The structure of ω_1-like orderings

by

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Abstract. We show that if $\mathfrak{A}$ is a structure having a binary relation $<$ that is an ω_1-like ordering, then the Scott-height of $\mathfrak{A}$ is $\leqslant \omega_1$. We obtain additional structural information for the case of $\mathfrak{A}$ which has, besides $<$, only unary relations.

To any structure one can attach its Scott-height, an ordinal described e.g. in [2] or [1] (c.f. also § 0 of this paper). The Scott-height is a significant measure of complexity of a structure and given a family K of structures it is natural to ask what is its "Scott-spectrum" i.e. the class of ordinals which are Scott-heights of elements of K. Very little is known about these spectra in general and we think it worthwhile to find the spectra of certain concrete families K. The initial aim of the investigation described here was to study the Scott-spectrum of the family of ω_1-like orderings. While attempting to do this we gathered much additional information about the structure of ω_1-like orderings. Our conjecture was that the Scott-height of these orderings is always $\leqslant \omega_1$. In § 1 we show by a direct proof that this conjecture holds true even for ω_1-like orderings with arbitrary extra-predicates. The additional structural information — which ellucidates in various ways the reason for the Scott-height being $\leqslant \omega_1$ — concerns the ω_1-like orderings with no extrapredicates (or, even, with unary extrapredicates); this is presented in §§ 2–6.

Consider these two familiar examples or orderings: the well ordered ω_1 and the dense $\eta\cdot\omega_1$. Both are ω_1-like but otherwise have quite different properties. For example, ω_1 is not $L_{\infty\omega}$-equivalent to any countable ordering while $\eta\cdot\omega_1$ is $L_{\infty\omega}$-equivalent to the countable η. In fact, ω_1 is not $L_{\infty\omega}$-equivalent to any ordering except (those isomorphic to) itself while $\eta\cdot\omega_1$ can be seen to be $L_{\infty\omega}$-equivalent to $2^{\aleph_1}$-many nonisomorphic ω_1-like orderings as well as to orderings of arbitrary cofinality. Every element of ω_1 is definable in $L_{\infty\omega}$ (actually in $L_{\omega_1\omega}$) while no element of $\eta\cdot\omega_1$ has such a definition, in fact, it has no nontrival $L_{\infty\omega}$-definable subsets.

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In a sense, the two examples considered are not as special as one might think. It turns out that there are just two types of ω_1-like orderings. Those of the first, which we call of "bounded" type share many properties with ω_1, those of the second, of "unbounded" type, behave just like $\eta\cdot\omega_1$. A few examples might clarify the meaning of the last two statements.

As in the case of ω_1, an ordering $\mathfrak{A}$ of bounded type is not $L_{\infty\omega}$-equivalent to any countable ordering; unlike ω_1, it may be $L_{\infty\omega}$-equivalent to other (non-isomorphic) orderings. However, $\mathfrak{A}$ is not $L_{\infty\omega}$-equivalent to any other ω_1-like ordering. The elements of $\mathfrak{A}$ are not necessarily $L_{\infty\omega}$-definable but each of them is contained in a definable countable set; in fact, $\mathfrak{A}$ is the union of a strictly increasing continuous ω_1-chain of $L_{\omega_1\omega}$-definable proper initial segments. This last property implies that all $L_{\infty\omega}$-equivalents of $\mathfrak{A}$ have cofinality ω_1. As to the ω_1-like orderings of unbounded type, they possess all the properties of $\eta\cdot\omega_1$ which we mentioned (save for the possibility of the existence of countably many $L_{\omega_1\omega}$-definable subsets); this follows from our Theorem 4.3 which describes very precisely the structure of the ω_1-like orderings of unbounded type.

The paper is organized as follows. § 0 contains preliminaries while § 1 contains the theorem about the Scott-height of ω_1-like orderings with extra predicates. The ω_1-like orderings with no (or only unary) extra predicates are shown in § 2 to have a rich group of automorphisms; in the same section the orderings of bounded and unbounded type are defined. § 3 deals with the ω_1-like ordering of bounded type, § 4 with those of unbounded and § 5 briefly sums up the results. In § 6 we use the techniques developed in §§ 3–4 to find out which uncountable orderings are $L_{\infty\omega}$-equivalent to ω_1-like ones. In § 7 we describe when an ω_1-like ordering has Scott height strictly less than ω_1.

§ 0. Preliminaries. We denote structures by capital gothic letters $\mathfrak{A}, \mathfrak{B}, \mathfrak{A}_0, \mathfrak{B}_0, \ldots$ and their universes by $A, B, A_0, B_0, \ldots$ resp. We write "$\vec{a}\in A$" or even "$\vec{a}\in\mathfrak{A}$" to mean that $\vec{a}$ is a finite sequence of elements in A. If $\psi(\vec{x})$ is a formula, we let $\psi(\mathfrak{A}) = \{\vec{a}\colon \vec{a}\in\mathfrak{A}\ \&\ \mathfrak{A}\models\psi[\vec{a}]\}$.

We say that the structures $\mathfrak{A}$ and $\mathfrak{B}$ are ∞-equivalent, α-equivalent or $\omega_1\omega$-equivalent, in symbols $\mathfrak{A}\underset{\infty}{\equiv}\mathfrak{B}$, $\mathfrak{A}\underset{\alpha}{\equiv}\mathfrak{B}$, $\mathfrak{A}\underset{\omega_1\omega}{\equiv}\mathfrak{B}$ if they satisfy the same $L_{\infty\omega}$-sentences, the same $L_{\infty\omega}$-sentences of quantifier depth $\leqslant\alpha$, the same $L_{\omega_1\omega}$-sentences resp. We will also speak of ∞-elementary substructures etc. with the symbols $\underset{\infty}{\prec}$, etc.

The Scott-height of a structure $\mathfrak{A}$, $\mathrm{SH}(\mathfrak{A})$ is the least ordinal α such that for all $\vec{a}, \vec{a}'\in\mathfrak{A}$, $(\mathfrak{A},\vec{a})\underset{\alpha}{\equiv}(\mathfrak{A},\vec{a}')$ implies that $(\mathfrak{A},\vec{a})\underset{\alpha+1}{\equiv}(\mathfrak{A},\vec{a}')$ (it then easily follows that $(\mathfrak{A},\vec{a})\underset{\alpha}{\equiv}(\mathfrak{A},\vec{a}')$ implies $(\mathfrak{A},\vec{a})\underset{\infty}{\equiv}(\mathfrak{A},\vec{a}')$). A Scott-sentence of $\mathfrak{A}$ is a sentence φ s.t. $\mathfrak{B}\models\varphi$ iff $\mathfrak{B}\underset{\infty}{\equiv}\mathfrak{A}$. Scott showed that every countable structure has an $L_{\omega_1\omega}$ Scott-sentence and the same argument shows that any structure has an $L_{\infty\omega}$ Scott-sentence. The ∞-type of a sequence $\vec{a}\in\mathfrak{A}$ is the set of all $L_{\infty\omega}$-formulas satisfied by $\vec{a}$ in $\mathfrak{A}$; the notions of α-type $\omega_1\omega$-type are defined similarly. We say that the α-type of $\vec{a}\in\mathfrak{A}$ is axiomatized by $\psi(\vec{x})$ if $\mathfrak{A}\models\psi[\vec{a}]$ and $\psi(\vec{x})$ implies all formulas in the α-type of $\vec{a}$. The α-type of $\vec{a}$ is always axiomatized by an $L_{\infty\omega}$-formula $\varphi_\alpha(\vec{x})$ of quantifier depth α and when α and $\mathfrak{A}$ are countable φ_α can be taken in $L_{\omega_1\omega}$. All the facts mentioned in this paragraph can be seen in, e.g., [2] or [6].

In this paper we are concerned with structures $\mathfrak{A}$ having a distinguished binary relation $<$ which is a linear ordering. When saying that $\mathfrak{A}'$ is an initial segment of $\mathfrak{A}$ we will mean that $(A', <)$ is an initial segment of $(A, <)$; for $a\in\mathfrak{A}$ we denote by $\mathfrak{A}\restriction a$ the initial segment of $\mathfrak{A}$ with universe $A\restriction a = \{b\colon b\in A\ \&\ b<a\}$. We use the name "ordering" for structures of the form $\mathfrak{A} = \langle A; <, P_0, P_1, \ldots\rangle$ with $P_0, P_1, \ldots$ unary predicates. The name "structure" will, of course, be used for $\mathfrak{A}$ which, besides $<$, may have predicates with more than one variable.

A structure (or an ordering) $\mathfrak{A}$ is called ω_1-like if it is uncountable but $\mathfrak{A}\restriction a$ is countable for all $a\in\mathfrak{A}$.

§ 1. ω_1-like structures. A useful characterization of ∞-equivalence says that $\mathfrak{A}\underset{\infty}{\equiv}\mathfrak{B}$ iff there is a back-and-forth relation between $\mathfrak{A}$ and $\mathfrak{B}$. A relation $\sim$ between finite sequences of $\mathfrak{A}$ and $\mathfrak{B}$ is called a back-and-forth relation if $\langle\,\rangle\sim\langle\,\rangle$ (the void sequences are related) and whenever $\vec{a}\sim\vec{b}$ we have $\forall a'\in A\ \exists b'\in B\ \vec{a}^\frown\langle a'\rangle\sim\vec{b}^\frown\langle b'\rangle$ and $\forall b'\in B\exists a'\in A\ \vec{a}^\frown\langle a'\rangle\sim\vec{b}^\frown\langle b'\rangle$ (see [1] for details).

In general it is known that $\mathrm{SH}(\mathfrak{A})<|\mathfrak{A}|^+$. For ω_1-like $\mathfrak{A}$, however, we can do better.

THEOREM 1.1. *If $\mathfrak{A}$ is an ω_1-like structure then $\mathrm{SH}(\mathfrak{A})\leqslant\omega_1$.*

Proof. Assume that $(\mathfrak{A},\vec{a})\underset{\omega_1}{\equiv}(\mathfrak{A},\vec{a}')$. We want to show that $(\mathfrak{A},\vec{a})\underset{\infty}{\equiv}(\mathfrak{A},\vec{a}')$; as $(\mathfrak{A},\vec{a})$ and $(\mathfrak{A},\vec{a}')$ are ω_1-like, this follows from a more general fact.

THEOREM 1.2. *If $\mathfrak{A}$ and $\mathfrak{B}$ are ω_1-like structures and $\mathfrak{A}\underset{\omega_1}{\equiv}\mathfrak{B}$, then $\mathfrak{A}\underset{\infty}{\equiv}\mathfrak{B}$.*

Proof. Letting $\vec{a}\sim\vec{b}$ iff $(\mathfrak{A},\vec{a})\underset{\omega_1}{\equiv}(\mathfrak{B},\vec{b})$, we want to show that $\sim$ is a back-and-forth relation. Obviously, it suffices to show:

LEMMA 1.3. *If $\mathfrak{A}$ and $\mathfrak{B}$ are ω_1-like and $\mathfrak{A}\underset{\omega_1}{\equiv}\mathfrak{B}$ then for each $a\in\mathfrak{A}$ there is a $b\in\mathfrak{B}$ such that $(\mathfrak{A},a)\underset{\omega_1}{\equiv}(\mathfrak{B},b)$.*

Proof. For all $\alpha<\omega_1$ pick $b_\alpha\in B$ s.t. $(\mathfrak{A},a)\underset{\alpha}{\equiv}(\mathfrak{B},b_\alpha)$. We claim that this can be so done as to have $\langle b_\alpha\colon \alpha<\omega_1\rangle$ bounded in $\mathfrak{B}$. Since $\mathfrak{B}$ is ω_1-like, this means that for some $b\in B$, $b = b_\alpha$ for arbitrarily large $\alpha<\omega_1$, hence, $(\mathfrak{A},a)\underset{\omega_1}{\equiv}(\mathfrak{B},b)$.

To complete the proof, assume that no bounded $\langle b_\alpha\colon \alpha<\omega_1\rangle$ as above can be found. Then an increasing sequence $\langle\alpha_\varkappa\colon \varkappa<\omega_1\rangle$ of countable ordinals can be found s.t.

$$(*)\qquad \mathfrak{B}\models\exists x\Big(\bigwedge_{i<\varkappa}\varphi_{\alpha_i}(x)\wedge\forall y\leqslant x\sim\phi_{\alpha_\varkappa}(y)\Big)$$

where $\varphi_\alpha(x)$ is an $L_{\infty\omega}$-formula of quantifier depth α axiomatizing the α-type of a. It follows that $\mathfrak{A}$ satisfies (*) as well so we can find $a_\varkappa \in A$ for which:

$$\mathfrak{A} \models \bigwedge_{i<\varkappa} \varphi_{\alpha_i}(a_\varkappa) \wedge \forall y \leqslant a_\varkappa \sim \varphi_{\alpha_\varkappa}(y).$$

As $\mathfrak{A} \models \varphi_\alpha(a)$ for all α, it easily follows that $\langle a_\varkappa\colon \varkappa < \omega_1\rangle$ is an increasing sequence bounded by a, a contradiction to the ω_1-likeness of $\mathfrak{A}$.

§ 2. **ω_1-like orderings.** In general, $(\mathfrak{A}, \vec{a}) \underset{\infty}{\equiv} (\mathfrak{A}, \vec{b})$ does not imply that $(\mathfrak{A}, \vec{a}) \cong (\mathfrak{A}, \vec{b})$, i.e. that $\mathfrak{A}$ has an automorphism mapping $\vec{a}$ to $\vec{b}$. However:

THEOREM 2.1. *If $\mathfrak{A}$ is an ω_1-like ordering and $(\mathfrak{A}, \vec{a}) \underset{\infty}{\equiv} (\mathfrak{A}, \vec{b})$ then $(\mathfrak{A}, \vec{a}) \cong (\mathfrak{A}, \vec{b})$.*

Proof. Assume first that $\vec{a}$ and $\vec{b}$ are singletons $\langle a_0\rangle$, $\langle a_1\rangle$ and w.l.o.g., $a_0 < a_1$. Then we can define by induction $a_2, a_3, \ldots$ s.t. for all n, $(\mathfrak{A}, a_n, a_{n+1}) \underset{\infty}{\equiv} (\mathfrak{A}, a_{n+1}, a_{n+2})$. Thus, in particular, we have $a_0 < a_1 < a_2 < \ldots$ Let $\mathfrak{A}_n = \mathfrak{A} \restriction a_n$. Then $(\mathfrak{A}, a_0) \underset{\infty}{\equiv} (\mathfrak{A}, a_1)$ implies that $\mathfrak{A}_0 \underset{\infty}{\equiv} \mathfrak{A}_1$ and as we are dealing with countable structures, this means that $\mathfrak{A}_0 \cong \mathfrak{A}_1$ by an isomorphism $f_0\colon A_0 \to A_1$. Likewise, $(\mathfrak{A}, a_n, a_{n+1}) \underset{\infty}{\equiv} (\mathfrak{A}, a_{n+1}, a_{n+2})$ means that $\mathfrak{A}_{n+1} - \mathfrak{A}_n \underset{\infty}{\equiv} \mathfrak{A}_{n+2} - \mathfrak{A}_{n+1}$, hence $\mathfrak{A}_{n+1} - \mathfrak{A}_n \cong \mathfrak{A}_{n+2} - \mathfrak{A}_{n+1}$ by an isomorphism $f_n\colon A_{n+1} - A_n \to A_{n+2} - A_{n+1}$. But then, as we are considering orderings (as opposed to arbitrary structures) we get that $f_\omega = \bigcup_{n<\omega} f_n$ is an automorphism of $\mathfrak{A}_\omega = \bigcup_{n<\omega} \mathfrak{A}_n$ mapping a_0 to a_1. We can extend it to an automorphism $f \supseteq f_\omega$ of all of $\mathfrak{A}$ by letting $f(a) = a$ whenever $a \in A - A_\omega$.

If $\vec{a} = (a_0, \ldots, a_n)$, $\vec{b} = (b_0, \ldots, b_n)$ with, say a_n and b_n the largest elements in the two sequences then we already know that $\mathfrak{A}$ has an automorphism g mapping a_n to b_n and as $(\mathfrak{A} \restriction a_n, a_0, \ldots, a_{n-1}) \underset{\infty}{\equiv} (\mathfrak{B} \restriction b_n, b_0, \ldots, b_{n-1})$, there is also an isomorphism h of $\mathfrak{A} \restriction a_n$ onto $\mathfrak{B} \restriction b_n$ mapping $a_0, \ldots, a_{n-1}$ onto $b_0, \ldots, b_{n-1}$. But then defining $f\colon A \to A$ by $f(x) = h(x)$ for $x < a_n$ and $f(x) = g(x)$ for $x \geqslant a_n$, we get an automorphism of $\mathfrak{A}$ mapping $\vec{a}$ onto $\vec{b}$.

We now distinguish between two types of ω_1-like orderings: bounded and unbounded. An ω_1-like ordering is said to be of *unbounded type* if it has an unbounded family of initial segments isomorphic to each other. In other words, there is a countable isomorphism type ξ such that every element of $\mathfrak{A}$ belongs to an initial segment of type ξ; the standard example is $\eta \cdot \omega_1$ (in which case, $\xi = \eta$). The ω_1-like orderings which do not have this property are called of *bounded type*; the standard example is ω_1 itself.

We devote the next section to the study of ω_1-like orderings of bounded type and § 4 to those of unbounded type.

§ 3. **ω_1-like orderings of bounded type.**

THEOREM 3.1. *If $\mathfrak{A}$ and $\mathfrak{B}$ are $\omega_1\omega$-equivalent ω_1-like orderings of bounded type then they are isomorphic to each other.*

Proof. With each $a \in A$ we naturally associate the structure $\mathfrak{A}(a)$ — the reduct of $\mathfrak{A}$ to the set $A(a) = \{b\colon b \in A \;\&\; \exists a' \geqslant b \;\; \mathfrak{A} \restriction a' \cong \mathfrak{A} \restriction a\}$. $\mathfrak{A}$ being of bounded type, $\mathfrak{A}(a)$ is one of its countable initial segments. $A(a)$ is definable by an $L_{\omega_1\omega}$-formula $\theta_a(x)$ saying that for some $y \geqslant x$, the set of elements less than y satisfies the Scott-sentence of $\mathfrak{A} \restriction a$. $\theta_a(x)$ enjoys a very special property: whenever a (not necessarily ω_1-like) ordering $\mathfrak{B}$ is $\omega_1\omega$-equivalent to $\mathfrak{A}$, we have: (a) $\theta_a(\mathfrak{B})$ is a proper initial segment of $\mathfrak{B}$, and, (b) for all initial segments $\mathfrak{B}'$ of $\mathfrak{B}$, if $\theta_a(\mathfrak{B}) \subseteq \mathfrak{B}'$, then $\theta_a(\mathfrak{B}') = \theta_a(\mathfrak{B})$. Let's call the $L_{\omega_1\omega}$-formulas with this property "nice". It is immediate that a countable disjunction of nice formulas is nice. So, we can easily find a strictly increasing continuous chain $\langle \mathfrak{A}_\alpha\colon \alpha < \omega_1\rangle$ of initial segments of $\mathfrak{A}$ such that the universe of $\mathfrak{A}_\alpha$ is $A_\alpha = \psi_\alpha(\mathfrak{A})$ for some nice $\psi_\alpha(x)$. Moreover, if $\mathfrak{B} \underset{\omega_1\omega}{\equiv} \mathfrak{A}$ then the sets $B_\alpha = \psi_\alpha(\mathfrak{B})$ are the universes of a strictly increasing continuous sequence $\langle \mathfrak{B}_\alpha\colon \alpha < \omega_1\rangle$ of initial segments of $\mathfrak{B}$; if $\mathfrak{B}$ is ω_1-like — in particular if $\mathfrak{B} = \mathfrak{A}$ — then, of course, $\mathfrak{B} = \bigcup_{\alpha<\omega_1} \mathfrak{B}_\alpha$.

We can now complete the proof of Theorem 3.1. If $\mathfrak{A}$, $\mathfrak{B}$ are both ω_1-like and are $\omega_1\omega$-equivalent then we decompose them as above and define by induction a continuous chain $\langle f_\alpha\colon \alpha < \omega_1\rangle$ of isomorphisms $f_\alpha\colon \mathfrak{A}_\alpha \to \mathfrak{B}_\alpha$. To begin with, notice that $\mathfrak{A}_\alpha \cong \mathfrak{B}_\alpha$ as the $\omega_1\omega$-equivalence implies that $\mathfrak{B}_\alpha = \psi_\alpha(\mathfrak{B})$ satisfies the Scott-sentence of $\mathfrak{A}_\alpha = \psi_\alpha(\mathfrak{A})$. In particular, $\mathfrak{A}_0 \cong \mathfrak{B}_0$ by a map f_0. For limit δ, set $f_\delta = \bigcup_{\alpha<\delta} f_\alpha$. Once f_α is defined, pick an isomorphism g between $\mathfrak{A}_{\alpha+1}$ and $\mathfrak{B}_{\alpha+1}$ and define $f_{\alpha+1}(x) = f_\alpha(x)$ for $x \in A_\alpha$ and $f_{\alpha+1}(x) = g(x)$ for $x \in A_{\alpha+1} - A_\alpha$ (that $f_{\alpha+1}$ is an isomorphism follows since $g(x)$ must take A_α onto B_α). We conclude that $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$ by the map $f = \bigcup_{\alpha<\omega_1} f_\alpha$.

Using the information contained in the previous proof, we infer additional results. In all statements throughout this section, $\mathfrak{A}$ *is assumed to be an ω_1-like ordering of bounded type.*

The example of $(\omega_2, <) \underset{\omega_1\omega}{\equiv} (\omega_1, <)$ (cf. [3]) shows that Theorem 3.1 cannot be extended to $\mathfrak{B}$ which is not ω_1-like. However:

THEOREM 3.2. *If $\mathfrak{B} \underset{\omega_1\omega}{\equiv} \mathfrak{A}$, then $\mathfrak{B}' \underset{\omega_1\omega}{\prec} \mathfrak{B}$ for some initial segment $\mathfrak{B}'$ which is ∞-equivalent to $\mathfrak{A}$.*

Proof. As in the proof of 3.1, set $\mathfrak{B}_\alpha = \psi_\alpha(\mathfrak{B})$ and take $\mathfrak{B}' = \bigcup_{\alpha<\omega_1} \mathfrak{B}_\alpha$. $\mathfrak{B}_\alpha$ satisfies the Scott-sentence of the countable structure $\mathfrak{A}_\alpha$, hence $\mathfrak{B}_\alpha \underset{\infty}{\equiv} \mathfrak{A}_\alpha$; moreover, since ψ_α defines $\mathfrak{A}_\alpha$ and $\mathfrak{B}_\alpha$ in $\mathfrak{A}_{\alpha+1}$ and $\mathfrak{B}_{\alpha+1}$ respectively, we also have $\mathfrak{B}_{\alpha+1} - \mathfrak{B}_\alpha \underset{\infty}{\equiv} \mathfrak{A}_{\alpha+1} - \mathfrak{A}_\alpha$. As we are dealing with orderings (rather than arbitrary structures) we can paste back-and-forth relations together to get $\mathfrak{B}' \underset{\infty}{\equiv} \mathfrak{A}$.

We have still to show that $\mathfrak{B}' \underset{\omega_1\omega}{\prec} \mathfrak{B}$. To this end we use:

LEMMA 3.3. *For all $\vec{a} \in A$, there is an $L_{\omega_1\omega}$-formula $\theta^*(\vec{x})$ which axiomatizes the ∞-type of $\vec{a}$ inside $\mathfrak{A}$ (i.e. $(\mathfrak{A}, \vec{a}) \underset{\infty}{\equiv} (\mathfrak{A}, \vec{a}')$ iff $\mathfrak{A} \models \theta^*(\vec{a}')$). Moreover, $\theta^*(\vec{x})$ is such that $\theta^*(\mathfrak{B}) = \theta^*(\mathfrak{B}')$ whenever $\mathfrak{B}, \mathfrak{B}' \underset{\omega_1\omega}{\equiv} \mathfrak{A}$ and $\mathfrak{B}' \subset \mathfrak{B}$.*

Proof. For some α, $\vec{a} \in \mathfrak{A}_\alpha = \psi_\alpha(\mathfrak{A})$. From the proof of Scott's theorem (in, e.g., [4]) we know that the ∞-type of $\vec{a}$ is axiomatized *inside* $\mathfrak{A}_\alpha$ by an $L_{\omega_1\omega}$-formula $\theta(\vec{x})$. If $\theta'(\vec{x})$ is the relativization of θ to ψ_α, let $\theta^*(\vec{x})$ be the formula stating that $\theta'(\vec{x})$ holds and all the elements of $\vec{x}$ satisfy ψ_α. We let the reader check that this formula satisfies the claims of the lemma (use the niceness of ψ_α for the second claim).

Concluding the Proof of 3.2. If $\vec{b} \in \mathfrak{B}'$ and $\mathfrak{B} \models \exists y \varphi(\vec{b}, y)$ for any $L_{\omega_1\omega}$-formula φ, we must show that $\mathfrak{B}' \models \exists y \varphi(\vec{b}, y)$ as well. Let $\vec{a} \in \mathfrak{A}$ be such that $(\mathfrak{A}, \vec{a}) \underset{\infty}{\equiv} (\mathfrak{B}', \vec{b})$. If $\theta^*(\vec{x})$ is as in 3.3 then $\mathfrak{B}, \mathfrak{B}' \models \theta^*[\vec{b}]$. As $\theta^*(\vec{x})$ is consistent with $\exists y \varphi(\vec{x}, y)$ in $\mathfrak{B}(\underset{\omega_1\omega}{\equiv} \mathfrak{A})$ it must be so in $\mathfrak{A}$ as well, hence $\mathfrak{A} \models \forall \vec{x}(\theta^*(\vec{x}) \to \to \exists y \varphi(\vec{x}, y))$. This last sentence is true also in $\mathfrak{B}'$, showing that $\mathfrak{B}' \models \exists y \varphi(\vec{b}, y)$, as required.

It is known that if $\mathrm{SH}(\mathfrak{A}) = \alpha$ then $\mathfrak{A} \underset{\infty}{\equiv} \mathfrak{B}$ whenever $\mathfrak{B} \underset{\alpha+\omega}{\equiv} \mathfrak{A}$. In our case, we can say more:

THEOREM 3.4. *If $\mathfrak{B} \underset{\omega_1+1}{\equiv} \mathfrak{A}$ then $\mathfrak{B} \underset{\infty}{\equiv} \mathfrak{A}$.*

Proof. The sentence saying that $\forall x \bigvee_{\alpha<\omega_1} \psi_\alpha(x) \wedge \bigwedge_{\alpha<\omega_1}$ "the Scott-sentence of $\mathfrak{A}_\alpha$ holds when relativized to ψ_α" is a Scott-sentence for $\mathfrak{A}$ and has rank $\leqslant \omega_1+1$.

Remark. Again, the example of $(\omega_1, <)$ shows that the bound ω_1+1 cannot improved. While $(\omega_1, <)$ has Scott-height ω_1 it should be remarked that there are bounded ω_1-like orderings with Scott-height as low as ω. E.g.: let $\langle P_\alpha: \alpha < \omega_1 \rangle$ be a sequence of distinct subsets of ω and let $P = \{\omega \cdot \alpha + n: \alpha < \omega_1 \ \& \, n \in P_\alpha\}$. Then $\mathfrak{A} = \langle \omega_1; <, P \rangle$ is a rigid structure in which every element is defined by an $L_{\omega_1\omega}$-formula of quantifier depth ω. In the above we could substitute orderings for the points and eliminate the colors.

In spite of the last mentioned fact, no ω_1-like ordering $\mathfrak{A}$ of bounded type has an $L_{\omega_1\omega}$-Scott sentences. Indeed, we have:

PROPOSITION 3.5. *$\mathfrak{A}$ is not ∞-equivalent to any countable structure.*

Proof. Suppose otherwise: $\mathfrak{A} \underset{\infty}{\equiv} \mathfrak{B}$, $\mathfrak{B}$ countable. Let L be a countable fragment of $L_{\omega_1\omega}$ containing the Scott-sentence of $\mathfrak{B}$. We can then represent $\mathfrak{A} = \bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha$ with $\langle \mathfrak{A}_\alpha: \alpha < \omega_1 \rangle$ a continuous increasing chain of proper initial segments which are L-substructures of $\mathfrak{A}$. As each $\mathfrak{A}_\alpha$ satisfies the Scott-sentence of $\mathfrak{B}$, they are all isomorphic to each other. This would mean that $\mathfrak{A}$ is unbounded.

Natural questions to ask are the following. When is $\mathfrak{A}$ ∞-equivalent to a non-isomorphic $\mathfrak{B}$ (by 3.1, $\mathfrak{B}$ has to be non ω_1-like)? To a $\mathfrak{B}$ of a given cardinality $\varkappa > \omega_1$? Similar questions for *countable* $\mathfrak{A}$ were settled by Landraitis in [5]. Using his results we can show the following (remember that $\mathfrak{A}$ is of bounded type).

THEOREM 3.6. *Precisely one of the following occurs*: (i) $\mathfrak{B} \cong \mathfrak{A}$ *whenever* $\mathfrak{B} \underset{\infty}{\equiv} \mathfrak{A}$. *This holds iff all orbits of $\mathfrak{A}$ are scattered*; (ii) *for every $\varkappa \geqslant \aleph_1$ there is a $\mathfrak{B}$ of power $\varkappa$ ∞-equivalent but nonisomorphic to $\mathfrak{A}$. This happens iff $\mathfrak{A}$ has a self additive interval*; (iii) *there is a $\mathfrak{B}$ of power $\varkappa$ ∞-equivalent but not isomorphic to $\mathfrak{A}$ iff $\aleph_1 \leqslant \varkappa \leqslant 2^{\aleph_0}$.*

Remark. The *orbit* of $a \in \mathfrak{A}$ is the set of all $b \in \mathfrak{A}$ with $(\mathfrak{A}, b) \cong (\mathfrak{A}, a)$. An interval is self additive iff its isomorphism type ξ satisfies $\xi + \xi = \xi$ (cf. [5]).

Proof. If $\mathfrak{B} \underset{\infty}{\equiv} \mathfrak{A}$ but $\mathfrak{B} \ncong \mathfrak{A}$ then, by 3.1, $\mathfrak{B}$ is not ω_1-like hence, some $\mathfrak{B}_\alpha$ is uncountable yet ∞-equivalent to the countable $\mathfrak{A}_\alpha$; by [5], some orbit of $\mathfrak{A}_\alpha$ (hence of $\mathfrak{A}$) is not scattered. Notice also that if $\mathfrak{B}$ has cardinality $> 2^{\aleph_0}$ so does some $\mathfrak{B}_\alpha$ (since their union is all of B) and, again by [5], the corresponding $\mathfrak{A}_\alpha$ has a self additive interval. The theorem follows now immediately from the results of [5] concerning possible cardinalities of the uncountable ∞-equivalents, of $\mathfrak{A}_\alpha$ and the "pasting" method we have used before.

§ 4. ω_1-like orderings of unbounded type. As we mentioned already, the simplest example of such an ordering is (the one with order type) $\eta \cdot \omega_1$. It is dense with no first or last element; there are $2^{\aleph_1}$ many nonisomorphic ω_1-like orderings having the same property (indeed, for all $S \subset \omega_1$, let $\mathfrak{A}_S = \bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha$ where $\langle \mathfrak{A}_\alpha: \alpha < \omega_1 \rangle$ is a strictly increasing chain of orderings of type η s.t. for all α, $\mathfrak{A}_{\alpha+1}$ endextends $\mathfrak{A}_\alpha$ and $\mathfrak{A}_{\alpha+1} - \mathfrak{A}_\alpha$ has a first element iff $\alpha \in S$; then $\mathfrak{A}_S \cong \mathfrak{A}_{S'}$ iff S and S' agree on a closed and unbounded set; the idea of this construction is due to J. Conway; see e.g. [7] for more details). All these $2^{\aleph_1}$-many structures are ∞-equivalent to each other. (In fact, $\infty\omega_1$-equivalent.)

Another example of an unbounded-type ω_1-like ordering is any "dense mixture of colors" i.e. $\mathfrak{A} = \langle A; <, P_0, P_1, \ldots \rangle$ where $(A, <)$ is ω_1-like dense with no endpoints and $P_0, P_1, \ldots$ are ($\leqslant \aleph_0$ many) mutually disjoint predicates whose union is all of A and each of which is dense in A. Again, there are $2^{\aleph_1}$-many nonisomorphic ω_1-like orderings ∞-equivalent to $\mathfrak{A}$ and we can easily describe them all.

A more complicated example is any "dense mixture of countable orderings". By this we mean a structure $\mathfrak{A}^*$ gotten from a dense color mixture $\mathfrak{A} = \langle A; <, P_0, P_1, \ldots \rangle$ in the following way: choose distinct isomorphism types $\xi_0, \xi_1, \ldots$ of countable (colored) orderings and replace every point $a \in A$ belonging to P_i by an ordering of type ξ_i; more formally, $\mathfrak{A}^* = \sum \langle \mathfrak{A}_a: a \in A \rangle$ where the disjoint orderings $\mathfrak{A}_a$ are so chosen as to have $\overline{\mathfrak{A}}_a = \xi_i$ whenever $a \in P_i$. The orderings $\mathfrak{A}_a$ will be called the *components* of the dense mixture $\mathfrak{A}^*$. Let us stress that the colors of the various ξ_i need not be distinct.

Finally, if $\mathfrak{A}'$ is any countable ordering and $\mathfrak{A}''$ any dense mixture of countable orderings then $\mathfrak{A} = \mathfrak{A}' + \mathfrak{A}''$ is also an unbounded ω_1-like ordering.

Can we find any other examples? The main result of this section is a negative answer to this question.

THEOREM 4.1. *Any ω_1-like ordering of unbounded type is $\mathfrak{A} = \mathfrak{A}'+\mathfrak{A}''$ where $\mathfrak{A}'$ is a proper initial segment and $\mathfrak{A}''$ a dense mixture of countable orderings.*

Proof. We define by induction an increasing sequence of equivalence relations R_α until a point is reached where $R_\alpha = R_{\alpha+1} = R^*$. The R^*-classes of sufficiently large elements will turn out to be the components of the dense mixture of orderings.

The definitions of R_α runs as follows:

(1) R_0 is the equality relation;

(2) for a limit δ, $R_\delta = \bigcup_{\alpha<\delta} R_\alpha$;

(3) $R_{\alpha+1} = R'$

where the operation $'$ attaching to an equivalence relation R a coarser one R' will be defined below. As we proceed with this inductive definition, we shall also prove that all the R_α have the property which we now define.

An equivalence relation R on A is called *good* if the following conditions are met:

(a) each R-class is convex and bounded (hence, countable);

(b) whenever $f\colon \mathfrak{A}_1 \to \mathfrak{A}_0$ is an isomorphism between the proper initial segments $\mathfrak{A}_0$ and $\mathfrak{A}_1$, f preserves R, i.e. for $a, b \in \mathfrak{A}_1$, aRb iff $f(a)Rf(b)$.

For the rest of this proof, *let $\mathfrak{A}_0$ be a fixed proper initial segment of $\mathfrak{A}$ such that every element of $\mathfrak{A}$ is in some initial segment isomorphic to $\mathfrak{A}_0$.*

The isomorphism type of any R-class will be simply called an R-type. Conditions (a) and (b) imply that any R-class is isomorphic to one included in $\mathfrak{A}_0$ and so, R has only $\leqslant\aleph_0$-many R-types. Indeed, given $a \in \mathfrak{A}$ take an initial segment $\mathfrak{A}_1$ isomorphic to $\mathfrak{A}_0$ and large enough to contain a and some $b > a$ with b non Ra. If $f\colon \mathfrak{A}_1 \to \mathfrak{A}_0$ is an isomorphism then we also have $f(a) < f(b) \in \mathfrak{A}_0$ and $f(a)$ non $Rf(b)$. It follows that the R-classes of a and $f(a)$ are included in $\mathfrak{A}_1$ and $\mathfrak{A}_0$ respectively and hence isomorphic to each other.

Obviously, R_0 (i.e. equality) is good and as soon as we define the operation $'$ we shall prove by induction that so are all R_α.

An R-type is called cofinal if it appears as the type of (the R-class of) arbitrarily large elements. We now define $aR'b$ to mean that (assuming e.g. $a \leqslant b$) there are finitely many elements $a = a_0 < a_1 < \ldots < a_n = b$ such that for each $i < k$, there is a cofinal R-type ξ_i which does not occur as the type of any R-class contained in the open interval (a_i, a_{i+1}); we shall call $\langle a_0, a_i, \ldots, a_{k-1}\rangle$ a *witnessing sequence for $aR'b$.*

LEMMA 4.2. *If R is good, so is R'.*

Proof of 4.2. Obviously, R' is an equivalence relation with convex classes. Each R'-class is bounded. Otherwise, we would have an a such that $aR'b$ for all $b > a$. Let k be the least number such that $aR'b$ has a wittnessing sequence of length $\leqslant k$ for unboundedly many (hence for all) $b > a$. For each such b, let $b' < b$ be the last element of a witnessing sequence of length $\leqslant k$. Then $aR'b'$ is wittnessed by a sequence of length $< k$, so — by the minimality of k — $\langle b'\colon b > a\rangle$ is bounded by an element c. This means that for each $b > c$ the interval (c, b) misses some cofinal R-type and this is easily seen to be a contradiction since there are only countably many R-types.

We leave the reader with the (simple) verification of the fact that R' satisfies condition (b) as well, thus completing the proof of 4.2.

From 4.2 and the obvious fact that the union of a countable increasing sequence of good relations is a good relation, we conclude by induction that R_α is good for all α. Next we claim that $R_\alpha = R_{\alpha+1}$ for some countable α. Otherwise, for each α we would have an a_α with $a_\alpha/R_\alpha \subsetneqq a_\alpha/R_{\alpha+1}$ and by the goodness of R_α and $R_{\alpha+1}$ we could insure $a_\alpha/R_{\alpha+1} \subseteq \mathfrak{A}_0$. This is impossible.

If $R_\alpha = R_{\alpha+1}$, the cofinal R_α-types are easily seen to be dense in the sense that they must occur in each open interval (a, b) with a non$R_\alpha b$. Taking $\mathfrak{A}'$ to be any proper initial sequence which is a union of R_α-classes and includes all occurrences of noncofinal R_α-types and letting $\mathfrak{A}'' = \mathfrak{A}-\mathfrak{A}'$, we get a decomposition $\mathfrak{A} = \mathfrak{A}'+\mathfrak{A}''$ as claimed except for the possibility that $\mathfrak{A}''$ has a first R_α-class, in which case we may transfer this class to $\mathfrak{A}'$ and get the desired decomposition. To see that there is no last R_α class we use the fact that A is uncountable but of cofinality ω_1 while each R_α-class is countable.

Remark. For $a < b$, let $\xi(a, b)$ be the isomorphism type of the structure $\mathfrak{A}\restriction(a, b)$ — the reduct of $\mathfrak{A}$ to the universe $(a, b) = \{c\colon c \in A \;\&\; a < c < b\}$; call ξ a *cofinal interval type* of $\mathfrak{A}$ if $\xi = \xi(a, b)$ for intervals (a, b) with arbitrarily large a. With the help of this notion, the final equivalence relation $R^*(= R_\alpha = R_{\alpha+1})$ can be described directly as follows. For $a \leqslant b$, aR^*b iff the interval (a, b) misses some cofinal interval type ξ (i.e. $\xi \neq \xi(c, d)$ whenever $a < c < d < b$). We could use this definition for an alternative, somewhat shorter, proof of 4.1. We think however, that the ordinal analysis of the structure of $\mathfrak{A}$ is of interest.

Theorem 4.1 has a number of immediate illuminating consequences. Throughout the rest of this section, *assume $\mathfrak{A}$ to be an unbounded ω_1-like ordering.*

THEOREM 4.3. *$\mathfrak{A}$ has a decomposition $\mathfrak{A} = \bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha$ with $\langle \mathfrak{A}_\alpha\colon \alpha < \omega_1\rangle$ a continuous increasing sequence of proper initial segments such that $\mathfrak{A}_\alpha \underset{\infty}{\prec} \mathfrak{A}_\beta \underset{\infty}{\prec} \mathfrak{A}$ whenever $\alpha < \beta < \omega_1$ (it follows that the $\mathfrak{A}_\alpha$ are isomorphic to each other).*

Proof. Any continuous decomposition into proper initial segments confirms the claim of the theorem provided that $\mathfrak{A}_\alpha = \mathfrak{A}'+\mathfrak{A}''_\alpha$ where $\mathfrak{A}''_\alpha$ is a union of a nonvoid collection of R_α-classes with no last such class. The ∞-inclusion follows by a standard (Cantor type) back-and-forth argument.

THEOREM 4.4. *$\mathfrak{A}$ has an $L_{\omega_1\omega}$-Scott-sentence, hence a countable Scott-height. Also, the ∞-type of any finite sequence in $\mathfrak{A}$ is axiomatized by an $L_{\omega_1\omega}$-formula.*

Proof. The Scott-sentence of $\mathfrak{A}_0$ (of Theorem 4.3) is also a Scott-sentence of $\mathfrak{A}$. The formulas axiomatizing the ∞-type of any finite sequence in $\mathfrak{A}_\alpha$ will do the same in $\mathfrak{A}$.

THEOREM 4.5. *$\mathfrak{A}$ is ∞-equivalent to $2^{\aleph_1}$-many nonisomorphic ω_1-like orderings (and we have a full description of each of these). Also, $\mathfrak{A}$ is ∞-equivalent to non ω_1-like orderings of any cardinality $\varkappa \geqslant \aleph_1$.*

Proof. Left to the reader.

§ 5. **Summing up results on ω_1-like orderings.** Call a formula $\varphi(x)$ *bounded* (in the ω_1-like $\mathfrak{A}$) iff $\varphi(\mathfrak{A})$ is bounded (i.e. countable). The various results of §§ 2–4 yield the following characterizations:

THEOREM 5.1. *An ω_1-like ordering $\mathfrak{A}$ is of the bounded type iff any of the following holds:*

(1) *every element of $\mathfrak{A}$ satisfies a bounded formula;*
(2) *every orbit of $\mathfrak{A}$ is bounded;*
(3) *$\mathfrak{A}$ has no $L_{\omega_1\omega}$-Scott-sentence;*
(4) *for any ω_1-like $\mathfrak{B}$, if $\mathfrak{B} \underset{\infty}{\equiv} \mathfrak{A}$ then $\mathfrak{B} \cong \mathfrak{A}$.*

The property of being of unbounded type is characterized by the negations of each of (1)–(3) as well as by the following strong negation of (4):

(5) *$\mathfrak{A}$ is ∞-equivalent to $2^{\aleph_1}$-many nonisomorphic ω_1-like orderings.*

From 3.3 and 4.4 we learn:

THEOREM 5.2. *If $\mathfrak{A}$ is an ω_1-like ordering then the ∞-type of any $\vec{a} \in \mathfrak{A}$ is axiomatized inside $\mathfrak{A}$ by an $L_{\omega_1\omega}$-formula.*

§ 6. **Orderings ∞-equivalent to ω_1-like ones.** Call an ordering $(\leqslant\omega_1)$-like if it is countable or ω_1-like. The methods of §§ 3–4 allow us to prove the following result.

THEOREM 6.1. *An uncountable ordering $\mathfrak{A}$ is ∞-equivalent to an $(\leqslant\omega_1)$-like one iff for all $a \in \mathfrak{A}$ the initial segment $\mathfrak{A} \restriction a$ is ∞-equivalent to a countable ordering. In fact, if $\mathfrak{A}$ satisfies this last condition then there is an $(\leqslant\omega_1)$-like $\mathfrak{A}' \underset{\infty}{\prec} \mathfrak{A}$.*

Proof. We analyse $\mathfrak{A}$ precisely as we analysed the ω_1-like orderings in §§ 3–4; the only difference being that wherever we mentioned isomorphisms between countable orderings we must now use ∞-equivalence of (not necessarily countable) orderings. We will stress the few modifications made in this spirit leaving the details to the reader. Before doing this, let us mention that *we may assume that $\mathfrak{A}$ has cofinality greater than ω*. Otherwise, $\mathfrak{A} = \bigcup_{n<\omega} (\mathfrak{A} \restriction a_n)$ where $\langle a_n : n<\omega \rangle$ is an increasing cofinal sequence. By replacing $\mathfrak{A} \restriction a_0$ and $\mathfrak{A} \restriction a_{n+1} - \mathfrak{A} \restriction a_n$, $n<\omega$, with countable ∞-substructures we get a countable ∞-substructure of $\mathfrak{A}$ itself.

Turning now to the analysis of $\mathfrak{A}$, we distinguish again between the bounded and unbounded type. $\mathfrak{A}$ is said to be *of unbounded type* if there is a countable isomorphism type ξ such that every element of a belongs to an initial segment ∞-equivalent to an ordering of type ξ; otherwise, $\mathfrak{A}$ is called *of bounded type.*

The bounded type case. If $\mathfrak{A}$ is of bounded type then, as in the Proof of 3.1 we attach to each $a \in \mathfrak{A}$ the structure

$$\mathfrak{A}(a) = \{b\colon b \in \mathfrak{A} \,\&\, \exists a' \geqslant b \;\; \mathfrak{A} \restriction a' \underset{\infty}{\equiv} \mathfrak{A} \restriction a\}.$$

Again, $\mathfrak{A}(a)$ is a proper initial segment definable by a *nice* formula. We can therefore define a sequence $\{\psi_\alpha(x)\colon \alpha<\omega_1\}$ of nice formulas s.t. $\mathfrak{A}_\alpha = \psi_\alpha(\mathfrak{A})$, $\alpha<\omega_1$ is a strictly increasing continuous chain of initial segments. If we had any $a \in \mathfrak{A} - \bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha$ then $\mathfrak{A} \restriction a$ would not be ∞-equivalent to any countable ordering, a contradiction. Thus, $\mathfrak{A} = \bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha$ and we get an ω_1-like ∞-substructure of $\mathfrak{A}$ by replacing $\mathfrak{A}_0$ and $\mathfrak{A}_{\alpha+1} - \mathfrak{A}_\alpha$ for all $\alpha<\omega_1$, by countable ∞-substructures.

The unbounded type case. We define equivalence relations R_α just as in the Proof of 4.1 with the following two slight modifications. First, the notion of *good* relation is now defined by:

(a) each R-class is convex, bounded and $L_{\omega_1\omega}$-definable with parameters (hence, ∞-equivalent to a countable ordering);

(b) whenever $\mathfrak{A}_0, \mathfrak{A}_1$ are proper initial segments and $a_0, b_0 \in \mathfrak{A}_0$, $a_1, b_1 \in \mathfrak{A}_1$ satisfy $(\mathfrak{A}_0, a_0, b_0) \underset{\infty}{\equiv} (\mathfrak{A}_1, a_1, b_1)$ then $a_0 R b_0$ iff $a_1 R b_1$.

Second, an R-type will now be the ∞-equivalence type of an R-class.

The proof now proceeds precisely as in 4.1. One point worth mentioning is why do we have $R_\alpha = R_{\alpha+1}$ for a countable α. If not, then either we would have an $a \in \mathfrak{A}_0$ s.t. $a/R_\alpha \subsetneqq a/R_{\alpha+1} \subset \mathfrak{A}_0$ for cofinally many $\alpha<\omega_1$, *or* we would have an increasing sequence α_i, $i<\omega_1$ of ordinals and elements $a_i \in \mathfrak{A}_0$ s.t. $a_i/R_{\alpha_i} \subsetneqq a_i/R_{\alpha_{i+1}} \subset \mathfrak{A}_0$ while for $j>i$ it is not the case that $a_j/R_{\alpha_i} \subsetneqq a_j/R_{\alpha_{i+1}} \subset \mathfrak{A}_0$. Both these possibilities contradict the assumption that $\mathfrak{A}_0$ is ∞-equivalent to a countable ordering.

Once we know that $R_\alpha = R_{\alpha+1}$ for a countable α we conclude that $\mathfrak{A}$ is a mixture of countable many R_α-types with all cofinal such types *dense*. For some $a \in \mathfrak{A}$ no non cofinal R_α-type occurs past a. Using the fact that $\mathfrak{A} \restriction a$ is ∞-equivalent to a countable ordering and so are all R_α-classes (as definable subsets of initial segments) we get an $\mathfrak{A}^* \underset{\infty}{\prec} \mathfrak{A}$ s.t. $\mathfrak{A}^*$ is a mixture of *the* countable representatives of the R_α-types and $\mathfrak{A}^* = \mathfrak{A}^{*\prime} + \mathfrak{A}^{*\prime\prime}$ where $\mathfrak{A}^{*\prime}$ is an initial segment and a countable union of (countable representatives of) R_α-types and $\mathfrak{A}^{*\prime\prime}$ a dense mixture of cofinal R_α-types. By further "cutting down" $\mathfrak{A}^{*\prime\prime}$ we can make it ω_1-like. This finishes (the sketch of) the proof.

The proof of 6.1 yields the following.

COROLLARY 6.2. *Let $\mathfrak{A}$ be an ordering of uncountable cofinality. $\mathfrak{A}$ is ∞-equivalent to an ω_1-like ordering iff for all $a \in \mathfrak{A}$, $\mathfrak{A} \restriction a$ is ∞-equivalent to a countable ordering. If this is the case then there is an ω_1-like $\mathfrak{A}' \underset{\infty}{\prec} \mathfrak{A}$.*

A natural question connected to 6.1 is: under what circumstances is an uncountable ordering $\mathfrak{A}$ equivalent to a countable one? Here is an answer involving the orbits of $\mathfrak{A}$ (in the uncountable case, by the orbit of an element a we mean the set $\{b: (\mathfrak{A}, b) \equiv_\infty (\mathfrak{A}, a)\}$).

THEOREM 6.3. *An ordering $\mathfrak{A}$ is ∞-equivalent to a countable one iff for all $a, b \in \mathfrak{A}$ if $a<b$ then each one of the structures $\mathfrak{A} \restriction a$, $\mathfrak{A} \restriction (a, b)$ and $\mathfrak{A} \restriction (b, \infty)$ has only countably many orbits.*

Proof. "Only if" is trivial. The "if" part is proven as follows: let $\mathfrak{B} \subseteq \mathfrak{A}$ be countable and taken so as to have every orbit $\mathfrak{A} \restriction a$, $\mathfrak{A} \restriction (a, b)$, $\mathfrak{A} \restriction (b, \infty)$ meet $\mathfrak{B} \restriction a$, $\mathfrak{B} \restriction (a, b)$, $\mathfrak{B} \restriction (b, \infty)$ respectively whenever $a, b \in \mathfrak{B}$ and $a<b$. We claim that $\mathfrak{B} \prec_\infty \mathfrak{A}$. To show this, it is enough to show that whenever $\mathfrak{A} \models \varphi(a, \vec{b})$ with $\vec{b} \in \mathfrak{B}$ and $\varphi(x, \vec{y})$ any $L_{\infty\omega}$-formula, then $\mathfrak{A} \models \varphi(a', \vec{b})$ for some $a' \in \mathfrak{B}$. Assume that $\vec{b} = \langle b_0, \ldots, b_{n-1}\rangle$ is increasing and that e.g. $b_i < a < b_{i+1}$. The orbit of a in $\mathfrak{A} \restriction (b_i, b_{i+1})$ meets $\mathfrak{B} \restriction (b_i, b_{i+1})$ in some element, say a'. This means that $\mathfrak{A} \restriction (b_i, a) \equiv_\infty \mathfrak{A} \restriction (b_i, a')$ and $\mathfrak{A} \restriction (a, b_{i+1}) \equiv_\infty \mathfrak{A}(a', b_{i+1})$ which yields, by a standard argument, $(\mathfrak{A}, a, \vec{b}) \equiv_\infty (\mathfrak{A}, a', \vec{b})$ showing that $\mathfrak{A} \models \varphi(a', \vec{b})$ as desired.

COROLLARY 6.4. *An uncountable ordering $\mathfrak{A}$ is ∞-equivalent to a $(\leqslant\omega_1)$-like one iff for all $a, b \in \mathfrak{A}$ if $a<b$ then $\mathfrak{A} \restriction a$ and $\mathfrak{A} \restriction (a, b)$ have only countably many orbits.*

Finally, to complete the picture, we must elucidate under what circumstances a linear ordering of cofinality ω is ∞-equivalent to an ω_1-like one. By our methods we have:

THEOREM 6.5. *Let $\mathfrak{A}$ be an ordering of cofinality ω such that for each $a \in \mathfrak{A}$, $A \restriction a$ is ∞-equivalent to a countable ordering. The following are equivalent:*

(1) *$\mathfrak{A}$ is ∞-equivalent to an ω_1-like ordering;*

(2) *some orbit of $\mathfrak{A}$ contains a set B of order type η such that for $b_1, b_2 \in B$, if $b_1 < b_2$ then (b_1, b_2) meets all cofinal orbits of $\mathfrak{A}$;*

(3) *$\mathfrak{A}$ has a proper initial segment $\mathfrak{A}_0$ such that $\mathfrak{A}_0 \prec_\infty \mathfrak{A}$;*

(4) *$\mathfrak{A}$ has an unbounded orbit and if $R_\alpha = R_{\alpha+1}$ with R_α defined as in the proof of 6.1 then R_α has no last equivalence class;*

(5) *$\mathfrak{A} = \mathfrak{A}' + \mathfrak{A}''$ where $\mathfrak{A}'$ is an initial segment ∞-equivalent to a countable ordering and for a dense linear ordering $(D, <)$ with no last element, $\mathfrak{A}''$ is a D-sum $\mathfrak{A}'' = \sum\{\mathfrak{A}_d: d \in D\}$ of convex subsets $\mathfrak{A}_d$ ∞-equivalent to countable orderings and such that $\{d': \mathfrak{A}_{d'} \equiv_\infty \mathfrak{A}_d\}$ is dense in D for all $d \in D$.*

Proof. By the proof of 6.1, $\mathfrak{A}$ is ∞-equivalent to a countable ordering. Thus, if condition (1) holds then $\mathfrak{A}$ must be ∞-equivalent to an ω_1-like ordering of unbounded type. It follows, by 4.1, that (1) implies all other conditions.

Assume (2). By the proof of 6.1, we can find a countable $\mathfrak{A}' \prec_\infty \mathfrak{A}$ with $B \subset A'$; obviously, $\mathfrak{A}'$ satisfies (2) as well, hence we may assume that $\mathfrak{A}$ is countable; also, it is easy to see that the orbit in which B lies in cofinal. Define equivalence relations R_α on $\mathfrak{A}$ as in the proof of 4.1. An induction on α shows that b_1 non $R_\alpha b_2$ whenever

b_1, b_2 are distinct elements of B; it follows that the equivalence classes of R_α are bounded in $\mathfrak{A}$. As $\mathfrak{A}$ is countable, $R_\alpha = R_{\alpha+1}$ for some $\alpha < \omega_1$ and we get that $\mathfrak{A} = \mathfrak{A}' + \mathfrak{A}''$ where $\mathfrak{A}'$ is an initial segment and $\mathfrak{A}''$ a dense mixture of cofinal R_α-types. Condition (1) follows at once.

For any countable fragment Δ of $L_{\omega_1\omega}$, condition (3) implies the existence of a countable $\mathfrak{A}_1' \equiv_\Delta \mathfrak{A}$ with a proper initial segment $\mathfrak{A}_0' \prec_\Delta \mathfrak{A}_1'$; remember that $\mathfrak{A}$ has a countable Scott-sentence and thus, if we take Δ large enough, we conclude that we have even $\mathfrak{A}_0' \prec_\infty \mathfrak{A}_1' \equiv_\infty \mathfrak{A}$. We can then construct a continuous chain $\{\mathfrak{A}_\alpha': \alpha < \omega_1\}$ with $(\mathfrak{A}_{\alpha+1}', \mathfrak{A}_\alpha') \cong (\mathfrak{A}_1', \mathfrak{A}_0')$; the union of this chain is the desired ω_1-like ordering ∞-equivalent to $\mathfrak{A}$.

Finally, assume (4). There must be an $a \in \mathfrak{A}$ such that no noncofinal R_α-type occurs past a; otherwise, $\mathfrak{A}$ would have no unbounded orbit. If so, then condition (5) follows immediately and this condition obviously implies (1).

§ 7. **ω_1-like orderings with countable Scott height.** We have already seen that an ω_1-like ordering has a Scott-sentence in $L_{\omega_1\omega}$ if and only if it is of unbounded type. We now ask and answer a related question: When does an ω_1-like ordering have countable Scott height?

THEOREM 7.1. *Let $\mathfrak{A}$ be an ω_1-like ordering. The following are equivalent:*

(i) $\mathrm{SH}(\mathfrak{A}) < \omega_1$.

(ii) *There is some fixed $\sigma < \omega_1$ such that for each $a \in \mathfrak{A}$, $\mathrm{SH}(\mathfrak{A} \restriction a) \leqslant \sigma$.*

(iii) *There is some fixed $\sigma < \omega_1$ and sequence $\langle a_\alpha: \alpha < \omega_1\rangle$ cofinal in $\mathfrak{A}$ such that for each $\alpha < \omega_1$, $\mathrm{SH}(\mathfrak{A} \restriction a_\alpha) \leqslant \sigma$.*

That (i) implies (ii) follows from the more general lemma.

LEMMA 7.2. *Let S be an interval of $\mathfrak{A}$ which is $L_{\infty\omega}$-definable from parameters $\vec{a} \in \mathfrak{A} \setminus S$. Then*

$$\mathrm{SH}(S) \leqslant \mathrm{SH}(\mathfrak{A}).$$

Proof. This is basically a Feferman-Vaught argument, but it seems simpler to give a direct proof. Suppose $\mathrm{SH}(\mathfrak{A}) = \sigma$ and $(S, \vec{b}) \equiv_\sigma (S, \vec{c})$. Then there is some sequence of partial isomorphisms witnessing this, say $\langle I_\alpha: \alpha \leqslant \sigma\rangle$. For each $\alpha < \sigma$ let J_α consist of all finite functions f from $\mathfrak{A}$ to $\mathfrak{A}$ so that $f \restriction S \in I_\alpha$ and f is the identity on $\mathrm{dom} f \setminus S$. $\langle J_\alpha: \alpha < \sigma\rangle$ is easily seen to witness the fact that $(\mathfrak{A}, \vec{b}, \vec{a}) \equiv_\sigma (\mathfrak{A}, \vec{c}, \vec{a})$. Since $\mathrm{SH}(\mathfrak{A}) = \sigma$, we have $(\mathfrak{A}, \vec{b}, \vec{a}) \equiv_\infty (\mathfrak{A}, \vec{c}, \vec{a})$, from which $(S, \vec{b}) \equiv_\infty (S, \vec{c})$ follows by relativization since S is $L_{\infty\omega}$-definable in $(\mathfrak{A}, \vec{a})$.

Remark 7.3. Lemma 7.2 clearly covers open intervals. To deal with closed intervals simply do an extra Feferman–Vaught argument for adding the endpoints.

The next result, despite its short proof, seemed quite surprising to us.

LEMMA 7.4. *Let $\mathfrak{A}$ be an ordering and I an interval in $\mathfrak{A}$. Suppose $S \subseteq \mathfrak{A}^k$ is $L_{\infty\omega}$-definable on $\mathfrak{A}$ with parameters. Then $S \cap I^k$ is $L_{\infty\omega}$-definable with parameters in I.*

Proof. Suppose $S = \{\vec{a}: \mathfrak{A} \models \varphi(\vec{a}, \vec{c}, \vec{d})\}$ where $\vec{c} \in I$ and $\vec{d} \in \mathfrak{A} \setminus I$. Notice first that if $\vec m, \vec n \in I$ and $(I, \vec m, \vec c) \underset{\infty}{\equiv} (I, \vec n, \vec c)$, then $(\mathfrak{A}, \vec m, \vec c, \vec d) \underset{\infty}{\equiv} (\mathfrak{A}, \vec n, \vec c, \vec d)$. This follows as above from another Feferman–Vaught style argument. Given a back and forth set witnessing the former equivalence, extend its functions by adding finite pieces of the identity on $\mathfrak{A}\setminus I$ to get a back and forth set for the latter equivalence.

In view of the above, for $\vec m, \vec n \in I$, if $(I, \vec m, \vec c) \underset{\infty}{\equiv} (I, \vec n, \vec c)$, then $\vec m \in S$ iff $\vec n \in S$. For each $\vec m \in S$ let $\theta_{\vec m}(\vec x, \vec c)$ be the canonical Scott formula characterizing $(I, \vec m, \vec c)$ up to $\infty\omega$-equivalence. Then it is clear that

$$S \cap I = \{\vec a \in I \colon I \models \bigvee_{\vec m \in S} \theta_{\vec m}(\vec a, \vec c)\}.$$

Remark 7.5. It follows from the above proof that if I is countable then $S \cap I$ is $L_{\omega_1\omega}$-definable with parameters.

We now return to the proof that (iii) implies (i). We will have to consider the cases of bounded and unbounded orderings separately.

First assume $\mathfrak{A}$ is of unbounded type. Fix any point a in the dense mixture component of $\mathfrak{A}$ and choose $a_\alpha > a$. Recall the final equivalence relation R^* of § 4. Then we know that if $S = \{b \leqslant a \colon b \text{ non } R^* a\}$, then $S \underset{\infty}{\equiv} \mathfrak{A}$. ($S$ is obtained by taking a principal initial segment of $\mathfrak{A}$ including some of the dense mixture and discarding the remnants of the final R^*-block in the segment). It can be checked by induction that R^* is $L_{\infty\omega}$-definable in $\mathfrak{A}$. Thus, by Lemma 7.4 S is definable in $\mathfrak{A} \restriction a_\alpha$. Finally, since $\mathrm{SH}(\mathfrak{A} \restriction a_\alpha) \leqslant \sigma$, by Lemma 7.2 $\mathrm{SH}(S) \leqslant \sigma$.

Now we assume that $\mathfrak{A}$ is of bounded type and, for the sake of contradiction, that $\mathrm{SH}(\mathfrak{A}) = \beta > \sigma$. First note that there is a sentence θ^β of $L_{\omega_1\omega}$ such that for any order $\mathfrak{B}$, $\mathfrak{B} \models \theta^\beta$ iff $\mathrm{SH}(\mathfrak{B}) \geqslant \beta$. (Note: this is a purely general fact not specific to orderings). Suppose θ^β is in some countable fragment L_B. By building a continuous unbounded chain of initial segments that are L_B-elementary submodels of $\mathfrak{A}$, we obtain a continuous unbounded sequence of initial segments each of whose Scott-height is $\geqslant \beta$. Since we are in the bounded case there is also a continuous unbounded sequence of $L_{\infty\omega}$-definable initial segments and therefore we can find some proper initial segment I which is both $L_{\infty\omega}$-definable and has Scott height $\geqslant \beta$. Now, choose some $a_\alpha \in \mathfrak{A}\setminus I$. Then, by Lemma 7.4 I is $L_{\infty\omega}$-definable in $\mathfrak{A} \restriction a_\alpha$. Thus by Lemma 7.2, $\mathrm{SH}(I) \leqslant \sigma$. This is the desired contradiction.

Remark 7.6. The above proof for the bounded case only depends upon the fact that $\mathfrak{A}$ has uncountable cofinality and does not use the full force of ω_1-likeness. Of course, we use the stronger notion of boundedness in terms of ∞-type rather than isomorphism type. We can even eliminate the requirement that $\mathfrak{A}$ have uncountable cofinality, but then our proof, which we give below, gives only $\sigma+\omega+2$ instead of σ, and we have no counterexample to the better bound.

Now for the proof. Suppose $(\mathfrak{A}, \vec m) \underset{\beta+\omega+2}{\equiv} (\mathfrak{A}, \vec n)$. Choose a_α above all members of $\vec m$ and $\vec n$. Define $I = \bigcup \{\mathfrak{A} \restriction a \colon \mathfrak{A} \restriction a \underset{\infty}{\equiv} \mathfrak{A} \restriction a_\alpha\}$. By our boundedness hypothesis,

I is a proper initial segment of $\mathfrak{A}$. Choose $a_\gamma > a_\alpha$ with $a_\gamma \notin I$. Let $\theta(\vec c)$ be the canonical Scott-sentence of $(\mathfrak{A} \restriction a_\gamma, \vec m)$ and ϱ the canonical Scott-sentence of $\mathfrak{A} \restriction a_\alpha$. Then θ and ϱ each have quantifier rank at most $\sigma+\omega$. Using ψ^u to denote the relativization of ψ to elements less than u, we define $\varphi(\vec x)$ as

$$\exists z\, [(\exists y < z)(y > \vec x \wedge \varrho^y) \wedge \theta^z(\vec x) \wedge \forall v\, (\varrho^v \to v < z)].$$

Clearly φ has quantifier rank at most $\sigma+\omega+2$, and $\mathfrak{A} \models \varphi(\vec m)$. Since $(\mathfrak{A}, \vec m) \underset{\sigma+\omega+2}{\equiv} (\mathfrak{A}, \vec n)$, $\mathfrak{A} \models \varphi(\vec n)$. Thus, there is a b such that $(\mathfrak{A} \restriction b, \vec n) \underset{\infty}{\equiv} (\mathfrak{A}|a_\gamma, \vec m)$. Then since I is $L_{\infty\omega}$-definable in both $\mathfrak{A} \restriction b$ and $\mathfrak{A} \restriction a_\gamma$ by the same definition in $L_{\infty\omega}$, $(I, \vec n) \underset{\infty}{\equiv} (I, \vec m)$. Now, pasting back $\mathfrak{A}\setminus I$ we have $(\mathfrak{A}, \vec n) \underset{\infty}{\equiv} (\mathfrak{A}, \vec m)$.

If we drop the assumption of boundedness without insisting that $\mathfrak{A}$ is ω_1-like then there are two cases to consider. The first case is that in which $\mathfrak{A}$ has an unbounded ∞-orbit. In this situation we get the analogue of a dense mixture and the result is as above. In the second case, every orbit is bounded but the ordering is still unbounded with respect to the ∞-type of its principal initial segments. Examples of this case include $\varkappa^*$ for $\varkappa$ a cardinal. We have not investigated this case. It may be that the result we showed above for ω_1-like orderings may hold for orderings in general, and perhaps even for a simpler, more uniform reason.

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