_n) \cdot c(X) \leq \kappa$

holds for each $n$ then $|X| \leq 2^\kappa$. 