Real closed rings
I. Residue rings of rings of continuous functions

by

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Abstract. We study the class of prime ideals P of the ring $C(X)$ of real-valued continuous functions on the space X determined by the condition: $C(X)/P$ is a real closed ring.

Introduction

This is the first of two articles dealing with real closed rings, which are ordered commutative domains satisfying either of the following equivalent conditions:

(1) R satisfies the "Intermediate Value Theorem" for polynomials in one variable.

(2) R is a convex subring of a real closed field.

Our point of departure consists of two well-known results:

FACT I. The theory of real closed fields is complete, model-complete and admits primitive recursive elimination of quantifiers in the language for ordered rings.

FACT II. If M is a maximal ideal in a ring of continuous functions $C(X)$, then $C(X)/M$ is a real closed field.

In the sequel [2] to the present article, we give a satisfactory generalization of Fact I. However Fact II does not generalize naively. There are accordingly two major problems:

PROBLEM A. Classify the *real closed ideals* of $C(X)$, that is the prime ideals P satisfying:

$$C(X)/P \text{ is a real closed ring}.$$

PROBLEM B. Investigate the structure of those rings of the form $C(X)/P$ which are not real closed.

We confine ourselves to Problem A. Our results include an explicit topological characterization of the real closed ideals in general, and a complete analysis of

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the situation when X is the one-point compactification of a discrete space, which is already nontrivial. We also analyse certain features of the case $X = [0, 1]$, but the prime ideal structure is exceedingly rich in this example and there is much that we do not know.

The organization of the paper is as follows. Preliminaries are presented in § 1 and the general theory in § 2. The abstract topological solution to Problem A is given in § 2.1, together with its immediate consequences. In § 2.2 we study the relationship between real closed ideals in metrizable spaces and P-point ultrafilters on a countable set. § 2.3 introduces a "transfer principle" based on an idea of Kohls which facilitates the consideration of various examples, notably the prime z-ideals immediately preceding a maximal ideal in the tree of prime z-ideals. In § 2.4 we study certain conditions under which the (unique) prime z-ideal immediately below a maximal ideal must be real closed.

In § 3 we analyse some concrete examples. We study the case in which $X = N^*$ is the one-point compactification of a countable discrete set in § 3.1. This automatically yields results in any space containing a copy of N^*, as discussed in § 3.2. The one-point compactification of an uncountable discrete set is analysed in § 3.3; here a partial, rather weak, "transfer principle" allows us to apply the results of § 3.1. In § 3.4 we want to consider the case $X = [0, 1]$; using the transfer principle of § 2.3 and the homeomorphism $[0, 1) \approx \boldsymbol{R}^+$, we work in $X = \boldsymbol{R}^+$. Our idea is to construct certain filters of closed sets of infinite Lebesgue measure ("fat" zero-sets) as opposed to the filters of discrete sets arising in § 3.2. The work accomplished in this section barely scratches the surface as far as knowing the prime ideal structure of $C([0, 1])$ is concerned.

Even in the simplest nontrivial case, $X = N^*$, there are many open questions connected with Problem B. For example:

Problem C. If P, Q are minimal prime ideals of $C(N^*)$ and neither is real closed, does it follow that $C(N^*)/P \equiv C(N^*)/Q$? (¹)

This appears to be a rather delicate question.

We wish to thank A. Louveau for calling our attention to the notion of P-point ultrafilter; he shall be credited for part of the authorship of Proposition 3.2.2.

§ 1. Preliminaries

A. *Real closed rings.*

Definition 1. A *real closed ring* A is a commutative ordered ring with unit which is not a field, satisfying the intermediate value property for polynomials:

if $Q \in A[Y]$, $a, b \in A$, $a < b$ and Q changes sign from a to b

(i.e., $Q(a) \cdot Q(b) < 0$), then Q has a zero c such that $a < c < b$.

We shall denote by RCVR the first order theory of real closed rings formulated in the language of ordered rings with unit. (The "V" in RCVR stands for "valuation"; the reasons for this name are made explicit in [2], but see also Definition 26, ff., p. 7, below.

(¹) Problem C has now been answered (in the affirmative, oddly enough) by J. Moloney.

The first order theory consisting of the axioms of RCVR except that which asserts the existence of a noninvertible element will be denoted by RCR; thus, the models of RCR are all real closed rings *and* fields.

In this paper we carry out an analysis of real closed rings among residue class rings of rings of continuous functions. The following is a brief review of the main notions, results and notation systematically used in this paper; these will frequently be used without explicit mention. Other results not appearing here will explicitly be referred to when they are used.

We denote by $C(X)$ the ring of all real-valued continuous functions defined on a topological space X. Our basic reference for the general theory of such rings is the classical book of Gillman and Jerison [3], particularly Chapters 13 and 14, which deal with the residue rings of $C(X)$ modulo maximal and prime ideals, respectively. Our notation and terminology follows theirs.

By [3; 3.9] in the study of rings of continuous functions it suffices to consider completely regular spaces. Accordingly, throughout this paper we make the blanket assumption that *all spaces are completely regular.*

B. *The Stone-Čech compactification.* For the definition and basic properties of the Stone-Čech compactification βX of a space X, see [3; Chapter 6]. We shall only recall that there is a bijective correspondence between the points of βX and the maximal ideals of $C(X)$; the maximal ideal corresponding to a point $p \in \beta X$ will be denoted by M^p; when $p \in X$ it is customary to write M_p instead.

The subring of $C(X)$ consisting of all bounded functions will be denoted by $C^*(X)$.

Theorem 6.5 of [3] gives several characterizations of the space βX; we shall need the following:

(2) Every $f \in C^*(X)$ has a unique extension, f^β, to a function in $C(\beta X)$.

(3) Every continuous function $f\colon X \to Y$, where Y is a *compact* space, has a unique continuous extension $\bar{f}\colon \beta X \to Y$.

In particular, since every $f \in C(X)$ can be considered as a continuous function with values in the one-point compactification $\boldsymbol{R}^* = \boldsymbol{R} \cup \{\infty\}$ of $\boldsymbol{R}$, it has a unique extension $f^*\colon \beta X \to \boldsymbol{R}^*$ given by (3). If f is bounded, then $f^* = f^\beta$, [3; 7.6]. The value of f^* at $p \in \beta X$ is determined by (cf. [3; 7.6]):

(4) (a) $f^*(p) = \infty$ iff $|f/M^p| > n$ for all $n \in \boldsymbol{N}$;

(b) $f^*(p) = r$ iff $|f/M^p - r| < 1/n$ for all $n \in \boldsymbol{N}$ $(r \in \boldsymbol{R})$.

In particular,

(5) $f \in M^p$ implies $f^*(p) = 0$.

Definition 6. A subspace Y of X is said to be C-*embedded* (resp. C^*-*embedded*) if every function in $C(Y)$ (resp. $C^*(Y)$) can be extended to a function in $C(X)$.

For more details, see [3; 1.16].

C. *Z-ideals.* The *zero-set* $f^{-1}[0]$ of a function $f \in C(X)$ is denoted by $Z(f)$; $Z \subseteq X$ is a zero-set if $Z = Z(f)$ for some $f \in C(X)$. $Z(X)$ denotes the lattice (under usual set-theoretic operations) of zero-sets of X. If $I \subseteq C(X)$ is an ideal, $Z(I)$ denotes the filter of $Z(X)$ consisting of zero-sets of members of I ([3; Chapter 2]). By a *z-filter* of X we mean a filter of the lattice $Z(X)$. A *cozero-set* is just the complement of a zero-set.

DEFINITION 7. An ideal $I \subseteq C(X)$ is a *z-ideal* iff for all $f \in C(X)$, $Z(f) \in Z(I)$ implies $f \in I$.

Every maximal ideal of $C(X)$ is a z-ideal ([3; 2.7]). Another important example is the ideal 0^p of germs of continuous functions at a point $p \in \beta X$ (cf. [3; 7.12]); this z-ideal is neither maximal nor prime, except in special cases. Further examples of prime z-ideals will be considered later.

D. *Prime ideals.*

(8) Every prime ideal of $C(X)$ lies between 0^p and M^p, for a unique $p \in \beta X$ ([3; 7.15]).

(9) The prime ideals of $C(X)$ containing a given prime ideal form a chain under inclusion ([3; 14.8]). In other words, the prime ideals contained in M^p for a given $p \in \beta X$ form a tree under the order of reverse inclusion (in the sense that the predecessors of a given element are totally ordered, but not necessarily well ordered).

(10) Every prime ideal contains a minimal prime ideal (under the order of inclusion), which is a z-ideal ([3; 14.7]).

Furthermore:

(11) Any two maximal chains of the tree of prime ideals contained in M^p ($p \in \beta X$) intersect, and their intersection has a minimal element which is a prime z-ideal ([3; 14.9 and p. 199]). In special cases this minimal element is M^p itself ([3; 14.G.5.]).

(12) 0^p is the intersection of all minimal prime ideals ([3; p. 199]). In particular, 0^p is a prime ideal iff the prime ideals contained in M^p form a chain under inclusion.

However, there can be prime z-ideals other than those mentioned above (the ideal M'_p of § 2.4 below is an example).

(13) There are prime ideals which are not z-ideals; in fact, above every nonmaximal prime there is a chain of type η_1 of pairs of consecutive prime ideals ([3; 14.19]), none of which is a z-ideal ([3; 14.10 and 14.D.4]).

Further examples of prime z-ideals will be introduced as we need them. For a comprehensive study of this subject, see [3; pp. 197–200 and 205 ff.] and Kohls [4], [5], [6], [7], where numerous examples are given.

E. *Properties of residue class rings.* The following simple properties of the rings $C(X)/P$, where P is a prime ideal of $C(X)$, will be assumed in many of the later arguments.

(14) $C(X)/P$ is a totally ordered integral domain containing the field $\boldsymbol{R}$ of real numbers ([3; 5.5]).

(15) Every prime ideal of $C(X)/P$ is convex ([3; 14.3(a)]).

(16) The prime ideals of $C(X)/P$ form a chain under inclusion; in particular, $C(X)/P$ is a local ring, i.e., it has only one maximal ideal ([3; 14.3(e)]).

(17) Every nonnegative element of C/P has nth-roots, for $n = 2, 3, \ldots$ ([3; 14.5]).

(18) If $a \in C/P$ is noninvertible and $b \in C/P$ is invertible, then $|ax| < |b|$ for every $x \in C/P$ ([3; 14.5(a)]).

(19) Every monic polynomial of odd degree with coefficients in C/P has a zero. Together with (17) this shows that C/M is a real closed field whenever M is a maximal ideal ([3; 13.4]).

F. *The divisibility property.* We define now a simple but fundamental notion used throughout the rest of this paper:

DEFINITION 20. Let A be a commutative (totally) ordered ring with unit. A is said to have the *divisibility property*, abbreviated DP, if

$$A \models \forall ab\ (0 < a < b \to b|a)\,.$$

The following basic result is proved in [2; Thm. 1]:

(21) For any commutative (totally) ordered ring with unit the following are equivalent:

(a) $A \models$ RCR.

(b) (i) $A \models$ DP;

(ii) $A \models$ every nonnegative element has a square root;

(iii) $A \models$ every monic polynomial of odd degree has a root.

In particular, together with (17) and (19), this proves:

(22) Let P be a prime ideal of $C(X)$. Then $C(X)/P$ is a model of RCR iff $C(X)/P \models$ DP. Another simple but important fact is the following:

(23) The divisibility property DP is preserved under homomorphic images; in other words, if A, B are ordered commutative rings with unit and $f\colon A \to B$ a surjective ring homomorphism which preserves order, then $A \models$ DP implies $B \models$ DP. We leave the proof of this as an exercise for the reader.

G. *Real closed rings vs. valuation rings.* The following results are proved in [2; Lemma 4 and Lemma 5]:

(24) Let K carry a valued field structure $\langle K, G, v\rangle$ and an ordered field structure $\langle K, <\rangle$. Then the following are equivalent:

1. K satisfies the compatibility condition

$$\forall ab\,(a \leqslant b \to v(a) \geqslant v(b))\,.$$

2. The valuation ring R of v is convex in K.
3. The maximal ideal M of R is convex in K.

4. The maximal ideal M of R is convex in R.
5. The maximal ideal M of R is bounded by ± 1.

(25) Let K be a real closed field carrying a valued field structure $\langle K, G, v\rangle$ and let $\langle K, <\rangle$ be its unique ordered field structure. Then the five conditions above are equivalent also to:

6. The valuation ring R of v is a real closed ring.
7. -1 is not a square in the residue class field $\bar{K}$ of K.
8. $\bar{K}$ is a real closed field.

DEFINITION 26. A ring A is called a *valuation ring* if

$$A \models \forall ab(a|b \vee b|a).$$

A totally ordered ring satisfying the divisibility property DP obviously is a valuation ring; in particular, so is any RCR.

It is well-known that a valuation ring gives rise to a canonical valuation structure, denoted $\langle A, S, \mathrm{val}\rangle$ with values on an ordered semigroup S. We shall only consider here valuation integral domains. In this case the canonical valuation structure of A extends uniquely, in an obvious way, to the field of fractions of A, $K = \mathrm{Quot}(A)$; let us denote $\langle K, G, v\rangle$ the resulting valued field structure. If, in addition, A carries an order structure, so does K, in the obvious way.

Let us now assume that the conditions above are met when A is a ring of the form $C(X)/P$. We obtain:

PROPOSITION 27. *Let $P \subseteq C(X)$ be a prime ideal. Then $C(X)/P$ is a valuation ring iff it is a real closed ring.*

Proof. The remark following Definition 26 proves the implication from right to left. ($\Rightarrow$) In Part II, § 1, Lemma 5, we prove that if A is a totally ordered integral domain satisfying properties (17) and (19) above, then the field of fractions K of A also satisfies them, i.e. K is a real closed field.

Let now $A = C(X)/P$ be a valuation ring. By the very definition of the canonical value structures in A and K, A is the valuation ring of $\langle K, G, v\rangle$. By (24) and (25) it suffices to check that the maximal ideal M of A is bounded by ± 1, i.e.

$$f \in C(X) \text{ and } f/P \geqslant 1 \Rightarrow f/P \text{ invertible in } C/P.$$

Since $f/P \geqslant 1$ we can assume, by changing f if necessary, that $f \geqslant 1$ on X. Then $f^{-1} \neq 0$ on X, whence $f^{-1} \in C(X)$ and $f/P \cdot f^{-1}/P = 1$. ∎

H. *Martin's Axiom.* In order to establish the existence of certain types of prime ideals of continuous functions we will need, in addition to the axioms of set theory (ZFC), the set-theoretic assumption known as Martin's Axiom. We now review the background connected with this axiom; for further details and applications, see [10].

DEFINITION 28. Let P be partially ordered set, let $A, D, G \subseteq P$, and let $\mathscr{D}$ be a family of subsets of P.

(a) Elements $p, q \in P$ are *compatible* iff there is $r \in P$ such that $r \leqslant p, q$.

(b) A is an *antichain* of P iff any two elements of A are incompatible.

(c) P satisfies the *countable antichain condition* (c.a.c.) iff every antichain in P is countable.

(d) D is *dense* in P iff every element of P lies above some element of D.

(e) G is *$\mathscr{D}$-generic* (where $\mathscr{D}$ is a family of dense subsets of P) iff any two elements of G are compatible and G meets each of the members of $\mathscr{D}$.

Martin's axiom is the following combinatorial principle:

(MA) Suppose that P is a partially ordered set satisfying the c.a.c. and that $\mathscr{D}$ is a family of (strictly) fewer than $2^{\aleph_0}$ dense subsets of P. Then P contains a $\mathscr{D}$-generic subset G.

When $\mathscr{D}$ is countable it is easy to prove this assertion in ZFC; thus (MA) is a consequence of the continuum hypothesis. However, (MA) does not imply the continuum hypothesis; cf. [10].

I. *P-points.* The notion of P-point in various spaces of ultrafilters will play a central role in §§ 2,3 below.

DEFINITION 29. A point x of a topological space X is a *P-point* iff every G_δ containing x is a neighborhood of x.

The P-points of the nonprincipal ultrafilter space $\beta N - N$ have been extensively investigated (cf. [12]).

We will need the following characterizations of P-points $\mathscr{U}$ of $\beta D - D$, where D is an infinite discrete space.

(30) For every sequence $\{U_n|\ n \in \omega\}$ of members of $\mathscr{U}$, there is $U \in \mathscr{U}$ such that $U - U_n$ is finite for all $n \in \omega$.

(31) For every partition $\{P_k|\ k \in \omega\}$ of N such that $P_k \notin \mathscr{U}$, there is $U \in \mathscr{U}$ such that $U \cap P_k$ is finite for all $k \in \omega$.

For the proof see [1; 4.7] and [3; 6.S].

In the case $D = N$ (i.e. $\overline{\overline{D}} = \aleph_0$) the following is also known:

(32) $\mathscr{U} \in \beta N - N$ is a *P-point* iff for each sequence $\langle x_n|\ n \in \omega\rangle$ in 2^N (endowed with the product topology) there is $A \in \mathscr{U}$ such that $\langle x_n|\ n \in A\rangle$ converges ([1; 4.7]).

(33) $\mathscr{U} \in \beta N - N$ is a P-point iff for each bounded sequence $f\colon N \to \boldsymbol{R}$ there is $A \in \mathscr{U}$ such that $\lim\limits_{\substack{n \to \infty \\ n \in A}} f(n)$ exists.

The last statement can be derived from (32); an alternative proof will be given in § 3 (Corollary 3.1.2). This characterization will be of fundamental importance in our later work.

Using (30) above it is easily checked:

(34) Let $\mathscr{U} \in \beta D - D$ and $A \in \mathscr{U}$. Then $\mathscr{U}$ is a P-point of $\beta D - D$ iff $\mathscr{U} \restriction A$ is a P-point of $\beta A - A$.

The existence of a P-point in $\beta N - N$ is not provable in ZFC. Indeed, we cite the following result due to Shelah (cf. [11; Part VI, §§ 4–5, pp. 213–232]):

(35) If ZF is consistent, then so is ZFC+"$2^{\aleph_0} = \aleph_2$"+"$\beta N - N$ does not contain P-points".

On the other hand, the existence of P-points in $\beta N - N$ — indeed, of a dense set of such points — can be proved in ZFC+Martin's Axiom (cf. [1; 4.13]).

J. *X-neighborhoods.*

DEFINITION 36. Let $p \in \beta X$ and $U \subseteq X$. U is called an *X-neighborhood* of p iff U has the form $V \cap X$ for some neighborhood V of p in βX.

In particular, when $p \in X$ the X-neighborhoods of p coincide with the neighborhoods of p in X.

We shall need the following characterization of X-neighborhoods:

LEMMA 37. *Let $p \in \beta X$ and $U \subseteq X$. U is an X-neighborhood of p iff there is $Z \in Z(X) - Z(M^p)$ such that $X - Z \subseteq U$.*

Proof. Routine, using:

(*) The sets $U_Z = \{q \in \beta X \mid Z \notin Z(M^q)\}$ for $Z \in Z(X)$ form a base of open sets for the topology of βX. Furthermore, $U_Z \cap X = X - Z$. ([3; p. 87]). ■

From this lemma and [3; 5.4(b)] it easily follows:

(38) Let $F \in C(X)$, $p \in \beta X$ and $r \in \boldsymbol{R}$; then:

$$f/M^p > r \text{ iff } f > r \text{ on an } X\text{-neighborhood of } p.$$

§ 2. General theory

2.1. Topological characterizations of real closed ideals.

A. *Global version.* We begin by giving (Theorem 1) a necessary and sufficient topological condition for a residue ring of the form $C(X)/P$ to be a real closed ring, whenever P is a prime z-ideal of $C(X)$. Combined with later results, this condition makes possible to detect in a quite efficient way the presence of prime ideals P such that $C(X)/P$ is a real closed ring. It is the main tool for the investigation carried out in the present paper.

We shall call an ideal P *real closed* iff $C(X)/P$ is a model of RCR. Real closed ideals are always prime, and P is real closed iff P is maximal or $C(X)/P \models$ RCVR.

THEOREM 1. *Let P be a prime z-ideal of $C(X)$ and let M^p, $p \in \beta X$, be the unique maximal ideal of $C(X)$ containing P. The following are equivalent:*

(1) *P is real closed.*

(2) *For every $Z \in Z(M^p) - Z(P)$ and every $l \in C(X - Z)$, $0 \leqslant l \leqslant 1$, there are $W \in Z(P)$ and $h \in C(X)$, $0 \leqslant h \leqslant 1$, such that $h \restriction (W - Z) = l \restriction (W - Z)$.*

Proof. (2) ⇒ (1). By 1.22 only the divisibility property (DP) needs to be verified. Let $0 < f/P < g/P$. Since (DP) holds automatically whenever the largest element in the inequality is invertible, we can assume g/P noninvertible. The ring $C(X)/P$ is a local ring whose maximal ideal is $\{h/P \mid h \in M^p\}$; hence $g \in M^p$. We can also assume without loss of generality that $0 \leqslant f \leqslant g$; in particular $Z(g) \subseteq Z(f)$.

Since P is a z-ideal, f/P, $g/P > 0$ imply $Z(f)$, $Z(g) \notin Z(P)$. Now apply (2) with $Z = Z(g)$ and $l = f/g$ (which is defined on $X - Z$) to get $W \in Z(P)$ and $h \in C^*(X)$, $0 \leqslant h \leqslant 1$, such that $h \restriction (W - Z) = l \restriction (W - Z)$. For $x \in W - Z$ we have $g(x)h(x) = f(x)$, and for $x \in Z$, $f(x) = g(x) = 0$ and hence $g(x)h(x) = f(x)$. This shows that $gh = f$ on W; since $W \in Z(P)$ and P is a z-ideal, we conclude $g/P\, h/P = f/P$, i.e., g/P divides f/P.

(1) ⇒ (2). Let $Z = Z(g) \in Z(M^p) - Z(P)$ and $l \in C(X - Z)$, $0 \leqslant l \leqslant 1$. Put:

$$f(x) = \begin{cases} 0 & \text{if } x \in Z, \\ g(x)l(x) & \text{if } x \in X - Z. \end{cases}$$

Since l is bounded and $g \in C(X)$, we have $f \in C(X)$. Also $0 \leqslant f \leqslant g$ (since $0 \leqslant l \leqslant 1$); hence $0 \leqslant f/P \leqslant g/P$. Now apply (1) to get $h \in C(X)$ such that $g/P \cdot h/P = f/P$. Put $W = Z(gh - f)$; then $W \in Z(P)$. For $x \in W - Z$ we have $g(x) \neq 0$ and $g(x)h(x) = g(x)l(x)$, whence $h(x) = l(x)$. By considering $(h \vee 0) \wedge 1$ instead of h, since $0 \leqslant l \leqslant 1$, the map h can be taken to be bounded between 0 and 1. ■

Remark. Under the assumptions of the theorem, condition (2) is equivalent to:

(3) For every $Z \in Z(X) - Z(P)$ and every $l \in C(X - Z)$, $0 \leqslant l \leqslant 1$, there are $W \in Z(P)$ and $h \in C(X)$, $0 \leqslant h \leqslant 1$, such that $h \restriction (W - Z) = l \restriction (W - Z)$.

Proof. Let $Z \in Z(X) - Z(M^p)$ and $l \in C(X - Z)$, $0 \leqslant l \leqslant 1$, be given.

We prove first that there is $W \in Z(P)$ such that $W \cap Z = \emptyset$. If this is not true, then $Z(P) \cup \{Z\}$ has the finite intersection property, and hence there is a z-ultrafilter containing it; let M be a maximal ideal on $C(X)$ such that $Z(P) \cup \{Z\} \subseteq Z(M)$ ([3; 2.5]). It follows that $P \subseteq M$, and since P is contained in a unique maximal ideal, then $M = M^p$; therefore $Z \in Z(M) = Z(M^p)$, contradicting our assumption.

Thus, W and Z are disjoint zero-sets, and hence completely separated ([3; 1.15]), i.e. there is $f \in C(X)$ such that $f \restriction W = 1$, $f \restriction Z = 0$ and $0 \leqslant f \leqslant 1$. Setting

$$h(x) = \begin{cases} f(x)l(x) & \text{for } x \in X - Z, \\ 0 & \text{for } x \in Z, \end{cases}$$

we conclude that $h \in C(X)$ (because l is bounded), $0 \leqslant h \leqslant 1$, and $h \restriction (W - Z) = l \restriction (W - Z)$ by the choice of f and $W = W - Z$.

As a corollary we obtain an extensive class of spaces where all prime z-ideals (indeed, all prime ideals; cf. Corollary 9 below) are real closed. An *F-space* is a space X in which the ideals 0^p are prime for all $p \in \beta X$ ([3; 14.25(1)]).

PROPOSITION 2. *If X is an F-space, every prime z-ideal of $C(X)$ is real closed.*

Proof. By [3; 14.25(6)] every cozero-set in X is C^*-embedded. Thus condition (2) of the preceding theorem is verified with $W = X$ (independently of the zero-set Z and the function l). ■

Remark. In the case of the ideals 0^p this result was proved by Kohls [4; 5.7] and [5; p. 529]. The restriction to z-ideals will be removed later (cf. Corollary 9).

A *P-space* is a space X such that 0^p is a maximal ideal for all $p \in \beta X$ [3; 14.29(1)], or, equivalently, every prime ideal is maximal. Proposition 2 is a source of models for the theory RCVR only in the case of F-spaces which are not P-spaces.

3. EXAMPLES of F-spaces which are not P-spaces. (a) $\beta X - X$, whenever X is a locally compact, σ-compact space which is not compact (cf. [3; 14.27]); this space is also compact. In particular, $\beta N - N$ and $\beta R - R$ are F-spaces but not P-spaces.

(b) βX, whenever X is an infinite P-space ([3; p. 212]). Thus βN is another example.

(c) $X \times \beta X$, whenever X is an infinite P-space ([12; 8.38]).

(d) Every infinite zero-set Z of βX such that $Z \cap X = \emptyset$ ([3; 14.0.1]).

(e) $\beta E - E$, whenever E is a cozero-set in a compact space and $\beta E - E$ is infinite ([3; 14.0.2]).

(f) $\beta X - X$, whenever X is a locally compact F-space which is not compact ([3; 14.0.3]).

(g) If X is a P-space, every subspace of βX containing X but not contained in νX (the real compactification of X) is an F-space but not a P-space ([3; 1.P.2]). ■

On the other side of the picture, most familiar spaces are not F-spaces. For example: no metrizable space is an F-space, unless it is discrete; furthermore: no nonisolated point of an F-space has a countable base of neighborhoods ([3; 14.N.2]).

B. *Local version.* We shall now show that a "local" version of the topological condition (2) of Theorem 1 is still sufficient to characterize real closed ideals. Accordingly, we will be able to sharpen Proposition 2 into a corresponding "local" version.

LEMMA 4. *Let $p \in \beta X$ and let U be an X-neighborhood of p. Then for every $g \in C(X)$ there is $g' \in C(X)$ such that $g/0^p = g'/0^p$ and $Z(g') \subseteq U$.*

Proof. Case 1. $g^*(p) \neq 0$. By 1.4 and 1.38 this assumption implies that there is an X-neighborhood of p on which $g > 0$ (if $g/M^p > 0$) or $g < 0$ (if $g/M^p < 0$). Considering $-g$ instead of g we reduce to the first of these alternatives. Replacing, if necessary, g by $\max\{g, 0\}$ and U by a smaller X-neighborhood of p, we can assume $g \geqslant 0$ on X. Furthermore, by considering $2/r \cdot g$ instead of g whenever $g^*(p) = r < 2$, we can assume that $g^*(p) \geqslant 2$; in particular, $p \notin Z(g)$.

Setting $f = 1 - \min\{g, 1\}$, we have $f \in C(X)$, $0 \leqslant f \leqslant 1$, $f = 1$ on $Z(g)$ and $(x) = 0$ whenever $g(x) \geqslant 1$. In particular, $Z(f) \supseteq g^{-1}[(1, \infty)]$, which proves that $Z(f)$ is an X-neighborhood of p (because $g^*(p) \geqslant 2$). Since g and $g+f$ coincide on $Z(f)$, we conclude that $g/0^p = g+f/0^p$. Next, observe that $Z(g+f) = Z(g) \cap Z(f)$ because $g, f \geqslant 0$; since g and f are never simultaneously zero (by the definition of f), we have $Z(g+f) = \emptyset \subseteq U$. Set $g' = g+f$.

Case 2. $g^*(p) = 0$. Under this assumption, the set $\{x \in X \mid |g(x)| \leqslant 1\}$ is an X-neighborhood of p on which g coincides with $\min\{g, 1\}$. Replacing, if necessary,

g by the latter and U by a smaller X-neighborhood of p, we can assume that $|g| \leqslant 1$ on X, and (by Lemma 1.37) that $U = X - Z$ for some zero-set Z of X.

Complete regularity implies the existence of a zero-set V of X which is an X-neighborhood of p with $V \subseteq U$ (cf. [3; 3.2(b)]). Therefore V and Z are disjoint zero-sets, and hence completely separated [3; 1.15]; let $f \in C(X)$ be such that $0 \leqslant f \leqslant 2$, $f = 0$ on V and $f = 2$ on Z. Set $g' = g+f$. Then g and g' coincide on the X-neighborhood V of p, whence $g/0^p = g'/0^p$; if $x \in X - U$ $(=Z)$, then $f(x) = 2$ and $|g(x)| \leqslant 1$, so that $g'(x) \neq 0$. This proves that $Z(g') \subseteq U$, and hence also the lemma. ■

PROPOSITION 5. *Let P be a prime z-ideal of $C(X)$ and let M^p, $p \in \beta X$, be the unique maximal ideal containing P. Let U be an X-neighborhood of p. Then the following are equivalent:*

(1) *P is real closed.*

(2) *For every $Z \in Z(M^p) - Z(P)$ such that $Z \subseteq U$ and every $l \in C(X-Z)$, $0 \leqslant l \leqslant 1$, there are $h \in C(X)$, $0 \leqslant h \leqslant 1$, and $W \in Z(P)$ such that $h \restriction (W-Z) = l \restriction (W-Z)$.*

Proof. (1) ⇒ (2) holds by the corresponding implication in Theorem 1. To prove the converse, if $C(X)/P \models 0 < f/P < g/P$, we can assume $Z(g) \subseteq U$, upon replacing, if necessary, the function g by the function g' given by the preceding lemma, and taking into account that $0^p \subseteq P$. The proof proceeds, then, as in Theorem 1. ■

PROPOSITION 6. *Let $p \in \beta X$ and assume p has an X-neighborhood which is an F-space. Then for every nonmaximal prime z-ideal $P \subseteq M^p$, $C(X)/P$ is a real closed ring.*

Proof. Let U be an X-neighborhood of p which is an F-space. By Lemma 1.37, there is $Z' \in Z(X) - Z(M^p)$ such that $X - Z' \subseteq U$; $X - Z'$ is also an X-neighborhood of p, and since a cozero-set in an F-space is again an F-space ([3; 14.26]), we can assume without loss of generality that $U = X - Z'$.

Let now $Z \subseteq U$ and $l \in C^*(X-Z)$. Since U is an F-space, the map $l \restriction (U-Z)$ has an extension $l' \in C^*(U)$, [3; 14.25(6)]; put

$$h(x) = \begin{cases} l'(x) & \text{for } x \in U = X - Z' \\ l(x) & \text{for } x \in X - Z. \end{cases}$$

Since l coincides with l' on $U - Z$, it follows that $h \in C^*(X)$. Hence condition (2) of Proposition 5 holds with $W = X$. ■

Remark. When p is a βF-point (with an X-neighborhood which is an F-space) and $P = 0^p$, this result was proved by Kohls [5; 4.1(1)].

C. *Preservation under extensions.* We prove now that the family of real closed prime ideals is closed under extensions (supersets). This applies to all prime ideals, not just z-ideals, and therefore extends considerably the scope of the preceding results.

This result shows that it should be considerably more difficult to find real closed ideals low in the tree of prime z-ideals — as is the case in the preceding results — than higher up in that tree, "near" the maximal ideal; the latter, at the top of the tree, is always real closed.

PROPOSITION 7 (extension property). *Let $Q \subseteq P$ be prime ideals of $C(X)$. If Q is real closed, then so is P.*

Proof. This follows at once from 1.23 because the natural map $\varphi(f/Q) = f/P$, $f \in C(X)$, from $C(X)/Q$ to $C(X)/P$ is a surjective ordered ring homomorphism. ■

COROLLARY 8. *Let X be any space.*

(1) *Let P be a prime ideal of $C(X)$. If there is a minimal prime ideal $J \subseteq P$ which is real closed, then $C(X)/P$ is a model of* RCR.

(2) *$C(X)/P$ is a real closed ring for every nonmaximal prime ideal P of $C(X)$ (resp. included in M^p, $p \in \beta X$) iff $C(X)/J$ is a real closed ring for every minimal prime ideal J of $C(X)$ (resp. included in M^p).* ■

COROLLARY 9. *If X is an F-space, then every prime ideal is real closed. Similarly, under the hypothesis of Proposition 6, every prime ideal contained in M^p is real closed.*

Proof. Immediate from Corollary 8 and Proposition 2 (resp. Proposition 6), taking into account that minimal prime ideals of $C(X)$ are z-ideals (1.10). ■

EXAMPLE 10. A space X where all prime ideals are real closed but which is not an F-space.

Let X_1, X_2 be F-spaces containing points p_1, p_2 such that 0_{p_i} is not maximal. Let X be the space obtained by "wedging" X_1, X_2 together through p_1 and p_2; i.e., X is the (disjoint) union of X_1 and X_2 with identification of p_1 and p_2 (into p, say); the neighborhoods of p are the unions of a neighborhood of p_1 in X_1 and a neighborhood of p_2 in X_2, while the open sets of X not containing p are those of X_1 and of X_2, not containing p_1 and p_2.

(a) X is not an F-space. For $i = 1, 2$, let φ_i: $C(X) \to C(X_i)$ be the map $\varphi_i(f) = f \restriction X_1$. Obviously φ_i is a surjective ring homomorphism, and if $0_{p_i} \subseteq P_i \subseteq M_{p_i}$, $\varphi_1^{-1}[P_1]$ and $\varphi_2^{-1}[P_2]$ are $\subseteq$-incomparable prime ideals contained in M_p. Indeed, if $g_i \in M_{p_i} - P_i$, then $g_1 \cup 0 \in \varphi_2^{-1}[P_2] - \varphi_1^{-1}[P_1]$ and $0 \cup g_2 \in \varphi_1^{-1}[P_1] - \varphi_2^{-1}[P_2]$; here, if f_i: $X_i \to \boldsymbol{R}$ are functions such that $f_1(p_1) = f_2(p_2)$, then $f_1 \cup f_2$: $X \to \boldsymbol{R}$ is the function defined by:

$$f_1 \cup f_2(x) = \begin{cases} f_i(x) & \text{if } x \in X_i - \{p_i\} \ (i = 1, 2) \\ f_1(x) & \text{if } x = p\,. \end{cases}$$

Let $J_i = \varphi_i^{-1}[0_{p_i}]$. The definition of the topology of X implies at once that $0_p = J_1 \cap J_2$. It follows that:

$$(*) \qquad \text{every prime ideal } P \subseteq M_p \text{ contains } J_1 \text{ or } J_2.$$

(Otherwise, let $f_i \in J_i - P$; then $f_1 f_2 \in J_1 \cap J_2 = 0_p \subseteq P$, implying that P is not prime). Thus, J_1, J_2 are the minimal prime ideals contained in M_p.

(b) In order to show that every prime ideal of $C(X)$ is real closed, it suffices to show that J_1 and J_2 are (Corollary 8(2)). Now, if

$$C(X) \xrightarrow{\varphi_i} C(X_i) \xrightarrow{\pi_i} C(X_i)/0_{p_i},$$

where π_i denotes the canonical projection, we have $\operatorname{Ker}(\pi_i \circ \varphi_i) = \varphi_i^{-1}[0_{p_i}] = J_i$. Hence $C(X)/J_i \simeq C(X_i)/0_{p_i}$ and J_i is real closed. ■

In this example, the maximal chain of prime ideals of $C(X)$ joining J_i to M_p is isomorphic to that joining 0_{p_i} to M_{p_i}. This is due to the fact that the map φ_i induces in the obvious way an order-preserving bijection between these chains (easy verification).

D. *Rings of bounded continuous functions.* We shall examine briefly the case of the ring $C^*(X)$ of bounded continuous functions on X. Since $C^*(X) = C(\beta X)$, the preceding theory does automatically apply but, as we shall see, it is simpler to consider directly the ring $C^*(X)$ rather than taking a detour via the space βX.

The isomorphism relation above implies that the results of § 1.E apply also to the ring $C^*(X)$; cf. [3; 6.6(c)]. To each $p \in \beta X$ there corresponds a unique maximal ideal of $C^*(X)$: $M^{*p} = \{f \in C^* |\ f^\beta(p) = 0\}$. We have that $M^p \cap C^* \subseteq M^p$, and p is the unique $q \in \beta X$ for which the relation $M^p \cap C^* \subseteq M^{*q}$ holds. Also, if $P \subseteq C(X)$ is a prime z-ideal, then so is $P \cap C^*$ and for $p \in \beta X$, $P \subseteq M^p$ if and only if $P \cap C^* \subseteq M^{*p}$. For further details see Mandelker [9].

PROPOSITION 11. *Let Q be a prime z-ideal of $C^*(X)$ and let M^{*p} $(p \in \beta X)$ be the unique maximal ideal of $C^*(X)$ containing Q. The following are equivalent:*

(1) *Q is real closed.*

(2) *For every $Z \in Z(M^p) - Z(Q)$ and every $l \in C(X-Z)$, $0 \leqslant l \leqslant 1$, there are $W \in Z(Q)$ and $h \in C(X)$, $0 \leqslant h \leqslant 1$, such that $h \restriction (W-Z) = l \restriction (W-Z)$.*

Proof. First notice that the proof of Theorem 1 carries over literally to the bounded case (that is, with C^* and M^{*p} replacing C and M^p, respectively); i.e., condition (1) is equivalent to:

(2′) for every $Z \in Z(M^{*p}) - Z(Q)$ and every $l \in C(X-Z)$, $0 \leqslant l \leqslant 1$, there are $W \in Z(Q)$ and $h \in C(X)$, $0 \leqslant h \leqslant 1$, such that $h \restriction (W-Z) = l \restriction (W-Z)$.
Next we remark that

$$Z(M^{*p}) = \begin{cases} Z(M^p) & \text{if } M^{*p} = M^p \cap C^*\,, \\ Z(X) & \text{if } M^{*p} \neq M^p \cap C^* \end{cases}$$

(by [9; Theorem 1] the second case occurs iff M^{*p} contains a unit of $C(X)$). In the first case (2) and (2′) are obviously equivalent. In the second case this equivalence follows from the condition:

(3) for every $Z \in Z(X) - Z(M^p)$ and every $l \in C(X-Z)$, $0 \leqslant l \leqslant 1$, there are $W \in Z(Q)$ and $h \in C(X)$, $0 \leqslant h \leqslant 1$, such that $h \restriction (W-Z) = l \restriction (W-Z)$.

This, in turn, is a consequence of:

(4) for every $Z \in Z(X) - Z(M^p)$ there is $W \in Z(Q) \cap Z(M^p)$ such that $W \cap Z = \emptyset$.

Proof of (4). Let $\mathcal{F} = Z(Q) \cap Z(M^p)$; then $\mathcal{F} = Z(M^p)$ if $M^p \cap C^* \subsetneq Q$, and $\mathcal{F} = Z(Q)$ if $Q \subseteq M^p \cap C^*$ (Q is comparable with $M^p \cap C^*$, [9; Theorem 1]); in particular, $\mathcal{F}$ is a proper prime z-filter. If (4) is false, let $Z \in Z(X) - Z(M^p)$ be such that $W \cap Z \neq \emptyset$ for all $W \in \mathcal{F}$. Then $\mathcal{F} \cup \{Z\}$ generates a z-ultrafilter, i.e., there is $q \in \beta X$ such that $\mathcal{F} \cup \{Z\} \subseteq Z(M^q)$. Since $Z \in Z(M^p)$, the prime z-filter $\mathcal{F}$ is contained in the distinct z-ultrafilters $Z(M^q)$, $Z(M^p)$, contradicting [3; 2.13].

Proof of (4) $\Rightarrow$ (3). Given Z and l as in (3), let $W \in Z(Q) \cap Z(M^p)$ be so that $W \cap Z = \emptyset$. By [3; Theorem 1.15] W and Z are completely separated, i.e. there is $f \in C(X)$ such that $f \restriction W = 1$, $f \restriction Z = 0$ and $0 \leqslant f \leqslant 1$. Set $h = fl$ on $X - Z$, $h = 0$ on Z. Since l is bounded and $f \restriction Z = 0$, we have $h \in C(X)$: also $0 \leqslant h \leqslant 1$. Furthermore, $W - Z = W$, and then $h \restriction (W-Z) = l \restriction (W-Z)$. ■

A corresponding "local" result is obtained by using Proposition 5 instead of Theorem 1.

Corollary 12. *Let P be a prime z-ideal of $C(X)$. Then*:

(1) *P is real closed iff $P \cap C^*$ is real closed.*

In particular,

(2) *$M^p \cap C^*$ is a real closed ideal of C^*.*

(3) *Any prime ideal Q of C^* containing $M^p \cap C^*$ is real closed.*

Proof. (2) follows from (1) and the fact that M^p is real closed. (3) is a consequence of (2) and the analogue of Proposition 7 for prime ideals of C^*. Remark that this analogue holds, as the fact (noted earlier) that the results of § 1.E carry over to the ring C^* entails at once that 1.22 is also true for ideals in C^*.

Statement (1) follows at once from Theorem 1 and Proposition 11, since the equality $Z(P) = Z(P \cap C^*)$ holds for any prime z-ideal P of C, as is readily verified. ■

The above and [3; 7.9(c)] imply that whenever X is a realcompact space (see the definition in [3; 5.9]), $M^p \cap C^*$ is a nonmaximal real closed ideal of C^*, for every $p \in \beta X - X$; this remark is due to the referee, improving an earlier observation by the authors. This contrasts with the results of the next section where we will prove that the existence of real closed ideals properly contained in M^p is not provable in ZFC, whenever X is metrizable and $p \in X$.

2.2. Real closed ideals and P-points. We shall now discuss the relationship between these two notions. Our analysis will be based on the characterization of P-points of $\beta N - N$ given in 1.33 and the following intermediate concept.

Definition 1. Let X be a topological space, $p \in X$, and $\mathcal{F}$ a filter of subsets of X. $\mathcal{F}$ is called a *$P(p)$-filter* iff for every function $f \in C(X - \{p\})$, $0 \leqslant f \leqslant 1$, there is $Y \in \mathcal{F}$ such that $\lim_{\substack{x \to p \\ x \in Y}} f(x)$ exists.

Clearly any filter containing a $P(p)$-filter is also a $P(p)$-filter.

We will prove first that the z-filter corresponding to any minimal prime ideal below a real closed ideal is a $P(p)$-filter for suitable $p \in X$. Next we show that in the case of a metrizable space X the existence of a nonmaximal real closed ideal contained in M_p, for $p \in X$, implies the existence of a P-point in $\beta N - N$.

Theorem 2. *Let $p \in X$, let $P \subsetneq M_p$ be a prime ideal, and let $Q \subseteq P$ be a minimal prime ideal. If P is real closed, then $Z(Q)$ is a $P(p)$-filter.*

Proof. Suppose $f \in C(X - \{p\})$, $0 \leqslant f \leqslant 1$, is such that $\lim_{\substack{x \to p \\ x \in Z}} f(x)$ does not exist for any $Z \in Z(Q)$. We shall show that $P = M_p$. Let $g \in M_p$. Replacing, if necessary, g by g^2 we can assume $0 \leqslant g$, so that $0 \leqslant fg \leqslant g$. Since f is bounded and $g(p) = 0$, the map fg (defined on $X - \{p\}$) extends in the obvious way to a map $C(X)$ also denoted by fg. Since P is real closed, there is $h \in C(X)$ such that

$$fg/P = h/P \cdot g/P.$$

Let $r = h(p)$, $f' = |f - r|$, $h' = |h - r|$. Clearly $h'(p) = 0$, $h' \in C(X)$ and defining $f'g$ on X as above, a routine computation shows that $f'g/P = h'/P \cdot g/P$. Let

$$Y_1 = \{x \in X - \{p\} \mid f'(x) \leqslant g(x) + h'(x)\} \cup \{p\},$$

$$Y_2 = \{x \in X - \{p\} \mid f'(x) \geqslant g(x) + h'(x)\} \cup \{p\}.$$

Clearly $\lim_{\substack{x \to p \\ x \in Y_1}} f' = 0$, whence $\lim_{\substack{x \to p \\ x \in Y_1}} f = r$. By assumption it follows that $Y_1 \notin Z(Q)$, and therefore $Y_2 \in Z(Q)$ (since Y_1, Y_2 are zero-sets and $Z(Q)$ is a prime z-filter). But on $Y_2 - \{p\}$ we have $f' - h' \geqslant g$, whence $(f' - h')g \geqslant g^2$ on Y_2; since Q is a z-ideal (1.10) and $P \supseteq Q$, it follows ([3; 5.4(a)]) that $(f' - h')g/P \geqslant g^2/P \geqslant 0$; since $(f' - h')g \in P$, we conclude that $g^2 \in P$; hence $g \in P$.

Theorem 3. *If X is a metrizable space, $p \in X$, and there is a prime $P(p)$-filter $\mathcal{F}$ of zero-sets such that $\mathcal{F} \subsetneq Z(M_p)$, then there is a P-point in $\beta N - N$.*

Proof. Let d denote a metric defining the topology of X. Fix $r_1 > r_2 > \ldots \to 0$ and let

$$A_0 = \{x \in X \mid d(x, p) \geqslant r_1\},$$

$$A_n = \{x \in X \mid d(x, p) \in [r_{n+1}, r_n]\} \quad \text{for } n \geqslant 1.$$

These are zero-sets of X ([3; 1.11]), and since p is nonisolated (because $\mathcal{F}$ is proper) we can assume without loss of generality that the A_n's are all nonempty. Set

$$Y_1 = \bigcup_n A_{2n} \cup \{p\},$$

$$Y_2 = \bigcup_n A_{2n+1} \cup \{p\}.$$

Clearly $Y_1 \cup Y_2 = X \in \mathcal{F}$ and we can assume $Y_1 \in \mathcal{F}$ without loss of generality. For $S \subseteq N$ define

$$Y(S) = \bigcup_{n \in S} A_{2n} \cup \{p\},$$

and let

$$\mathcal{U} = \{S \subseteq N \mid Y(S) \in \mathcal{F}\}.$$

It is easily checked that $\mathcal{U}$ is an ultrafilter on N ($\emptyset \notin \mathcal{U}$ follows from $\{p\} \notin \mathcal{F}$ which, in turn, is a consequence of $\mathcal{F} \neq Z(M_p)$). Moreover, $\mathcal{U}$ is nonprincipal, for $\{k\} \in \mathcal{U}$ would imply $A_{2k} \cup \{p\} \in \mathcal{F}$, and (since $\mathcal{F}$ is a $P(p)$-filter) also $\{p\} \in \mathcal{F}$, a contradiction.

Now we use the characterization (1.33) to show that $\mathcal{U}$ is a P-point. Let $f \in C^*(N)$ be given. It is easy to construct a function $f' \in C^*((0, \infty))$ such that

$$f'(r) = f(n) \quad \text{for } r \in [r_{2n+1}, r_{2n}],\ n \in N.$$

Then define

$$f^*(x) = f'(d(x, p))$$

Thus, $f^* \in C^*(X - \{p\})$. Fix $Z \in \mathcal{F}$ so that $\lim\limits_{\substack{x \to p \\ x \in Z}} f^*(x)$ exists; we may suppose $Z \subseteq Y_1$.

Set

$$S = \{n \in N \mid Z \cap A_{2n} \neq \emptyset\}.$$

Then $Z \subseteq Y(S)$, implying that $S \in \mathcal{U}$ and

$$\lim_{\substack{n \to \infty \\ n \in S}} f = \lim_{\substack{x \to p \\ x \in Z}} f^*;$$

this shows that $\lim\limits_{\substack{n \to \infty \\ n \in S}} f$ exists. ■

COROLLARY 4. *Let X be a metrizable space, $p \in X$, and assume M_p contains a nonmaximal real closed ideal; then there is a P-point in $\beta N - N$. In particular, the existence of such ideals is not provable in* ZFC.

2.3. The operation γ. In [6; § 2] Kohls introduced a one-to-one correspondence γ between prime z-ideals of $C(X)$ contained properly in M^p — where p is a nonisolated G_δ-point of the space X — and certain prime z-ideals of $C(X - \{p\})$.

Our main result in this section (Theorem 2) gives an exact characterization of real closed ideals of the form $\gamma(Q)$, in terms of the ideal Q. This characterization is essential for later work in § 3.

1. Definition and basic properties. Let X be an arbitrary (completely regular) space and $p \in X$ be a nonisolated G_δ-point. Let i: $X - \{p\} \to X$ be the inclusion, let X_1 be the largest subspace of $\beta(X - \{p\})$ on which i admits a continuous X-valued extension (for its existence, see [3; 10.13]), and let ϕ be the (unique) continuous extension of i to X_1. Then ϕ is a closed map and $\phi^{-1}(p) = X_1 - (X - \{p\})$ is nonempty (cf. [3; 10.13]).

The closure operator cl_X maps $Z(X - \{p\})$ into $Z(X)$ ([6; 2.1]) and therefore it induces a map γ with the following properties:

(1) γ is a bijection of the family $\mathcal{Q}$ of all prime z-ideals Q of $C(X - \{p\})$ such that $Z(Q)$ converges to a point of $\phi^{-1}(p)$, onto the family $\mathcal{P}$ of all nonmaximal prime z-ideals of $C(X)$ contained in M^p.

(2) For $Q \in \mathcal{Q}$, $\gamma(Q)$ is defined to be the (unique) prime z-ideal such that

$$Z(\gamma(Q)) = \{\mathrm{cl}_X(Y) \mid Y \in Z(Q)\}.$$

(3) The inverse map γ^{-1} of γ is determined by the condition:

$$Z(\gamma^{-1}(P)) = \{Z - \{p\} \mid Z \in Z(P)\}, \quad \text{for } P \in \mathcal{P}.$$

(4) For $Q \in \mathcal{Q}$,
Q is maximal $\Leftrightarrow \gamma(Q)$ is an immediate prime z-ideal predecessor of M^p.

For the proof of these properties and some examples of the effect of the map γ, see [6; § 2].

We now turn to the proof of:

THEOREM 2. *Let p be a nonisolated G_δ-point of X, and $Q \in \mathcal{Q}$. The following are equivalent:*

(1) *The ideal $\gamma(Q)$ is real closed.*

(2) (a) *Q is real closed, and*

(b) *for every $f \in C(X - \{p\})$, $0 \leqslant f \leqslant 1$, there are $Y \in Z(Q)$ and $h \in C(X)$, $0 \leqslant h \leqslant 1$, such that $h \restriction Y = f \restriction Y$.*

Remark. We give two rather different proofs. They illustrate dual aspects — set-theoretical and algebraic — of the same subject.

First proof. The following two facts [6; p. 451] will be needed in the proof; they can be easily verified by the reader:

(i) If F is a closed subset of X, then $\gamma(F - \{p\})$ is either $F - \{p\}$ or F.

(ii) If F is a closed subset of $X - \{p\}$, then $\gamma(F) - \{p\} = F$.

From (i) and (ii) it follows:

(iii) If $F \in Z(X)$, then $F \in Z(\gamma(Q)) \Leftrightarrow F - \{p\} \in Z(Q)$.

Since $\{p\}$ is a zero-set iff it is G_δ (cf. [3; 3.11(b)]), the assumption gives:

(iv) $\{p\} \in Z(M^p)$ and every member of $Z(M^p)$ contains p.

(1) $\Rightarrow$ (2). Assume $\gamma(Q)$ is real closed. By Theorem 2.1.1 the following holds:

(*) for every $Z \in Z(M^p) - Z(\gamma(Q))$ and every $l \in C(X - Z)$, $0 \leqslant l \leqslant 1$, there are $W \in Z(\gamma(Q))$ and $h \in C(X)$, $0 \leqslant h \leqslant 1$, such that $h \restriction (W - Z) = l \restriction (W - Z)$.

Let q be the unique point of $\beta(X - \{p\})$ such that $Q \subseteq M^q$; necessarily $q \in \phi^{-1}(p)$.

Proof of (2.a). By Theorem 2.1.1 again, Q is real closed iff:

(**) for every $Y \in Z(M^q) - Z(Q)$ and every $m \in C((X - \{p\}) - Y)$, $0 \leqslant m \leqslant 1$, there are $V \in Z(Q)$ and $g \in C(X - \{p\})$, $0 \leqslant g \leqslant 1$, such that $g \restriction (V - Y) = m \restriction (V - Y)$.

To prove (**), let such Y, m be given. From 1.(1) and (iii) we get at once:

(v) $\gamma(Y) \in Z(M^p) - Z(\gamma(Q))$;

hence by (iv),

(vi) $p \in \gamma(Y)$.

Now, setting $Z = \gamma(Y)$ and $l = m$ in (*), get $W \in Z(\gamma(Q))$ and $h \in C(X)$, $0 \leqslant h \leqslant 1$, such that $h \upharpoonright (W-Z) = m \upharpoonright (W-Z)$. Further, we set $V = W - \{p\}$ and prove:

(vii) $V - Y = W - Z$.

Clearly $V - Y = (W - \{p\}) - Y = W - (Y \cup \{p\})$, and by (ii), $\gamma(Y) - \{p\} = Y$; hence $Y \cup \{p\} = \gamma(Y) = Z$.

Put $g = h \upharpoonright (X - \{p\})$, and (**) follows at once from (iii) and (vii).

Proof of (2.b). By (iv) and $Z(\gamma(Q)) \subsetneqq Z(M^p)$ we get $\{p\} \in Z(M^p) - Z(\gamma(Q))$.

Setting $Z = \{p\}$ and $l = f$ in (*) gives $W \in Z(\gamma(Q))$ and $h \in C(X)$, $0 \leqslant h \leqslant 1$, such that $h \upharpoonright (W - \{p\}) = f \upharpoonright (W - \{p\})$. Putting $Y = W - \{p\}$, (2.b) follows at once from (iii).

(2) $\Rightarrow$ (1). We assume (**) and (2.b) and prove (*).

By the remark following Theorem 2.1.1, (**) is equivalent to:

(***) for every $Y \in Z(X - \{p\}) - Z(Q)$ and every $m \in C((X - \{p\}) - Y)$, $0 \leqslant m \leqslant 1$, there are $V \in Z(Q)$ and $g \in C(X - \{p\})$, $0 \leqslant g \leqslant 1$, such that $g \upharpoonright (V - Y) = m \upharpoonright (V - Y)$.

Let $Z \in Z(M^p) - Z(\gamma(Q))$ and $l \in C(X - Z)$, $0 \leqslant l \leqslant 1$, be given. Set $Y_0 = Z - \{p\}$. Then $Y_0 \in Z(X - \{p\}) - Z(Q)$ (by (iii)), and l is defined on $(X - \{p\}) - Y_0$ $(= X - Z)$. Using (***) with $Y = Y_0$ and $m = l$ we get $V \in Z(Q)$ and $g \in C(X - \{p\})$, $0 \leqslant g \leqslant 1$, such that

$$(+) \qquad g \upharpoonright (V - Y_0) = l \upharpoonright (V - Y_0).$$

Setting $W_0 = \gamma(V)$, we have $W_0 \in Z(\gamma(Q))$, $V = W_0 - \{p\}$ (by (ii)), and

$$(++) \qquad V - Y_0 = (W_0 - \{p\}) - Y_0 = W_0 - Z.$$

Now, using the assumption (2.b) with $f = g$, get $Y_1 \in Z(Q)$ and $h \in C(X)$, $0 \leqslant h \leqslant 1$, such that $g \upharpoonright Y_1 = h \upharpoonright Y_1$. If $W_1 = \gamma(Y_1)$, then as above we conclude that $W_1 \in Z(\gamma(Q))$ and $Y_1 = W_1 - \{p\}$. Hence:

$$(+++) \qquad g \upharpoonright (W_1 - \{p\}) = h \upharpoonright (W_1 - \{p\}).$$

Let $W = W_0 \cap W_1$; then $W \in Z(\gamma(Q))$. Since $p \in Z$, we obtain:

$$h \upharpoonright (W - Z) = g \upharpoonright (W - Z) \quad (\text{by } (+++)),$$

and

$$g \upharpoonright (W - Z) = l \upharpoonright (W - Z) \quad (\text{by } (+) \text{ and } (++)).$$

Thus (*) is verified. ■

Second proof. Set $R = C(X)/\gamma(Q)$, $R' = C(X - \{p\})/Q$. The restriction map $\varrho\colon C(X) \to C(X - \{p\})$ induces a monomorphism $\varrho^*\colon R \to R'$, because $\varrho[\gamma(Q)] \subseteq Q$ and $\varrho^{-1}[Q] \subseteq \gamma(Q)$ as is readily verified.

Condition (2.b) is equivalent to:

(2.b′) If $0 < a \leqslant 1$ in R', then $a \in \operatorname{Range}(\varrho^*)$.

Now, since R, R' both satisfy the condition:

$$(*) \qquad \text{If } 1 \leqslant b, \text{ then } b \text{ is invertible (cf. 1.18)},$$

it follows that the divisibility property (DP) for R, R' is equivalent to:

$$(**) \qquad \text{If } 0 < a < b < 1, \text{ then } b \text{ divides } a.$$

Hence, if (2.b′) holds, then the divisibility property for R is equivalent to the divisibility property for R', or in other words:

$$(2.\text{b}') \Rightarrow ((1) \Leftrightarrow (2.\text{a})).$$

Hence to prove the theorem it suffices to show:

(1) $\Rightarrow$ (2.b′). Fix $0 < a \leqslant 1$ in R'. Since $\gamma(Q) \neq M_p$, we can choose $f \in M_p - \gamma(Q)$. Let $f' = \varrho^*(f/\gamma(Q))$. We can find a function $g \in CX()$ satisfying:

(i) $0 \leqslant g \leqslant f$;

(ii) $g' = af'$, where $g' = \varrho^*(g/\gamma(Q))$.

Indeed, let $\alpha \in C(X - \{p\})$ be so that $0 \leqslant \alpha \leqslant 1$ and $\alpha/Q = a$, and set

$$g = \begin{cases} \alpha f & \text{on } X - \{p\} \\ 0 & \text{at } p. \end{cases}$$

Now apply the divisibility property in $R = C(X)/\gamma(Q)$ to conclude that there is a function $h \in C(X)$ satisfying:

$$f/\gamma(Q) \cdot h/\gamma(Q) = g/\gamma(Q).$$

Set $h' = \varrho^*(h/\gamma(Q))$ and compute:

$$f'h' = g' = af';$$

since $f' \neq 0$ we obtain $a = h'$, as desired. ■

Applying Theorem 2 to the case where Q is maximal, gives:

COROLLARY 3. *Let X and p be as in Theorem 2, and let M^q be a maximal ideal of $C(X - \{p\})$ such that $q \in \phi^{-1}(p)$. Then the following are equivalent:*

(1) *$\gamma(M^q)$ is real closed;*

(2) *for every $f \in C(X - \{p\})$, $0 \leqslant f \leqslant 1$, there are $Y \in Z(M^q)$ and $h \in C(X)$, $0 \leqslant h \leqslant 1$, such that $f \upharpoonright Y = h \upharpoonright Y$.*

Condition (2) implies:

(3) *every bounded real-valued function on $X - \{p\}$ has a limit at p over a set of $Z(M^q)$; i.e., $Z(M^q)$ is a $P(p)$-filter.*

If X is normal or p has a compact neighborhood, then (2) and (3) are equivalent.

Proof. Since a maximal ideal is real closed, the equivalence between (1) and (2) follows at once from Theorem 2. Likewise, (2) ⇒ (3) is obvious since $\lim\limits_{\substack{x\to p\\ x\in Y}} f(x) = h(p)$.

(3) ⇒ (2). Let $f'\colon Y\cup\{p\}\to \boldsymbol{R}$ be defined by $f'\restriction Y = f\restriction Y$ and $f'(p) = \lim\limits_{\substack{x\to p\\ x\in Y}} f(x)$. Obviously $f'\in C(Y\cup\{p\})$ and we can assume, without loss of generality, that $0\leqslant f'\leqslant 1$. Also $Y\cup\{p\}$ is a closed subset of X. If X is a normal space, $Y\cup\{p\}$ is C^*-embedded in X ([3; 3.D]), i.e. there is $h\in C^*(X)$ such that $h\restriction(Y\cup\{p\}) = f'$, whence $h\restriction Y = f\restriction Y$.

If p has a compact neighborhood, V, then by [3; 3.2(b)] we can choose V to be, in addition, a zero-set neighborhood of p. By [3; 7.15], $V\in Z(\gamma(M^p))$, so that $V-\{p\}\in Z(M^q)$ and $V\cap Y\in Z(M^q)$. The set $(V\cap Y)\cup\{p\}$ is compact too, and therefore C-embedded ([3; 3.11(c)]). Defining as before the map f' on this set, we have $f'\in C^*((V\cap Y)\cup\{p\})$ and the proof proceeds as in the preceding case. ∎

In the next section we will meet situations where $X-\{p\}$ is C^*-embedded in X; under this assumption Theorem 2 gives:

COROLLARY 4. *Let X, p and Q be as in Theorem 2, and assume that $X-\{p\}$ is C^*-embedded in X. Then $\gamma(Q)$ is real closed iff Q is real closed. In particular, if $q\in\phi^{-1}(p)$, then $\gamma(M^q)$ is an immediate prime z-ideal predecessor of M^p which is real closed.*

Proof. Immediate from Theorem 2 and 1.(4) above. ∎

2.4. βF-points. The notion of a βF-point of βX was introduced by Kohls in [4]. The simplest way of defining this notion is the following:

DEFINITION 1. A point $p\in\beta X$ is a *βF-point* iff the ideal O^p is prime.

In this section we study the behaviour of prime ideals contained in M^p, when p is a (nonisolated) βF-point. The key to this study is Lemma 2. Using a refinement of a technique already employed by Kohls (cf. [4, p. 46] and [5, pp. 530–531]), we prove that if the space X is normal or the point p has compact X-neighborhood, then the immediate prime z-ideal predecessor of M^p (whenever it exists) is real closed.

As an application of the results of § 2.3 we consider next the case where p is, in addition, a G_δ-point belonging to X; no extra assumptions on the space X are made here. We reprove a result of Kohl's ([4, Theorem 5.7] and [6, p. 453]), which says that in this case M_p does have an immediate prime z-ideal predecessor which is real closed, and give an explicit characterization of it.

Recall that each of the following conditions on a point $p\in\beta X$ is equivalent to it being a βF-point:

(1) the set of prime ideals below M^p is totally ordered by inclusion (1.12).

(2) for every $f, g\in C(X)$ such that $f^*(p) = g^*(p) = 0$, there is an X-neighborhood U of p such that $Z(f)\cap U\subseteq Z(g)\cap U$ or $Z(g)\cap U\subseteq Z(f)\cap U$ ([5; Theorem 2.20 (6)], [4; Theorem 5.3]);

(3) for every $f, g\in M^p$, either $g\in\langle O^p, f\rangle$ or $f\in\langle O^p, g\rangle$ (where $\langle A\rangle$ denotes the *z-ideal* generated by $A\subseteq C(X)$); [5; Theorem 2.20 (6)];

(4) for every $f\in C(X)$ such that $f^*(p) = 0$, there is an X-neighborhood of p on which one of the relations $f\geqslant 0$ or $f\leqslant 0$ holds; [4; Theorem 5.2].

We begin by proving the following fundamental:

LEMMA 2. *Let $p\in\beta X$ be a βF-point. Then for every zero-set Z of X, $X-Z$ is C^*-embedded in the space $(X-Z)\cup\{p\}$ endowed with the topology induced by βX.*

Remark. The open subsets of the space $(X-Z)\cup\{p\}$ contained in $X-Z$ those induced by X; the open subsets containing p are of the form $(U\cap(X-Z))\cup\{p\}$, where U is an open X-neighborhood of p.

Proof. Let $Z = Z(g)$, $g\in C(X)$, and $l\in C^*(X-Z)$. We can assume without loss of generality that $0\leqslant l\leqslant 1$ and $g\geqslant 0$. Set

$$f(x) = \begin{cases} 0 & \text{for } x\in Z\\ g(x)l(x) & \text{for } x\in X-Z.\end{cases}$$

Clearly $0\leqslant f\leqslant g$ and $f\in C(X)$ (since l is bounded).

For $r\in\boldsymbol{R}^+$, let $\mu_r = f-rg$. Then $\mu_r\in C(X)$ and

(a) $r\leqslant s\Rightarrow \mu_s\leqslant\mu_r$.

(b) $\mu_0 = f\geqslant 0$ on X.

(c) $\mu_1\leqslant 0$ on X.

Set

$$r_0 = \operatorname{Sup}\{r\in[0,1] \mid \mu_r/O^p\geqslant 0\}$$

and

$$l'(p) = r_0, \quad l'\restriction(X-Z) = l.$$

We show that l' is continuous at p. This amounts to the following: for every $\varepsilon > 0$ there is an X-neighborhood U of p such that $|l(x)-r_0|\leqslant\varepsilon$ for $x\in U-Z$.

Since $r_0-\varepsilon < r_0$, there is r so that $r_0-\varepsilon < r < r_0$ and $\mu_r/O^p\geqslant 0$, i.e., $\mu_r\geqslant 0$ on some X-neighborhood U_1 of p [3; 5.4(a) and 7.12(a)]; for $x\in U_1$ we have:

$$0\leqslant\mu_r(x) = f(x)-rg(x)\leqslant f(x)-(r_0-\varepsilon)g(x);$$

dividing by $g(x)>0$ this gives:

$$(*)\qquad -\varepsilon\leqslant l(x)-r_0 \quad \text{for } x\in U_1-Z.$$

In order to prove the other inequality we consider two cases:

CASE 1. $r_0 = 1$. Then $\mu_1/O^p = 0$ and $\mu_1\in O^p$. Hence there is an X-neighborhood U_2 of p such that $\mu_1 = 0$ on U_2, i.e., $f(x) = g(x)$ for $x\in U_2$. It follows that

$$(**)\qquad l(x)-r_0 = 0 < \varepsilon \quad \text{for} \quad x\in U_2-Z.$$

CASE 2. $r_0 < 1$. This implies that $\mu_{r_0+\varepsilon}$ cannot be $\geqslant 0$ on an X-neighborhood of p. Then, property (4) above implies the existence of an X-neighborhood U_2

of p such that $\mu_{r_0+\varepsilon} \leqslant 0$ on U_2. As before we conclude that

$$(**) \qquad l(x)-r_0 \leqslant \varepsilon \quad \text{for } x \in U_2 - Z.$$

By (*) and (**), $|l(x)-r_0| \leqslant \varepsilon$ for $x \in (U_1 \cap U_2)-Z$. ■

LEMMA 3. *Let $Q \subsetneqq P$ be z-ideals of $C(X)$. Then the following are equivalent:*

(1) *Q is an immediate z-ideal predecessor of P;*

(2) *for every $f, g \in P-Q$ there is $h \in Q$ such that*

$$Z(f) \cap Z(h) = Z(g) \cap Z(h).$$

Proof. Routine, using that the z-ideal of $C(X)$ generated by a z-ideal I and a function f is given by: $\langle I, f\rangle = \{g \in C(x) |$ there is $h \in I$ so that $Z(f) \cap Z(h) \subseteq Z(g) \cap Z(h)\}$. ■

Our main result is the following:

THEOREM 4. *Let $p \in \beta X$ be a βF-point. Assume that either,*

(a) *the space X is normal and M^p is a real ideal, or*

(b) *p has a compact X-neighborhood.*

If $P \subset M^p$ is an immediate prime z-ideal predecessor of M^p, then P is real closed.

Proof. We want to prove that in the case (a), condition (2) of Theorem 2.1.1. is satisfied; in the case (b) we will verify the corresponding local condition (2) of Proposition 2.1.5.

We shall first prove:

CLAIM 1. *Assumption* (b) *implies that the ideal M^p is real.*

Proof of claim 1. By [3; Theorem 5.14] it suffices to show that the intersection of countably many members of $Z(M^p)$ is nonempty.

Let $\{f_n\}_{n\geqslant 1}$ be a countable set of functions in M^p. Considering, if necessary, $\inf\{f_n^2, 1\}$ instead of f_n we can assume that $0 \leqslant f_n \leqslant 1$ without altering the corresponding zero-sets. Furthermore, replacing f_n by $\sum_{i=1}^{n} f_i (\in M^p)$ we can assume that the zero-sets $Z(f_n)$ are decreasing (that is, $Z(f_m) \subseteq Z(f_n)$ for $m \geqslant n$) without changing the set $\bigcap_{n\geqslant 1} Z(f_n)$.

Now fix a compact X-neighborhood C of p; by 1.37 there is $W \in Z(X)-Z(M^p)$ such that $X-W \subseteq C$.

In this situation, the proof will be finished if we show that $Z(f_n) \cap C \neq \emptyset$ for all $n \geqslant 1$, for the compactness of C implies $\bigcap_{n\geqslant 1} Z(f_n) \cap C \neq \emptyset$, and hence $\bigcap_{n\geqslant 1} Z(f_n) \neq \emptyset$.

Thus, we have to show:

$$f \in M^p \Rightarrow Z(f) \cap C \neq \emptyset.$$

But notice that, in fact, we have:

$$f \in M^p \Rightarrow Z(f) \cap (X-W) \neq \emptyset.$$

Otherwise, we would have $Z(f) \subseteq W$ which, together with $Z(f) \in Z(M^p)$ and $W \in Z(X)$ imply at once $W \in Z(M^p)$, a contradiction. ■

Let $Z \in Z(M^p)-Z(P)$ and $l \in C^*(X-Z)$; for the case (b) fix a compact X-neighborhood C of p and assume (using [3; 3.2(b)] if necessary) that $Z \subseteq C$.

Henceforth we assume that M^p is real. We shall establish the next result under this assumption but no other assumption on the space X or on the point p:

CLAIM 2. *There are $W \in Z(P)$ and $l_0 \in C^*(W)$ such that $l_0 \restriction (W-Z) = l \restriction (W-Z)$.*

Proof of claim 2. Using Lemma 2 extend l to a continuous function l' on $(X-Z) \cup \{p\}$; then extend l' further to a (not necessarily continuous) function l'' on $X \cup \{p\}$ by setting $l''(x) = l'(p)$ for $x \in Z$.

Recall that the oscillation of a real-valued function F at a point p is defined as:

$$\operatorname{osc}(F, p) = \limsup_{x\to p} F(x) - \liminf_{x\to p} F(x).$$

Since l'' is continuous at p, for every $n \geqslant 1$ we can find an x-neighborhood U_n of p such that $\operatorname{osc}(l'', x) < 1/n$ for every $x \in U_n$.

By Lemma 2.1.4 we can find functions $k_n \in C(X)$, $0 \leqslant k_n \leqslant 1$, such that $Z(k_n) \subseteq U_n$ and $k_n^\beta(p) = 0$ for all $n \geqslant 1$. Since M^p is real, by [3; 7.2 and 7.9(c)] we conclude that $k_n \in M^p$; furthermore, by [3; Theorem 5.14], there is a function $s \in M^p$ such that:

$$(*) \qquad \bigcap_n U_n \supset \bigcap_n Z(k_n) = Z(s).$$

Now we show:

$$(**) \qquad \text{there is } k \in P \text{ such that } \bigcap_n U_n \supseteq Z \cap Z(k).$$

Indeed, if $s \in P$, then (**) follows from (*) by taking $k = s$. If $s \notin P$, let $g \in M^p - P$ be such that $Z(g) = Z$; since P is an immediate prime z-ideal predecessor of M^p, by Lemma 3 there is $k \in P$ so that $Z(s) \cap Z(k) = Z(g) \cap Z(k)$; (**) follows at once from this and (*).

Condition (**) says that at each point of $Z \cap Z(k)$ the oscillation of l'' is zero. Since l'' $(= l)$ is continuous on $X-Z$, at each point of $Z(k)-Z$ its oscillation is also zero. Therefore l'' is continuous on $Z(k)$.

The claim follows by setting $W = Z(k)$ and $l_0 = l'' \restriction W$. ■

Returning to the proof of Theorem 4, when X is normal, W is C^*-embedded ([3; 3.D]). Under assumption (b) notice that we can choose $W \subseteq C$; hence W is compact and by [3; 3.11(c)] W is C^*-embedded.

Hence in both cases there is $h \in C^*(X)$ such that $h \restriction W = l_0$. Clearly h and l coincide on $W-Z$. Therefore the ideal P is real closed. ■

The proof of Theorem 4 also yields:

COROLLARY 5 (Kohls). *Let $p \in \beta X$ be a βF-point such that M^p is a real ideal. If 0^p is an immediate prime z-ideal predecessor of M^p, then 0^p is real closed.*

Proof. Apply Claim 2 in the proof of Theorem 4 with $P = 0^p$; then W contains an open X-neighborhood U of p. By complete regularity there is $f \in C(X)$ such that

$$0 \leqslant f \leqslant 2, \quad f \restriction (X-U) = 0 \quad \text{and} \quad f^{\beta}(p) = 2.$$

Let $f' = \min\{f, 1\}$. Then $0 \leqslant f' \leqslant 1$, $f' = 0$ on $X-U$, and $f' = 1$ on some X-neighborhood of p. Setting $W' = Z(f'-1)$ we thus have $W' \in Z(0^p)$. Let

$$h(x) = \begin{cases} 0 & \text{for } x \in X-U \\ f'(x)l_0(x) & \text{for } x \in U. \end{cases}$$

Since l_0 is bounded and $f' = 0$ on $X-U$, it follows that $h \in C(X)$; also $h(x) = l_0(x)$ for $x \in W'$. We conclude then:

$$h \restriction ((W \cap W')-Z) = l_0 \restriction (W \cap W')-Z) = l \restriction ((W \cap W')-Z).$$

Since $W \cap W' \in Z(0^p)$, this proves that 0^p satisfies condition (2) of Theorem 2.1.1, and therefore that it is real closed. ■

Remarks. (1) Theorem 4 is inspired by Kohl's proof of Corollary 5 (see [5; Theorem 4.1]) and uses some of the same technique. Kohl's original statement of Corollary 5 makes no explicit assumption about M^p, but the fact that it is real is actually used in the proof. Corollary 8 below is an application of Theorem 4.

(2) Note that the assumption "M^p is real" is automatically verified when $p \in X$.

(3) The space $\Sigma_{\mathscr{U}}$ considered in § 3.1 below is an example where the assumptions of Corollary 5 are verified (for $p = \infty$); cf. [3; 4.M.9]. More generally, assume that the point $p \in \beta X$ is such that M^p is real, and in addition satisfies:

for every $f \in C(X)$ such that $f^*(p) = 0$, there is a (deleted) X-neighborhood of p where one of the relations $f > 0$, $f < 0$ or $f = 0$ holds.

Then 0^p is an immediate prime z-ideal predecessor of M^p (this point is checked using Lemma 3 above and Lemma 2.1.4; we leave it as an exercise). This condition is fulfilled, for example, when p is a βF-point and a P-point of βX. For more details, see [4; § 5] and [5; § 4].

We shall now impose further restrictions on the βF-point p (but none on the space X); namely, we shall require that $p \in X$ and that p be a nonisolated G_δ-point. As remarked in [5; p. 529], the class of points p satisfying all these requirements is rather restricted, but by no means empty.

Note that $\{p\}$ is G_δ iff it is a zero-set (cf. [3, 3.11(b)]). Then from Lemma 2 we obtain:

COROLLARY 6. *Let $p \in X$ be a G_δ- and βF-point. Then $X-\{p\}$ is C^*-embedded in X.*

Remark. Using Urysohn's theorem, [3; 1.17], one can easily give a proof of this result which does not use Lemma 2.

From Corollary 6 we infer the following:

COROLLARY 7. *Let $p \in X$ be a nonisolated, G_δ-, βF-point, and let ϕ be the map defined in 2.3.1. Then $\phi^{-1}(p)$ has cardinality 1.*

Proof. If $q_1, q_2 \in \phi^{-1}(p)$ and $q_1 \neq q_2$, then by 2.3.1(1) we would have $\gamma(M^{q_1}) \neq (M^{q_2})$; by 2.3.1(4) both would be immediate prime z-ideal predecessors of M_p, and hence incomparable under inclusion, which contradicts (1) above.

COROLLARY 8. *Let $p \in X$ be a nonisolated, G_δ- and βF-point. Then M_p has an immediate prime z-ideal predecessor, M'_p, which is real closed.*

Proof. Let $q \in \beta X$ be the unique member of $\phi^{-1}(p)$, and set $M'_p = \gamma(M^q)$. The result follows at once from 2.3.4. ■

Now we give a more precise characterization of the prime z-ideal M'_p.

PROPOSITION 9. *Let $p \in X$ be a nonisolated, G_δ- and βF-point. Then*

$$M'_p = \{f \in C(X) \mid p \text{ is a limit point of } Z(f)\}.$$

Proof. Call I the right hand side of the equality.

(a) I is a prime z-ideal.

The only nontrivial point to check is that I is closed under addition. Let $f, g \in I$. Since $I \subseteq M_p$, by condition (2), p. 027, there is an X-neighborhood of p (and hence a neighborhood of p, since $p \in X$), U, such that $Z(f) \cap U \subseteq Z(g) \cap U$ or $Z(g) \cap U \subseteq Z(f) \cap U$. Assume, for instance, that the first inclusion holds. Let V be an arbitrary neighborhood of p, and $V' = V-\{p\}$. Since $f \in I$, then $Z(f) \cap U \cap V' \neq \emptyset$; whence $\emptyset \neq Z(f) \cap Z(g) \cap U \cap V' \subseteq Z(f-g) \cap V'$. This proves that $f-g \in I$.

Note that the assumption that $\{p\}$ is G_δ is not used in (a).

(b) $I \subseteq M'_p$.

Since $\{p\}$ is G_δ, it is a zero-set; clearly $\{p\} \notin Z(I)$. Hence $\{p\} \in Z(M_p)-Z(I)$, and thus $I \subsetneq M_p$.

Since M'_p is the (unique) immediate prime z-ideal predecessor of M_p, then $I \subseteq M'_p$.

(c) $M'_p \subseteq I$.

Let $f \in M'_p$ and let V be a neighborhood of p. Using Lemma 2.1.4 choose $g \in 0_p$ such that $Z(g) \subseteq V$. Since I is a prime ideal, by considering f^2 instead of f we can assume $f, g \geqslant 0$ without loss of generality. Now we have $f+g \in M'_p$ (since $0_p \subseteq M'_p$), and therefore $Z(f+g) = Z(f) \cap Z(g) \in Z(M'_p)$. Since $M'_p \subsetneq M_p$, then $\{p\} \notin Z(M'_p)$, and we conclude:

$$\emptyset \neq (Z(f) \cap Z(g))-\{p\} = Z(f) \cap (Z(g)-\{p\}) \subseteq Z(f) \cap (V-\{p\}),$$

which shows that $f \in I$. ■

J. Chaize (unpublished) has considerably extended Corollary 8 and Proposition 9.

§ 3. Concrete cases

In this section we will apply the general results proved in § 2 to give precise characterizations of the real closed members of certain families of prime ideals of continuous functions in some specific spaces.

In § 3.1 we study the case of N^*, the one-point compactification of the countable discrete space N; we obtain an elegant algebraic characterization of P-points of $\beta N-N$ in terms of residue rings of $C(N^*)$: $\mathcal{U}\in\beta N-N$ is a P-point iff $C(N^*)/P_{\mathcal{U}}$ is a real closed ring (iff $C(N^*)/P_{\mathcal{U}}$ is a valuation ring, by Proposition 1.27). Here $P_{\mathcal{U}}$ is the minimal prime ideal of $C(N^*)$ canonically determined by $\mathcal{U}$ (see below).

This result makes it possible to identify the real closed members of a certain family of prime ideals in $C(R)$ — called ideals "of type E" — introduced by Kohls in [6]. This is the subject of § 3.2.

In § 3.3 we carry out a detailed analysis of prime ideals in $C(D^*)$ contained in M_∞; D^* denotes the one-point compactification of an uncountable discrete space D and ∞ its "point at infinity".

We show that the class of these ideals is similar in some important respects to the corresponding one in the ring $C(N^*)$. We obtain a complete classification of the minimal prime ideals, and a characterization of the P-points of $\beta D-D$ analogous to that of the countable case.

In § 3.4 we introduce a class of prime ideals in $C((R^+)^*)$ ($\cong C([0,1])$) naturally defined in terms of the map γ (cf. § 2.3) and the Lebesgue measure on R^+. We characterize exactly the real closed members of this class, and using Martin's axiom we prove that they form a nonempty subclass.

3.1. P-points of $\beta N-N$ and real closed ideals in $C(N^*)$. We shall denote by N^* the one-point compactification of the countable discrete space N and by ∞ its "point at infinity". To each nonprincipal ultrafilter $\mathcal{U}$ on N we associate the set

$$P_{\mathcal{U}}=\{f\in C(N^*)|\ Z(f)-\{\infty\}\in\mathcal{U}\}.$$

It is easily seen [3; 14.G] that $P_{\mathcal{U}}$ is a minimal prime ideal contained in M_∞ and that the correspondence $\mathcal{U}\mapsto P_{\mathcal{U}}$ establishes bijection between $\beta N-N$ and the minimal prime ideals contained in M_∞.

Recall that $\Sigma_{\mathcal{U}}$ denotes the set $N\cup\{\infty\}$ with the topology for which the points of N are isolated and the neighborhoods of ∞ are the sets $U\cup\{\infty\}$, $U\in\mathcal{U}$. $\Sigma_{\mathcal{U}}$ is an F-space; the ideal 0_∞ is prime but not maximal; for more details, cf. [3; 4.M].

We shall prove something more precise than the equivalence announced above.

THEOREM 1. *The following are equivalent*:

(1) *$\mathcal{U}$ is a P-point of $\beta N-N$.*

(2) $C(\Sigma_{\mathcal{U}})/0_\infty\simeq C(N^*)/P_{\mathcal{U}}$.

(3) *$C(N^*)/P_{\mathcal{U}}$ is a real closed ring.*

Proof. (2) ⇒ (3) follows immediately from Proposition 2.1.2 ($\Sigma_{\mathcal{U}}$ is an F-space).

(1) ⇒ (2). We show first that (1) implies:

(*) for every $f\in C(\Sigma_{\mathcal{U}})$ there is $g\in C(N^*)$ such that $f/0_\infty=g/0_\infty$.

Let $r=f(\infty)$. Since $f\in C(\Sigma_{\mathcal{U}})$, for every $n\geqslant 1$ we have $U_n=f^{-1}[(r-1/n,r+1/n)]\cap N\in\mathcal{U}$. By 1.30 there is $U\in\mathcal{U}$ such that $U-U_n$ is finite for all $n\geqslant 1$. Let $g\colon N\cup\{\infty\}\to R$ be defined by

$$g\restriction U=f\restriction U,$$
$$g\restriction(N^*-U)=r.$$

To see that $g\in C(N^*)$ it suffices to prove that $W_n=g^{-1}[(r-1/n,r+1/n)]$ is cofinite for all $n\geqslant 1$. Indeed, it is easily verified that $N^*-W_n\subseteq U-U_n$, a finite set.

Since $f=g$ on $U\cup\{\infty\}$ and $U\in\mathcal{U}$, it follows that $Z(f-g)\in Z(0_\infty)$ and $f/0_\infty=g/0_\infty$. By (*) the canonical map $f\mapsto f/0_\infty$ from $C(N^*)$ into $C(\Sigma_{\mathcal{U}})/0_\infty$ is surjective, and it is easily checked that it induces the desired isomorphism.

(3) ⇒ (1). We use here the characterization of P-points given in 1.31. Since $P_{\mathcal{U}}$ is a prime z-ideal, Theorem 2.1.1 gives:

(**) for every $Z\in Z(M_\infty)-Z(P_{\mathcal{U}})$ and every $l\in C(N^*-Z)$, $0\leqslant l\leqslant 1$, there are $W\in Z(P_{\mathcal{U}})$ and $h\in C(N^*)$, $0\leqslant h\leqslant 1$, such that $h\restriction(W-Z)=l\restriction(W-Z)$.

Consider any such Z and a partition $\{P_k|\ k\geqslant 1\}$ of N such that $P_k\notin\mathcal{U}$ for $k\geqslant 1$; let $l_0\colon N\to R$ be defined by

$$l_0(n)=\frac{1}{k}\Leftrightarrow n\in P_k.$$

Since $\infty\in Z$, then $l_0\in C(N^*-Z)$, and obviously $0\leqslant l_0\leqslant 1$. Notice that $\lim\limits_{\mathcal{U}} l_0=0$ because $P_k\notin\mathcal{U}$ for all k. Let W_0, h be obtained by applying (**) to Z and l_0, and let $W=W_0-Z$. Then $h(\infty)=0$ and since h is continuous at ∞, $h^{-1}[(-1/k,1/k)]$ is cofinite. Hence

$$\begin{aligned}W\cap(N-h^{-1}[(-1/k,1/k)])&=W-\{n\in N|\ l_0(n)<1/k\}\\&=\{n\in W|\ l_0(n)\geqslant 1/k\}=W\cap\bigcup_{i=1}^{k}P_i,\end{aligned}$$

is finite. Thus, we have $W\in\mathcal{U}$ and $W\cap P_k$ finite for all k, showing that $\mathcal{U}$ is a P-point of $\beta N-N$. ■

In Kohls [6; 3.4] it is proved that in this case (i.e., when $X=N^*$, $p=\infty$) one has $\phi^{-1}(\infty)=\beta N-N$. Since N $(=X-\{p\})$ is discrete, the only prime z-filter of N converging to the point $\mathcal{U}\in\phi^{-1}(\infty)=\beta N-N$ is the nonprincipal ultrafilter $\mathcal{U}$ itself. It is easily verified that $\gamma(\{f\in C(N)|\ Z(f)\in\mathcal{U}\})=P_{\mathcal{U}}$; by (2.3.1(4) there are no prime z-ideals strictly between $P_{\mathcal{U}}$ and M_∞. The preceding theorem and the equivalence between (1) and (3) of Corollary 2.3.3 imply at once:

COROLLARY 2. *Let $\mathcal{U}\in\beta N-N$. Then $\mathcal{U}$ is a P-point of $\beta N-N$ iff for every bounded sequence f of real numbers there is $Y\in\mathcal{U}$ such that $\lim\limits_{n\to\infty}f\restriction Y$ exists.* ■

Conversely, it is quite easy to derive the equivalence (1) $\Leftrightarrow$ (3) of Theorem 1 from Corollary 2 and Corollary 2.3.3.

The next two statements summarize the situation concerning the distribution and existence of nonmaximal real closed ideals of $C(N^*)$ contained in M_∞.

PROPOSITION 3. *Let P be a prime ideal of $C(N^*)$, $P_{\mathcal{U}} \subseteq P \subseteq M_\infty$ for $\mathcal{U} \in \beta N - N$. Then P is real closed iff $\mathcal{U}$ is a P-point or $P = M_\infty$.*

Proof. ($\Leftarrow$). By Theorem 1 and Proposition 2.1.7.

($\Rightarrow$). By Theorem 2.2.1 and Corollary 2. ■

PROPOSITION 4 (1) *The existence of nonmaximal real closed ideals in $C(N^*)$ is not provable in* ZFC.

(2) *Martin's axiom implies the existence of (infinitely many) nonmaximal real closed z-ideals in $C(N^*)$.*

Proof. An immediate consequence of the above and the results mentioned in § 1.1. ■

3.2. Ideals of type E. In [6] Kohls introduced a special type of prime z-ideal on $C(X)$, for a wide class of spaces X including, among many others, the metric spaces.

DEFINITION 1. Let p be a fixed nonisolated point of a space X, S a sequence of distinct points of $X - \{p\}$ having p as its only limit point and $\mathcal{U}$ a non-principal ultrafilter on S. The ideal $I_{S,p,\mathcal{U}}$($I_{\mathcal{U}}$ or $I_{S,\mathcal{U}}$ for short), called the *ideal of type E associated with S, p and $\mathcal{U}$*, is defined by:

$$I_{\mathcal{U}} = \{f \in C(X) |\ p \in Z(f) \text{ and } Z(f) \cap S \in \mathcal{U}\}.$$

These ideals enjoy the following properties:

(1) $I_{\mathcal{U}}$ is a prime z-ideal contained in M_p, [6; 3.2], and if $\{p\}$ is G_δ, then $I_{\mathcal{U}}$ is properly contained in M_p.

(2) If p is a nonisolated G_δ-point of X, and either $X - \{p\}$ is normal or S converges to p, then any ideal of type E associated with S and p is the image by the map γ of a maximal ideal belonging to the family $\mathcal{Q}$ of 2.3.1(1), and therefore is maximal among prime z-ideals properly contained in M_p; [6; 3.5].

(3) Let S converge to the G_δ-point p and let

$$J = \{f \in C(X) |\ Z(f) \cap S \text{ is cofinite in } S\}.$$

If X is normal, or p has a compact neighborhood, or each point of S is G_δ, then the ideals of type E associated with S constitute the class of minimal prime ideals of $C(X)$ containing J; [6; 3.6].

Ideals of type E (and therefore also the quotient rings modulo such ideals) are obviously determined by the ultrafilter $\mathcal{U}$ and the behaviour of continuous functions on the subspace $S \cup \{p\}$. Whenever S converges to p this subspace is homeomorphic to N^*. Calling φ this homeomorphism (canonically defined by $\varphi(s_n) = n$ and $\varphi(p) = \infty$, where $S = \{s_n |\ n \in \omega\}$), we have:

PROPOSITION 2 *Let the sequence S converge to p, and let $\mathcal{U}$ be a nonprincipal ultrafilter on S. Then $C(X)/I_{\mathcal{U}} \simeq C(N^*)/P_{\varphi\mathcal{U}}$. In particular, under this assumptions $I_{\mathcal{U}}$ is real closed iff $\mathcal{U}$ is a P-point of $\beta S - S (\approx \beta N - N)$.*

Proof. Put $S' = S \cup \{p\}$ and

$$P'_{\mathcal{U}} = \{f \in C(S') |\ p \in Z(f) \text{ and } Z(f) \cap S \in \mathcal{U}\}.$$

Obviously, it suffices to show that $C(X)/I_{\mathcal{U}}$ is isomorphic to $C(S')/P'_{\mathcal{U}}$. The map $\psi(f/I_{\mathcal{U}}) = f \restriction S'/P'_{\mathcal{U}}$ is a well defined ring monomorphism between the preceding quotient rings. In order to verify that it is surjective, recall that S' is a compact — and therefore C-embedded — subspace of X ([3; 3.11(c)]). Then, given $f \in C(S')$ there is $g \in C(X)$ such that $g \restriction S' = f$; hence $\psi(g/I_{\mathcal{U}}) = f/P'_{\mathcal{U}}$. ■

3.3. The one-point compactification of uncountable discrete spaces. In this section D shall denote an uncountable discrete space of arbitrary cardinality, and D^* its one-point compactification; the "point at infinity" of D^* will be denoted ∞.

We carry out a study of the structure of prime ideals of $C(D^*)$ contained in M_∞. We shall see that there are significant analogies to the case of $C(N^*)$. For example, we obtain results similar to Theorem 3.1.1 and Proposition 3.1.3, and show that also in $C(D^*)$ nonmaximal prime z-ideals coincide with the minimal prime ideals. We classify these and show that there are $2^{(\varkappa^{\aleph_0})}$ of them ($\bar{\bar{D}} = \varkappa$).

The first four statements in the following list of preliminary results can be easily proved by the reader.

(1) $Z \in Z(D^*) \Leftrightarrow Z \subseteq D$ is finite, or $\infty \in Z$ and $D - Z$ is countable.

(2) Let $A \subseteq D^*$, $\infty \in A$; then

$$A \text{ is open} \Leftrightarrow A \text{ is cofinite},$$

$$A \text{ is } G_\delta \Leftrightarrow A \text{ is co-denumerable}.$$

(3) $f \in M_\infty \Leftrightarrow \infty \in Z(f)$ and $D - Z(f)$ is countable;

$f \in 0_\infty \Leftrightarrow \infty \in Z(f)$ and $D - Z(f)$ is finite.

(4) If P is a nonmaximal prime ideal of $C(D^*)$, then $P \subset M_\infty$ (and hence $0_\infty \subseteq P$).

(5) If P is a prime z-ideal of $C(D^*)$ such that $0_\infty \subseteq P \subseteq M_\infty$, then there is a nonprincipal ultrafilter $\mathcal{V}$ on D^* such that $Z(P) = Z(D^*) \cap \mathcal{V}$ ([3; 14.F]). An ultrafilter $\mathcal{V}$ with these properties will be called *associated with P*.

In the theorems below P will be a fixed nonmaximal prime ideal of $C(D^*)$ and b a fixed function in $M_\infty - P$. Let $Z = Z(b)$. By (3), $\infty \in Z$ and $D - Z$ is countably infinite. Let $\varphi: N \to D - Z$ be a bijection.

(6) We associate to each map $f \in C(D^*)$ a function $f_*: N^* \to R$ defined as follows:

$$f_*(n) = f(\varphi(n))$$

$$f_*(\infty) = f(\infty).$$

Since the topology induced by D^* on $(D-Z) \cup \{\infty\}$ makes it the one-point compactification of the countable discrete space, it follows that $f_* \in C(N^*)$.

(7) Conversely, given $g \in C(N^*)$, let $g^*\colon D^* \to \boldsymbol{R}$ be the map

$$g^*(d) = \begin{cases} g(\varphi^{-1}(d)) & \text{if } d \in D-Z \\ g(\infty) & \text{if } d \in Z. \end{cases}$$

Again, it is seen without difficulty that $g^* \in C(D^*)$ and $(g^*)_* = g$.

Now we define:

(8) $P_* = \{f_* \mid f \in P\}$.

Lemma 1. (1) *P_* is a prime ideal of $C(N^*)$ contained in M_∞.*

(2) *If P is a z-ideal, so is P_*.*

(3) $P_* = \{f \in C(N^*) \mid f^* \in P\}$.

(4) *$f \in C(D^*)$ and $f_* \in P_*$ imply $f \in P$.*

Proof. (1) This is obvious since the ring operations in $C(N^*)$ and $C(D^*)$ are defined pointwise, and P is prime.

(2) Let $h \in P_*$ and $g \in C(N^*)$ be such that $Z(h) = Z(g)$. Hence $h = f_*$ for some $f \in P$. One checks without difficulty the equality $Z(f) \cup Z = Z(g^*) \cup Z$, i.e., $Z(fb) = Z(g^*b)$. Since $fb \in P$ and P is a z-ideal, $g^*b \in P$, and then (since $b \notin P$), $g^* \in P$. Hence $(g^*)_* = g \in P_*$.

(3) If $f \in C(N^*)$ and $f^* \in P$, then $(f^*)_* = f \in P_*$. Conversely, if $f \in P_*$, then $f = g_*$ for some $g \in P$. Since $gb(d) = 0$ for $d \in Z$, this equality implies $f^*b = gb$; hence $f^*b \in P$, and since $b \notin P$, $f^* \in P$.

(4) We have $f_* = g_*$ for some $g \in P$. Then $gb \in P$ and $(fb)_* = (gb)_*$. Since $fb = gb = 0$ on Z, the preceding equality gives $fb = gb \in P$; hence $f \in P$. ■

Proposition 2. *Let P be a nonmaximal prime ideal in $C(D^*)$. Then $C(D^*)/P \simeq C(N^*)/P_*$.*

Proof. Let $\psi\colon C(D^*) \to C(N^*)/P_*$ be defined by

$$\psi(f) = f_*/P_*.$$

Clearly ψ is a ring homomorphism. Moreover, ψ is surjective: $\psi(g^*) = (g^*)_*/P_* = g/P_*$ for $g \in C(N^*)$. Finally, $\operatorname{Ker}(\psi) = P$. Let $f \in \operatorname{Ker}(\psi)$; then $f_* \in P_*$, and by Lemma 1.(4), $f \in P$. Thus, $\operatorname{Ker}(\psi) \subseteq P$, and the reverse inclusion is clear. ■

Remark. The construction of the prime ideal P_* *is not* independent of the choice of the function $b \in M_\infty - P$. Thus, the map $P \mapsto P_*$ becomes unambiguously defined only after a choice of maps $bp \in M_\infty - P$ for $P \subsetneq M_\infty$ has been fixed. In Proposition 3.(1) below we fix a nonmaximal prime ideal Q, a function $b \in M_\infty - Q$, and assume that the ideals P_* are uniformly constructed from the same map b, for all $P \subseteq Q$.

Proposition 3. (1) *Let Q be a fixed nonmaximal prime ideal. The map $P \mapsto P_*$ is an order monomorphism of the family of prime ideals contained in Q into that of prime ideals contained in Q_*.*

(2) (a) *Every nonmaximal prime z-ideal P of $C(D^*)$ is minimal, and $P_* = P_{\mathscr{U}}$ for some $\mathscr{U} \in \beta N - N$* (cf. § 3.1 above).

(b) *Conversely, for every $P_{\mathscr{U}}$ there is a minimal prime ideal $J \subseteq M_\infty$ of $C(D^*)$ such that $J_* = P_{\mathscr{U}}$.*

(3) *With the notation of* (2.a), *if $\mathscr{V}$ is an ultrafilter associated with P, then $\mathscr{U}$ and $\mathscr{V}$ are related as follows: $\varphi\mathscr{U} = \mathscr{V} \restriction (D-Z)$.*

Proof. (1) We prove that $P \subseteq P'$ iff $P_* \subseteq P'_*$ (where $P, P' \subseteq Q$). This shows that the map $P \mapsto P_*$ is injective and order preserving. Lemma 1.(4) proves at once the implication from right to left; the other implication follows from the definition (8) of P_*.

(2.a) follows from the corresponding fact about $C(N^*)$ by (1) and Lemma 1.(2)

(3) Since $Z(b) = Z \notin Z(P)$ and $Z(P) = Z(D^*) \cap \mathscr{V}$, we have $Z \notin \mathscr{V}$; also $\infty \in Z$ gives $D^* - Z = D - Z \in \mathscr{V}$. Hence $\mathscr{V} \restriction (D-Z)$ is a nonprincipal ultrafilter on $D-Z$.

Let now $X \in \mathscr{U}$, so that $\varphi(X) \in \varphi\mathscr{U}$ and $\varphi(X) \subseteq D-Z$. Setting $f(n) = 0$ for $n \in X \cup \{\infty\}$ and $f(n) = 1/n$ for $n \in N - X$, we obtain $f \in C(N^*)$ and $Z(f) - \{\infty\} = X$; hence $f \in P_{\mathscr{U}} = P_*$. By Lemma 1.(3), $f^* \in P$, and we have $Z(f^*) = Z(f \circ \varphi) \cup \cup Z \in Z(P)$. Since $Z \notin Z(P)$, we conclude that $Z(f \circ \varphi) = \varphi(X) \in Z(P)$; whence $\varphi(x) \in \mathscr{V} \restriction (D-Z)$. This proves the inclusion $\varphi\mathscr{U} \subseteq \mathscr{V} \restriction (D-Z)$; since both members are ultrafilters on $D-Z$, the equality follows.

(2.b) Let $A \subseteq D$, $\bar{A} = \aleph_0$, and let $\varphi\colon N \to A$ be a bijection. By (1), p. 37, $D^* - A \in Z(D^*)$, i.e., there is $a \in C(D^*)$ such that $Z(a) = D^* - A$. Clearly $a \in M_\infty$, and since A is infinite, $a \notin 0_\infty$. As 0_∞ is the intersection of all minimal prime ideals (cf. 1.12), there is one such ideal, J, such that $a \notin J$. Since $\mathscr{V} \cap Z(D^*) = Z(J)$ it follows that $A \notin \mathscr{V}$. By (3), $\varphi\mathscr{U} = \mathscr{V} \restriction A$.

Let now J_* be constructed as above from the ideal J and the function a. By Lemma 1.(4), $a_* \notin J_*$ and hence (Lemma 1.(1)) J_* is a nonmaximal prime z-ideal of $C(N^*)$; it follows that the inclusion $P_{\mathscr{U}} \subseteq J_*$ automatically implies the equality $P_{\mathscr{U}} = J_*$.

Let $f \in P_{\mathscr{U}}$, i.e., $\infty \in Z(f)$ and $Z(f) \cap N \in \mathscr{U}$. By (3), $\varphi(Z(f) \cap N) \in \mathscr{V} \restriction A$. Using (7) one easily checks that $Z(f^*) = \varphi(Z(f) \cap N) \cup D^* - A$; this obviously implies $Z(f^*) \in \mathscr{V}$, and then $Z(f^*) \in Z(J)$; since J is a z-ideal, $f^* \in J$; hence $f = (f^*)_* \in J_*$. ■

Proposition 3, (1) and (2), shows — in analogy to the case of $C(N^*)$ — that each nonmaximal prime ideal of $C(D^*)$ contains exactly one minimal prime ideal, and that distinct maximal chains of prime ideals included in M_∞ meet only at M_∞ (cf. 1.11). Our next result is the analog for $C(D^*)$ of Proposition 3.1.1.

Theorem 4. *Let P be a nonmaximal prime ideal of $C(D^*)$, J be the unique minimal prime ideal contained in P, and $\mathscr{V}$ be an ultrafilter on D^* associated with J. Then the following are equivalent:*

(1) *P is real closed;*

(2) *$\mathscr{V} \restriction D$ is a P-point of $\beta D - D$.*

Proof. By Proposition 3.(2.a), $J_* = P_{\mathscr{U}} \subseteq P_*$ for some $\mathscr{U} \in \beta N - N$. Since P_* is a nonmaximal prime ideal in $C(N^*)$, Proposition 2, Proposition 3.1.3, Proposition 3.(3) and 1.34 applied in order prove the following equivalences:

P is real closed $\Leftrightarrow$
$C(D^*)/P \models \mathrm{RCR}$ $\Leftrightarrow$
$C(N^*)/P_* \models \mathrm{RCR}$ $\Leftrightarrow$
P_* is real closed $\Leftrightarrow$
$\mathscr{U}$ is a P-point of $\beta N - N$ $\Leftrightarrow$
$\mathscr{V} \upharpoonright D$ is a P-point of $\beta D - D$. ■

Corollary 5. *The existence of nonmaximal real closed ideals in $C(D^*)$ is independent from* ZFC.

Proof. Immediate from the proof of Theorem 4 and the results mentioned in § 1.I. ■

6. Classifying minimal prime ideals in $C(D^*)$. The preceding results show some similarities between the order structure of the tree of prime ideals of $C(D^*)$ contained in M_∞, and that of the corresponding tree in $C(N^*)$.

There are some important differences too. For example, while the classifying object of minimal prime ideals of $C(N^*)$ contained in M_∞ is $\beta N - N$ (cf. § 3.1), the corresponding classifying object in $C(D^*)$ is not $\beta D - D$ but a suitable quotient of it.

For $\mathscr{V}_1, \mathscr{V}_2 \in \beta D$, let:

$$\mathscr{V}_1 \equiv \mathscr{V}_2 \Leftrightarrow \mathscr{V}_1 \text{ and } \mathscr{V}_2 \text{ contain the same countable subsets of } D.$$

This is an equivalence relation; each principal ultrafilter is equivalent only to itself, while all ultrafilters which do not contain countable subsets of D form a single equivalence class, $\aleph$. Setting $\beta_\omega D = \beta D/\equiv$, D is then embedded in $\beta_\omega D$.

Lemma 7. *The map $J \mapsto \mathscr{V}_J \upharpoonright D/\equiv$, where $\mathscr{V}_J$ is any ultrafilter associated with J, is a bijection of the family of minimal prime ideals of $C(D^*)$ contained in M_∞, onto $\beta_\omega D - (D \cup \aleph)$.*

Proof. To begin with, note that if $\mathscr{V}, \mathscr{V}'$ are nonprincipal ultrafilters on D^*:

$$(*) \qquad \mathscr{V} \upharpoonright D \equiv \mathscr{V}' \upharpoonright D \quad \text{iff } \mathscr{V} \cap Z(D^*) = \mathscr{V}' \cap Z(D^*).$$

This is easily derived from the equality:

$$\mathscr{V} \cap Z(D^*) = \{D^* - A \mid A \subseteq D, \bar{A} \leqslant \aleph_0, A \notin \mathscr{V}\};$$

in turn, this in an easy consequence of (1), (p. 37), because $\mathscr{V}, \mathscr{V}'$ are nonprincipal. If $\mathscr{V}, \mathscr{V}'$ are ultrafilters on D^* associated with J, they are nonprincipal (since $J \subseteq M_\infty$) and $\mathscr{V} \cap Z(D^*) = Z(J) = \mathscr{V}' \cap Z(D^*)$. It follows from $(*)$ that the map of the statement is well defined. Using $(*)$ from left to right we obtain that it is injective. Since $Z(J)$ contains no finite set, its range is included in $\beta_\omega D - D$, and since $J \subsetneq M_\infty$, it cannot take on the value $\aleph$; hence, its range is contained in $\beta_\omega D - (D \cup \aleph)$.

Finally, in order to show that this map is surjective, let $\mathscr{V}$ be a nonprincipal ultrafilter on D containing at least one countable set (i.e., $\mathscr{V}/\equiv \in \beta_\omega D - (D \cup \aleph)$), and let us construct a prime z-ideal $J \subsetneq M_\infty$ such that $\mathscr{V}_J \upharpoonright D \equiv \mathscr{V}$. It is easily seen that the set

$$\mathscr{V} \cup \{D^* - A \mid A \subseteq D, \bar{A} \leqslant \aleph_0, A \notin \mathscr{V}\}$$

has the finite intersection property; let $\mathscr{V}'$ be an ultrafilter on D^* containing it, and let J be the unique prime z-ideal of $C(D^*)$ such that $Z(J) = \mathscr{V}' \cap Z(D^*)$ (cf. [3; 14.F]); clearly $Z(J) = \{D^* - A \mid A \subseteq D, A \leqslant \aleph_0, A \notin \mathscr{V}\}$. Now (3) of p. 37 implies at once $J \subseteq M_\infty$, and since $\mathscr{V}$ contains at least one countable set, it follows that $J \neq M_\infty$. Finally, in order to check that $\mathscr{V}_J | D \equiv \mathscr{V}$, notice that $\mathscr{V}' \cap Z(D^*) = Z(J) = \mathscr{V}_J \cap Z(D^*)$, $\mathscr{V}' \upharpoonright D = \mathscr{V}$, and apply $(*)$ ■

Dualizing, the map

$$J \mapsto \{A \subseteq D \mid \bar{A} \leqslant \aleph_0 \text{ and } A \notin \mathscr{V}_J\}$$

establishes a classification of the minimal prime ideals of $C(D^*)$ contained in M_∞ by the set of nonprincipal maximal ideals of the lattice (without 1) of countably subsets of D, which will be denoted by $L_\omega(D)$. This shows at once that the cardinality of the set of minimal prime ideals of $C(D^*)$ contained in M_∞ is at most $2^{(\varkappa^{\aleph_0})}$, where $D = \varkappa$. We shall now show:

Proposition 8. *There are exactly $2^{(\varkappa^{\aleph_0})}$ minimal prime ideals of $C(D^*)$ contained in M_∞ (where $D = \varkappa$).*

Proof. Put $D = \varkappa$. Let us call a set $\mathscr{S} \subseteq L_\omega(\varkappa)$ *independent* iff whenever $A_1, \ldots, A_n, B_1, \ldots, B_k$ are pairwise distinct members of $\mathscr{S}$, $\bigcap_i A_i \cap \bigcap_j ((\varkappa - B_j) \neq \emptyset$. Each subset $\mathscr{S}$ of an independent set $\mathscr{S}_0 \subseteq L_\omega(\varkappa)$ gives rise to an ultrafilter $\mathscr{U}_{\mathscr{S}}$ on $\varkappa$ because the set $\mathscr{S} \cup \{\varkappa - B \mid B \in \mathscr{S}_0 - \mathscr{S}\}$ has the finite intersection property. Furthermore, for distinct $\mathscr{S}$'s the ultrafilters obtained in whis way are pairwise nonequivalent modulo $\equiv$ (the countable members of $\mathscr{U}_{\mathscr{S}}$ are those of $\mathscr{S}$ and, possibly, some members of $L_\omega(\varkappa) - \mathscr{S}_0$).

In view of Lemma 7, in order to prove the theorem it suffices to show that there is an independent set $\mathscr{S}_0$ of cardinality $\varkappa^{\aleph_0}$. We shall prove this by a variant of a technique due to Ketonen (cf. Kunen [8]).

Let $\mathscr{M}(\varkappa, \aleph_0) = \{f \in \aleph_0 \mid \overline{\varkappa - f^{-1}(0)} \leqslant \aleph_0\}$. A set $\mathscr{F} \subseteq \mathscr{M}(\varkappa, \aleph_0)$ is *independent* iff given distinct $f_1, \ldots, f_n \in \mathscr{F}$ and arbitrary $i_1, \ldots, i_n \geqslant 1$, $\bigcap_j f_j^{-1}[i_j] \neq \emptyset$. Observe that an independent set $\mathscr{F} \subseteq \mathscr{M}(\varkappa, \aleph_0)$ gives rise to an independent subset $\mathscr{S}_0$ of $L_\omega(\varkappa)$. Indeed, it suffices to put $\mathscr{S}_0 = \{f^{-1}[I] \mid f \in \mathscr{F}\}$, where I is any set such that $\emptyset \neq I \subsetneq \aleph_0 - \{0\}$. Moreover, the correspondence $f \mapsto f^{-1}[I]$ is injective; for considering $i_0 \in I$, $j_0 \notin I$, $j_0 \neq 0$, and distinct $f_1, f_2 \in \mathscr{F}$ we have $\emptyset \neq f_1^{-1}[i_0] \cap$

$\cap f_2^{-1}[j_0] \subseteq f_1^{-1}[I] - f_2^{-1}[I]$. The theorem follows, then, from the existence of an independent $\mathscr{F} \subseteq \mathscr{M}(\varkappa, \aleph_0)$ of cardinality $\varkappa^{\aleph_0}$, which we now prove.

Let $\{\langle s_\alpha, r_\alpha\rangle \mid \alpha < \varkappa\}$ enumerate the set $\{\langle s, r\rangle \mid s \in P_\omega(\varkappa),\ r \in {}^{s}(\aleph_0 - \{0\})\}$. For $A \in L_\omega(\varkappa)$ let $f_A\colon \varkappa \to \aleph_0$ be defined by:

$$f_A \restriction (\varkappa - A) = 0$$
$$f_A(\alpha) = r_\alpha(A \cap s_\alpha) \quad \text{for } \alpha \in A.$$

Set $\mathscr{F} = \{f_A \mid A \in L_\omega(\varkappa)\}$. If $A_1, \ldots, A_n \in L_\omega(\varkappa)$ are pairwise distinct and $i_1, \ldots, i_n \geqslant 1$, for $1 \leqslant k < l \leqslant n$ let $\alpha_{kl} \in (A_k - A_l) \cup (A_l - A_k)$; put $s = \{\alpha_{kl} \mid 1 \leqslant k < l \leqslant n\}$. Clearly $A_k \cap s \neq A_j \cap s$ for $k < l$. The correspondence $r(A_l \cap s) = i_l$ is then well defined and it can be extended to a map in ${}^{s}(\aleph_0 - \{0\})$. If $s_\alpha = s$, $r_\alpha = r$, we have α $\alpha \in \bigcap_j f_{A_j}^{-1}[i_j]$. ■

3.4. Measure ultrafilters. In the present section we will discuss some prime z-ideals in $C(R^{+*})$, where R^{+*} is the one-point compactification of R^+, the non-negative reals; R^{+*} is, of course, homeomorphic to $[0, 1]$.

It is clear that $S(R^{+*})$ contains many ideals of type E; there are, in fact, $2^{2^{\aleph_0}}$ such ideals associated with each convergent sequence in R^{+*}.

We now want to consider ideals which are in some sense at the opposite extreme. A filter of zero-sets ($=$ losed sets) in R^+ will be called a *measure filter* iff all of its elements are of infinite Lebesgue measure. In particular, a *measure z-ultrafilter* on R^+ contains all closed sets whose complements are of finite measure (and conversely). By abuse of notation we shall identify here z-ultrafilters $\mathscr{U}$ of R^+ with the corresponding maximal ideals M of $C(R^+)$ such that $Z(M) = \mathscr{U}$.

Applying Corollary 2.3.3 we obtain a characterization of those z-ultrafilters $\mathscr{U}$ on R^+ such that the corresponding prime z-ideal $\gamma(\mathscr{U})$ of $C(R^{+*})$ is real closed.

PROPOSITION 1. *Let $\mathscr{U}$ be a z-ultrafilter on R^+. Then $\gamma(\mathscr{U})$ is defined and the following are equivalent*:

(1) *$\gamma(\mathscr{U})$ is a real closed ideal of $C(R^{+*})$.*

(2) *$\mathscr{U}$ is a $P(\infty)$-filter.*

Proof. That $\gamma(\mathscr{U})$ is defined follows from the fact that the map ϕ is defined on βR^+, [3; 6.5(I)]. The rest follows from the equivalence between (1) and (3) of Corollary 2.3.3. ■

Our goal in the rest of this section is to prove in a suitable extension of ZFC that measure $P(\infty)$-filters do exist. As in the theorem of existence of P-points in $\beta N - N$, this can be done under the additional assumption of Martin's axiom.

THEOREM 2. *Assuming* MA, *then measure $P(\infty)$-filters exist.*

Proof. We will show that it is possible to associate to each continuous function $f\colon R^+ \to [0, 1]$ a pair $p(f) = \langle f', r_f\rangle$ consisting of:

(a) a strictly increasing function $f'\colon N \to N$,

(b) a real number r_f,

and satisfying a further condition (c) which we now describe.

For $x \in R^+$, set $i_f(x) = \operatorname{Sup}\{n \in N \mid |f(x) - r_f| \leqslant 1/n\}$. Then $i_f(x) \in N \cup \{\infty\}$. Let

$$X(f) = \{x \in R^+ \mid i_f(x) \in N \wedge x \leqslant f'(i_f(x)+1)\} \cup f^{-1}[r_f].$$

Since f is continuous, the set $X(f)$ is measurable.

The condition on $p(f)$ is then:

(c) for all $k \geqslant 1$ and all continuous $f_1, \ldots, f_k\colon R^+ \to [0, 1]$, the set

$$X(\vec{f}) = X(f_1) \cap \ldots \cap X(f_k)$$

has infinite measure.

Notice that we will then have:

$$\lim_{X \to \infty} f \restriction X(f) = r_f.$$

Indeed, for given $n \geqslant 1$, if $x \in X(f)$, $x > f'(n)$, then $n \leqslant i_f(x)$, and hence $|f(x) - r_f| \leqslant 1/n$.

If C is a closed set such that $m(R^+ - C) < \infty$ (where m denotes the Lebesgue measure), then from $m(X(\vec{f})) = m(X(\vec{f}) - C) + m(X(\vec{f}) \cap C)$ and (c) we conclude that $X(\vec{f}) \cap C$ has infinite measure. It follows that the family:

$$\{X(f) \mid f\colon R^+ \to [0, 1] \text{ continuous}\} \cup \{C \mid C \text{ closed}, m(R^+ - C) < \infty\}$$

has the finite intersection property, and any z-filter containing it is a measure $P(\infty)$-filter.

It remains then to construct a suitable map p. This is done inductively, using some list $\{f \mid \alpha < 2^{\aleph_0}\}$ of all continuous functions from R^+ to $[0, 1]$. In order to carry out this induction we need only prove the following:

LEMMA 3. *Let F be a family of fewer than $2^{\aleph_0}$ continuous functions from R^+ to $[0, 1]$, and let $f\colon R^+ \to [0, 1]$ be another continuous function. Assume that a map $p\colon F \to N^N \times [0, 1]$ has been defined satisfying conditions* (a)–(c) *above for $f_1, \ldots, f_k \in F$. Then there is a pair $\langle f', r_f\rangle$ such that the extension of p by $f \mapsto \langle f', r_f\rangle$ defines a map $p'\colon F \cup \{f\} \to N^N \times [0, 1]$ satisfying conditions* (a)–(c) *for $f_1, \ldots, f_k \in F \cup \{f\}$.*

Proof. Let F, p be as specified. We will define a partially ordered set P whose elements are, intuitively speaking, finite approximations to the desired pair $\langle f', r_f\rangle$. Let P be the set of all pairs (s, δ) such that $s = \delta = \emptyset$ or there is $n \geqslant 1$ such that:

(i) $s\colon \{1, \ldots, n\} \to N$ is a strictly increasing function;

(ii) $\delta\colon \{1, \ldots, n\} \to \{0, 1\}$;

(iii) if $r_\delta = \sum_{i=1}^{n} \delta(i)/2^i$, then for all $k \geqslant 1$ and all $f_1, \ldots, f_k \in F$, the set $X(\vec{f}) \cap f^{-1}[[r_\delta;\ r_\delta + 1/2^n]]$ has infinite measure. We call n the *length* of (s, δ); $(\emptyset, \emptyset)$ has length 0; note setting $r_\emptyset = 0$, $n = 0$, condition (iii) is verified by $(\emptyset, \emptyset)$. The ordering of P is that of restriction on both coordinates: $(t, \varepsilon) < (s, \delta)$ iff length $(s, \delta) = n <$ length (t, ε) and

$$t \restriction n = s, \quad \varepsilon \restriction n = \delta.$$

Since P is countable, it satisfies the c.a.c.

Let D_n be the set of elements of P of length at least n. We claim:

(1) D_n is dense in P.

This is obvious for $n = 0$, and if $(s, \delta) \in P$ is of length $n \in \omega$ we obtain an extension (s', δ') of (s, δ) of length $n+1$ by defining $s'(n+1)$ arbitrarily and checking at once that one of the two possible values of $\delta'(n+1)$ makes condition (iii) hold. The result follows by induction on n.

Now associate to each $(s, \delta) \in P$ a set $X(s, \delta)$ as follows. For $x \in \boldsymbol{R}^+$ define:

$$i(x, s, \delta) = \sup\left\{k \in N \cup \{\infty\} \,\middle|\, \sup\left\{|f(x)-r| \,\middle|\, r_\delta \leqslant r \leqslant r_\delta + \frac{1}{2^n}\right\} \leqslant \frac{1}{k}\right\}.$$

Set:

$$X(s, \delta) = \{x \leqslant s(i(x, s, \delta)+1) \mid i(x, s, \delta) < n\} \cup \{x \leqslant s(n) \mid i(x, s, \delta) \geqslant n\},$$

where $n = \text{length } (s, \delta)$.

For $g_1, \ldots, g_k \in F$ and $l \in N$, define

$$D_{\bar{g},l} = \{(s, \delta) \mid m(X(\bar{g}) \cap X(s, \delta)) \geqslant l\}.$$

We now claim:

(2) $D_{\bar{g},l}$ is dense in P.

Fix $(s, \delta) \in P$. Fix an extension δ' of δ of length $n+1$ so that condition (iii) above is satisfied; in particular, $X(\bar{g}) \cap f^{-1}[[r_{\delta'}, r_{\delta'}+1/2^{n+1}]]$ has infinite measure. Choose an integer j so that

$$m(X(\bar{g}) \cap f^{-1}[[r_{\delta'}, r_{\delta'}+1/2^{n+1}]] \cap [0, j) \geqslant l.$$

Let now s' be an extension of s of length $n+1$ so that $s'(n+1)\ Tj$. We claim that $(s', \delta') \in D_{\bar{g},l}$. This clearly follows from:

$$f^{-1}[[r_{\delta'}, r_{\delta'}+1/2^{n+1}]] \cap [0, j] \subseteq X(s', \delta').$$

In order to prove this, fix $x \leqslant j$ satisfying $f(x) \in [r_{\delta'}, r_{\delta'}+1/2^{n+1}]$. Then $i(x, s', \delta') \geqslant 2^{n+1} > n+1$ and $x \leqslant j \leqslant s'(n+1)$; so $x \in X(s', \delta')$, as desired.

Now apply Martin's Axiom to obtain a $\mathscr{D}$-generic subset G of P with respect to

$$\mathscr{D} = \{D_n \mid n \in N\} \cup \{D_{\bar{g},l} \mid l \in N,\ \bar{g} \subseteq F \text{ finite } \neq \emptyset\}.$$

Since the elements of G are compatible and G meets each D_n, there are unique functions f', ϱ such that

$$G = \{(f' \restriction n, \varrho \restriction n) \mid n \geqslant 1\}.$$

We now extend the function $p\colon F \to N^N \times [0, 1]$ by setting:

$$p'(f) = \langle f', r_f \rangle \quad \text{with } r_f = \sum_{i=1}^{\infty} \varrho(i)/2^i.$$

It remains to be seen that condition (c) is satisfied by p', or in other words that for $g_1, \ldots, g_k \in F$, $l \in N$ arbitrary, we have: $m(X(\bar{g}) \cap X(f)) \geqslant l$.

Consider accordingly an integer n such that $(f' \restriction n, \varrho \restriction n) \in D_{\bar{g},l}$. Let $s = f' \restriction n$, $\delta = \varrho \restriction n$. It suffices to prove that $X(s, \delta) \subseteq X(f)$. Fix $x \in X(s, \delta)$; then there are two cases to consider:

Case 1. $i(x, s, \delta) < n$, $x \leqslant s(i(x, s, \delta)+1)$. Since

$$\sup\left\{|f(x)-r| \,\middle|\, r_\delta \leqslant r \leqslant r_\delta + \frac{1}{2^n}\right\} \geqslant |f(x) - r_f|,$$

we have $i(x, s, \delta) \leqslant i_f(x)$; since f' is increasing,

$$s(i(x, s, \delta)+1) \leqslant f'(i_f(x)+1);$$

hence $x \in X(f)$.

Case 2. $i(x, s, \delta) \geqslant n$, $x \leqslant s(n)$. Since $i(x, s, \delta) \leqslant i_f(x)$, it follows that

$$s(n) \leqslant s(i_f(x)) < s(i_f(x)+1) = f'(i_f(x)+1),$$

so $x < f'(i_f(x)+1)$; hence again $x \in X(f)$.

This completes the proof of the lemma, and hence of Theorem 2. ∎

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