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Received 12 September 1983

A correction of two papers concerning Hilbert manifolds

by

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In [8] the author has claimed the following result: if X is a complete separable ANR and there is a Z-set K in X such that $X\setminus K$ is a manifold modelled on the Hilbert space l_2, then X is a Hilbert manifold itself. This result is false as will be seen from an example due to P. Bowers and J. Walsh and presented in §A. The author would like to express his sincere thanks to his colleagues from the University of Tennessee, and particularly to M. Bestvina, P. Bowers and J. Walsh, for referring this example to him.

The purpose of this corrigendum is to prove in §B a modified version of the result claimed in [8] and to introduce in §C necessary changes in another author's paper [9] where that result has been used as a technical tool in establishing a characterization of l_2-manifolds. In the appendix we include a version of Bing's shrinking criterion which allows to consider in §B certain manifolds modelled on spaces different from l_2, as is done in [8].

All the unexplained notions have the meaning of [8] or of [9]. Function spaces are considered in the limitation topology and we say that f is approximable by functions in a set S if $f \in \operatorname{cl} S$.

§A. Z-sets versus $\bar{Z}$-sets. Let us say that K is a Z-set in X (resp. a $\bar{Z}$-set in X) if it is closed in X and id_X is approximable by maps $f\colon X \to X$ satisfying $f(X) \cap K = \emptyset$ (resp. $\operatorname{cl} f(X) \cap K = \emptyset$, cl denoting closure). In [6] D. W. Henderson has given characterizations of Z-sets in ANR's (cf. also [4], [1]) and, in connection with an application in [2] and [10] and a question posed on a conference in Oberwolfach in 1970, has established the following fact:

A1. Let X be a manifold modelled on a locally convex metric linear space E such that $E \cong E^\infty$ or $E \cong E^\infty_f = \{(x_i) \in E^\infty\colon x_i = 0$ for almost all i's$\}$. Then the family of $\bar{Z}$-sets in X coincides with that of Z-sets in X.

It was also known to Henderson that these families may differ if X is merely an ANR (oral communication to the author from 1972). The following example illustrating this is due to P. Bowers and J. Walsh:

A2. Example. Let $X = [0, 1] \times \{0\} \cup \{2^{-n}\colon n \in N\} \times [0, 1] \subset I^2$ and $p = (0, 0)$. Then $K = \{p\}$ is a Z-set in X but is not a $\bar{Z}$-set in X.

(In fact, K is a Z-set by [6] and each map $f\colon X \to X$ with $d(f(a,b),(a,b)) < a/2$ for $(a,b) \in X$ necessarily satisfies $p \in \mathrm{cl} f(X)$).

A3. Example (M. Bestvina and J. Walsh). Let X be a complete separable ANR and K a Z-set which is not a $\bar{Z}$-set in X. Let $M = (X \times l_2)_K$, i.e. $M = K \cup (X \setminus K) \times l_2$ equipped with the weakest topology in which the projection $\pi\colon M \to X$ is continuous. Then K is a Z-set but not a $\bar{Z}$-set in the separable complete ANR M. By A1, M fails to be a manifold and π is not approximable by homeomorphisms, contrary to the result claimed in [8].

§B. Enlarging a manifold. A corrected version of Theorem 5.2 of [8] is as follows (with $E = [0,1]^\infty$ the version of [8] and its proof remain valid):

B1. Theorem. *Let X be a neighbourhood retract of a locally convex metric linear space E such that $E \cong E^\infty$ or $E \cong E_f^\infty$, and let K be a $\bar{Z}$-set in X. If $X \setminus K$ is an E-manifold then the projection $\pi\colon X \times E \to (X \times E)_K$ is approximable by homeomorphisms and X is an E-manifold.*

Proof. As in [8] it suffices to prove the first statement. The most important special case is that when E is complete (i.e. homeomorphic to a Hilbert space); then the proof may be performed using the techniques employed in [9], as follows.

Let $C = ([0,1] \times E)_{\{0\}}$, the metric cone over E; then $C \cong E$ by [5]. Take a map $u\colon X \to X$ which is close to id_X and satisfies $\mathrm{cl}\, u(X) \cap K = \emptyset$. Since $E \times X$ embeds as a Z-set in $E \times [1/2, 1]$, we get a Z-embedding $v\colon X \times E \times \{1\} \to X \times C$ with $v = up_X$. ($X \times E \times \{1\}$ is considered as a subset of $X \times C$). Then, v is close to π and $\mathrm{im}(v)$ misses $N \times C$, for some closed neighbourhood N of K. By the homeomorphism extension theorem ([0], [2], [10]) for Z-sets in the E-manifold $X \times C$, there is a homeomorphism $g\colon X \times C \to X \times C$ such that $g|\mathrm{im}(v) = v^{-1}$ and πg is close to π, say $\mathscr{V}$-close where $\mathscr{V}$ is a pre-assigned open cover of X. Since $g(N) \cap X \times E \times \{1\} = \emptyset$, we may insert between $g(N)$ and $X \times E \times \{1\}$ a graph of a map $\alpha\colon X \times E \to (0,1)$ and use a homeomorphism h of $X \times C$ moving points along the I-axis of C to push that graph as close to $X \times \{0\}$ as we wish. Then, for each $x \in K$ there is a neighbourhood $W(x)$ in $Y = (X \times C)_K$ such that $\pi^{-1}(W(x)) = W(x) \times C$ is squeezed by hg to a set of small size, and by the definition of Y this is true also of points of $Y \setminus K$. (We omit easy details which are the same as some of those in the proof of Lemma 2.3 in [9]). Thus we may apply Bing's shrinking criterion in the version of [9] (cf. Appendix, case 1) to deduce that π is approximable by homeomorphisms.

This completes the proof in the case where E is complete. In the general case two points need additional comments. Firstly, it is then necessary to use the version of Bing's shrinking criterion stated in the appendix. This can be done since the homeomorphisms g and h above may be constructed so as to be identities on $A = K \times \{0\} \subset X \times C$ and on $(X \setminus U) \times C$, where U is a given neighbourhood of K in X. Secondly, the homeomorphism extension theorem employed in the proof has in the published literature been stated only for homeomorphisms between the so called "deficient sets", rather than between Z-sets. (See [0], [3], [7]. A set P

in an E-manifold M is said to be *deficient* if there is a homeomorphism $w\colon M \to M \times E$ with $w(P) \subset M \times \{0\}$). Therefore, in the proof above it is preferable to use the fact that $E \times \{1\}$ is a deficient set in C and $X \times E$ admits a deficient embedding into $E \times [1/2, 1]$. With these modifications the previous proof can be applied.

B2. Remark. Using the additional statement of Bing's shrinking criterion it follows from A1 and the proof above that if K is a Z-set in an E-manifold X and E is as in B1, then there is a homeomorphism $w\colon X \to X \times E$ with $w(x) = (x, 0)$ for $x \in K$, i.e. K is deficient. This establishes the homeomorphism extension theorem for Z-sets in X, discussed in the proof.

§C. A corrigendum to [9]. The false result claimed in [8] has been employed in [9]. The necessary corrections are localized in two places:

1. In §2 of [9] one should replace Z by $\bar{Z}$ and use B1 in place of Lemma 2.2. Therefore, the corrected version of Proposition 2.1 of [9] requires that maps $Y \to X$ be approximable by embeddings whose images are $\bar{Z}$-sets, rather than Z-sets.

2. In §3 of [9] the above modification of the assumptions of 2.1 effects the proof of the main theorem 3.1, which should therefore be changed as follows:

Proof of 3.1. Let Y be a space as in (*); we show first as on p. 255 that

(i) any map $Y \to X$ is approximable by embeddings.

Using this and the continuity of the composition operation it remains to show that id_X is approximable by $\bar{Z}$-embeddings.

Write $C = C(X \times (0,1], X \times (0,1])$. Using 1.3, 1.4 and (i), it follows that any of the sets

$$G_n = \{f \in C\colon p_X f | X \times [n^{-1}, 1] \text{ is an embedding}\}$$

is a dense G_δ-set in C. Therefore, the Baire property of C implies that $G = \bigcap\{G_n\colon n \in N\}$ is dense in the open set

$$U = \{f \in C\colon d(p_X f(x,t), x) < t \quad \text{for} \quad (x,t) \in X \times (0,1]\}.$$

Choose $u \in U \cap G$. Then

(ii) $p_X u$ is one-to-one, and

(iii) for any map $\alpha\colon X \to (0,1]$ the formula

$$h_\alpha(x) = p_X u(x, \alpha(x))$$

defines an embedding $X \to X$.

(In fact, if $(h_\alpha(x_n))$ converges then using $u \in U$ it follows that (x_n) converges in case $\inf \alpha(x_n) = 0$, and using $u \in G$ we deduce the same in the other case).

Thus $\mathrm{im}(h_\alpha) \cap \mathrm{im}(h_\beta) = \emptyset$ if $\beta < \alpha$; also h_α is close to id_X if α is close to $x \mapsto 0$. Accordingly, the h_α's are $\bar{Z}$-embeddings which may be taken as the desired approximations to id_X.

Appendix: A version of Bing's shrinking criterion for incomplete spaces.

Proposition. *Let $\pi\colon Y \to Z$ be a map of metric spaces satisfying the following condition:*

(bi) *given* $\mathscr{U} \in \operatorname{cov}(Y)$ *and* $\mathscr{V} \in \operatorname{cov}(Z)$ *there are* $\mathscr{W} \in \operatorname{cov}(Z)$ *and a homeomorphism* $f\colon Y \to Y$ *with* πf $\mathscr{V}$*-close to* π *and* $f\pi^{-1}(\mathscr{W}) \prec \mathscr{U}$.

In any of the following cases π *is approximable by homeomorphisms*:

(1) Y *and* Z *are topologically complete and* $\pi(Y)$ *is dense in* Z, *or*

(2) $\pi(Y) = Z$ *and there are closed sets* $K \subset Z$ *and* $A \subset Y$ *such that* $\pi(A) = K$ *and for every neighbourhood* U *of* K *in* Z *the homeomorphism* f *of* (bi) *may be taken to satisfy* $f(x) = x$ *if either* $x \in A$ *or* $\pi(x) \notin U$.

ADDITIONAL CLAIM. *In case* (2), *the mapping* π *is approximable by homeomorphisms* g *satisfying* $g(x) = \pi(x)$ *for* $x \in A$.

Proof. Given $\mathscr{V} \in \operatorname{cov}(Z)$ we shall construct a homeomorphism $g\colon Y \to Z$ $\mathscr{V}$-close to π. To this end we fix metrics ϱ of Z and d of Y such that $d < 1/2$ and the cover of Z by closed ϱ-balls of radius 1 refines $\mathscr{V}$ (cf. [1], p. 63); in case (1) require also that ϱ and d be complete. We put $f_0 = \mathrm{id}_Y$ and $\mathscr{W}_0 = \{Z\}$ and we construct inductively $\mathscr{W}_n \in \operatorname{cov}(Z)$ and homeomorphisms $f_n\colon Y \to Y$ satisfying for $n \geqslant 1$ the following conditions

$$(\mathrm{a}_n) \qquad d(f_n, f_{n-1}) \leqslant 2^{-n};$$

$$(\mathrm{b}_n) \qquad \varrho(\pi f_n^{-1}, \pi f_{n-1}^{-1}) \leqslant 2^{-n};$$

$$(\mathrm{c}_n) \qquad \operatorname{diam}_d f_n \pi^{-1}(W) \leqslant 2^{-n-1} \quad \text{for} \quad W \in \mathscr{W}_n.$$

If (2) holds then we require additionally that

$$(\mathrm{d}_n) \qquad f_n(y) = f_{n-1}(y) \text{ if either } y \in A \text{ or } \operatorname{dist}_\varrho(\pi(y), K) > 1/n \text{ or } \pi(y) \notin \operatorname{st}(K, \mathscr{W}_{n-1}).$$

The inductive step. If f_{n-1} and $\mathscr{W}_{n-1}$ satisfy (c_{n-1}) then, refining $\mathscr{W}_{n-1}$ if necessary, we may assume in addition that $\operatorname{diam}_\varrho W < 2^{-n}$ for $W \in \mathscr{W}_{n-1}$. By (bi) there are $\mathscr{W}_n \in \operatorname{cov}(Z)$ and a homeomorphism u of Y such that

$$u \text{ is } \pi^{-1}(\mathscr{W}_{n-1})\text{-close to } \mathrm{id}_Y \quad \text{and} \quad \operatorname{diam}_d f_{n-1} u \pi^{-1}(W) < 2^{-n-1} \text{ for } W \in \mathscr{W}_n.$$

We let $f_n = f_{n-1} u$; then (c_r) holds and applying (c_{n-1}) we get (a_n). Also $\varrho(\pi u^{-1}, \pi) \leqslant \sup\{\operatorname{diam} W\colon W \in \mathscr{W}_{n-1}\} \leqslant 2^{-n}$, which gives (b_n). If we require that u be the identity on an appropriate set we get (d_n).

Observe that $(f_n(y))$ converges, for each $y \in Y$. This is clear if d is complete, by (a_n), or else if $y \in \pi^{-1}(Y \setminus K)$, by (d_n). If $y \in \pi^{-1}(K)$ and (2) holds then choose $a \in \pi^{-1}\pi(y) \cap A$ and observe that $\lim f_n(y) = a$, by (c_n) and (d_n). Let $f = \lim f_n$; then

$$(\mathrm{i}) \qquad d(ff_m^{-1}, \mathrm{id}_Y) = d(f, f_m) \leqslant 2^{-m} \quad \text{for} \quad m \geqslant 0.$$

Similarly,

$$\varrho(\pi f_m^{-1} f_n, \pi) = \varrho(\pi f_m^{-1}, \pi f_n^{-1}) \leqslant 2^{-m} \quad \text{if} \quad m \geqslant n,$$

whence

$$(\mathrm{ii}) \qquad \varrho(\pi f_m^{-1} f, \pi) \leqslant 2^{-m} \quad \text{for} \quad m \geqslant 0.$$

By (i) and (c_n), the filter $\mathscr{F}(z)$ generated by $\{f\pi^{-1}(W)\colon W \text{ is a neighbourhood of } z\}$ is Cauchy, for each $z \in Z$, whence $z \mapsto \lim \mathscr{F}(z)$ properly defines a map $h\colon Z \to Y$ satisfying $h\pi = f$. (This is because either $\pi(Y) = Z$ or Y is complete and $\operatorname{cl}\pi(Y) = Z$.)

Write $F = \{y \in Y\colon (\pi f_n^{-1}(y)) \text{ converges}\}$; then $F \supset \operatorname{im}(f)$ by (ii). In case (2), it follows from (c_n) and (d_n) that $f_n(y) = f_{n-1}(y)$ if $\operatorname{dist}(f_{n-1}(y), A) > 2^{-n}$, whence in this case $F \supset \operatorname{im}(f) \supset A \cup (Y \setminus A) = Y$; in the other case we have $F = Y$ by (b_n). Thus in either case there is a well-defined function $g = \lim \pi f_n^{-1}\colon Y \to Z$. By (b_n), (i) and (ii), g is continuous and

$$hg = \lim ff_n^{-1} = \mathrm{id}_Y \quad \text{and} \quad gh\pi = \pi.$$

Since $\pi(Y)$ is dense in Z, g and h are mutually inverse; moreover

$$\varrho(g, \pi) = \lim \varrho(\pi f_n^{-1}, \pi f_0^{-1}) \leqslant 1.$$

Thus g is $\mathscr{V}$-close to π, as desired.

The additional claim follows from the above proof.

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Received 4 June 1984