Note 1. Theorems 1, 2 and 3 are also valid changing $c_0(I)$ by $l^p(I)$, $0<p<\infty$, with the same proofs.

Note 2. In [1] the construction of Fleissner and Kunen [3] and the method of "coordinates" are used also to obtain dense Baire subspaces of $l^2(\omega_1)$ whose product is not a Baire space, but unfortunately the example given there does not satisfy the required properties.

Note 3. Independently of us R. Pol and J. van Mill have obtained normed Baire spaces with product non-Baire. They have proven in [7] that in a F-space of weight $\aleph_1$ there are dense Baire subspaces such that their product is not Baire. The reasoning used in [7] is based on a general version of the construction of Fleissner and Kunen [3] given in [6].

Open question. Let E be a nonseparable Fréchet space. Does there exists a dense Baire subspace F of E such that $F\times F$ is not Baire?

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Extensions of functions from products with compact or metric factors

by

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Abstract. This paper is a continuation of an earlier paper [3]. A subspace A of a space X is $\pi_{\mathscr{X}}$-embedded in X, where $\mathscr{X}$ denotes a non-empty class of spaces, if for every $Z\in\mathscr{X}$ and for every continuous function $f\colon A\times Z\to I$ there exists an extension of f over $X\times Z$. We shall prove that $\pi_{\mathscr{M}\cup\mathscr{C}}$-embedding is equivalent to $\pi_{\mathscr{M}\times\mathscr{C}}$-embedding, where $\mathscr{M}\cup\mathscr{C}$ denotes the class consisting of all metric and all compact spaces, and $\mathscr{M}\times\mathscr{C}$ — the class of products of a metric and a compact space. We shall also give an example which shows that $\pi_{\mathscr{M}\times\mathscr{C}}$-embedding does not imply $\pi_{\mathscr{P}}$-embedding where $\mathscr{P}$ denotes the class of all paracompact p-spaces.

Throughout this paper by a topological space we mean a completely regular space, by a function or an extension — a continuous function or a continuous extension. Symbols $\mathscr{M}$, $\mathscr{C}$, $\mathscr{M}\times\mathscr{C}$, and $\mathscr{P}$ denote the classes of metric spaces, compact spaces, products of a metric and a compact space, and paracompact p-spaces, respectively. By P we shall denote the set of all irrational numbers and by Q — the set of all rational numbers in the unit interval I.

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Let us recall (cf. [2]) that a subspace A of a space X is $\pi_{\mathscr{X}}$-embedded in X, where $\mathscr{X}$ denotes a non-empty class of spaces, if for every $Z\in\mathscr{X}$ and for every function $f\colon A\times Z\to I$ there exists an extension of f over $X\times Z$. Obviously, if $\mathscr{X}\subset\mathscr{X}'$ then $\pi_{\mathscr{X}'}$-embedding implies $\pi_{\mathscr{X}}$-embedding. Thus $\pi_{\mathscr{P}}$-embedding implies $\pi_{\mathscr{M}\times\mathscr{C}}$-embedding, which implies $\pi_{\mathscr{M}\cup\mathscr{C}}$-embedding. For dense subsets of topological spaces the inverse implications are also true: T. C. Przymusiński showed in [2] that for a dense subset of a topological space $\pi_{\mathscr{P}}$-embedding is equivalent to $\pi_{\mathscr{M}\cup\mathscr{C}}$-embedding. It turns out that in the case of closed subsets this theorem does not hold. We shall give an example of a space Y and its closed subspace X which is $\pi_{\mathscr{M}\cup\mathscr{C}}$-embedded but is not $\pi_{\mathscr{P}}$-embedded in Y. We shall also prove that $\pi_{\mathscr{M}\cup\mathscr{C}}$-embedding is equivalent to $\pi_{\mathscr{M}\times\mathscr{C}}$-embedding; thus our example will show that $\pi_{\mathscr{M}\times\mathscr{C}}$-embedding does not imply $\pi_{\mathscr{P}}$-embedding. These results will also provide a negative answer to the problem raised by K. Morita in [1], whether every Čech-complete, paracompact space is absolutely P-embeddable (a space X is absolutely

P-embeddable if, whenever X is P-embedded in a space Y, $X\times Z$ is P-embedded in $Y\times Z$ for any space Z).

THEOREM. *If A is $\pi_{\mathcal{M}\cup\mathcal{C}}$-embedded in X, then A is $\pi_{\mathcal{M}\times\mathcal{C}}$-embedded in X.*

Proof. It suffices to show that $A\times M$ is P-embedded in $X\times M$ for every metric space M. Naturally, $A\times M$ is C-embedded in $X\times M$. Let $\mathcal{U}=\{U_s: s\in S\}$ be a locally finite cozero covering of $A\times M$. Find cozero subsets U_s^* in $X\times M$ so that $U_s^*\cap(A\times M)=U_s$. There exist cozero subsets $U_{s,n}$ and zero subsets $F_{s,n}$ of $A\times M$ such that $U_{s,n-1}\subset F_{s,n-1}\subset U_{s,n}\subset U_s$ and $\bigcup_{n\in N} U_{s,n}=U_s$. Let K_n be a zero subset of $A\times M$ such that $\bigcup_{s\in S} F_{s,n-1}\subset K_n\subset\bigcup_{s\in S} U_{s,n}$ and let $\mathcal{B}$ be a σ-locally finite base of M; for each $B\in\mathcal{B}$ choose a point $z_B\in B$. For every $\langle s,n,B\rangle$ define $U_{s,n,B}=\{x\in A: \langle x,z_B\rangle\in U_{s,n}\}$ and for every $\langle n,B\rangle$ let $K_{n,B}=\{x\in A: \langle x,z_B\rangle\in K_n\}$. The family $\{U_{s,n,B}\}_{s\in S}$ is a locally finite cozero family in A and $K_{n,B}\subset\bigcup_{s\in S} U_{s,n,B}$. Since $K_{n,B}$ is a zero set, there exists a locally finite family $\{W_{s,n,B}\}_{s\in S}$ of cozero subsets of X such that $W_{s,n,B}\cap A\subset U_{s,n,B}$ and $K_{n,B}\subset\bigcup_{s\in S} W_{s,n,B}$. Define $\mathcal{W}=\{(W_{s,n,B}\times B)\cap U_s^*: s\in S,\ n<\omega,\ B\in\mathcal{B}\}$. Obviously, $\mathcal{W}$ is a σ-locally finite in $X\times M$ family of cozero sets and $\mathcal{W}\restriction(A\times M)\prec\mathcal{U}$. It remains to show that $A\times M\subset\bigcup\mathcal{W}$. Let $\langle x,z\rangle\in A\times M$. Choose all such $s_1,\ldots,s_k$ that $\langle x,z\rangle\in U_{s_i}$ and let n be such that $\langle x,z\rangle\in\operatorname{Int}K_n$. There is a B containing z and such that $\{x\}\times B\subset\operatorname{Int}K_n\cap\bigcap_{i=1}^{k} U_{s_i}\setminus\bigcup\{\overline{U}_{s,n}: s\neq s_i\}$. Thus, $x\in K_{n,B}$ and $x\in W_{s,n,B}$ for some $s\in S$. Consequently, $x\in W_{s,n,B}\subset U_{s,n,B}$ and $\langle x,z_B\rangle\in U_{s,n}$. This implies, that $s=s_i$ for some $i\leqslant k$. Thus $\langle x,z\rangle\in(W_{s,n,B}\times B)\cap U_s^*$, which completes the proof. ■

Remark. The above theorem actually proves that if $A\times M$ is C-embedded in $X\times M$ and $A\times C$ is C-embedded in $X\times C$ for a metric M and a compact C, then $A\times M\times C$ is C-embedded in $X\times M\times C$. It is a natural question whether the assumption that $A\times M\underset{C}{\subset}X\times M$ can be relaxed to $A\times M\underset{C^*}{\subset}X\times M$, for a non-discrete M. In other words: Does $A\times M\underset{C^*}{\subset}X\times M$ imply $A\times M\underset{C}{\subset}X\times M$ for a non-discrete M?

EXAMPLE. $\pi_{\mathcal{M}\times\mathcal{C}}$-embedding does not imply $\pi_{\mathcal{P}}$-embedding.

Proof. According to the above theorem, it suffices to give an example of a space Y and its closed subset X which is $\pi_{\mathcal{M}\cup\mathcal{C}}$-embedded but is not $\pi_{\mathcal{P}}$-embedded.

Let us represent the interval I as the union of two disjoint sets B' and C' such that $|B'|=|C'|=\mathfrak{c}$ and every compact space contained either in B' or in C' is countable. Let $B=B'\setminus Q$ and $C=C'\setminus Q$. For each $q\in Q\setminus\{0\}$ choose a sequence $\{p_{q,n}\}_{n\in N}$ of points of B converging to q, and open intervals $I_{q,n}$ with rational end-points such that $q\in I_{q,n}$, $p\notin I_{q,n}$ and the family $\{I_{q,n}\times\{p_{q,n}\}: q\in Q,\ n\in N\}$ is discrete in $I^2\setminus\Delta$.

Let us generate a topology on $Z=(B\cup C)\times I=P\times I$ by declaring that the points of the form $\langle b,0\rangle$, where $b\in B$, have neighbourhoods as in the Niemytzki plane and all remaining points have usual Euclidean neighbourhoods.

Define $Y=(B\cup C)\times I\times B\subset I^3$, $T_{q,n}=\{\langle a,0,p_{q,n}\rangle\in Y: a\in I_{q,n}\}$ and $\Delta=\{\langle b,0,b\rangle\in Y: b\in B\}$. Let us generate the topology on Y by declaring that:

(1) The points of $Y\setminus(\Delta\cup\bigcup\{T_{q,n}: q\in Q,\ n\in N\})$ have usual Euclidean neighbourhoods;

(2) Neighbourhoods U_b of the point $\langle b,0,b\rangle\in\Delta\subset Y$ are of the form $U_b=V_b\cup W_b$, where V_b is a neighbourhood of the point $\langle b,0,b\rangle$ in the product topology on $Z\times B$ and W_b is a neighbourhood of the set

$$(V_b\cap[(B\cup C)\times\{0\}\times B])\setminus\{\langle b,0,b\rangle\}$$

in the usual Euclidean topology on Y;

(3) Intervals $T_{q,n}$ have the usual topology and they are open and closed in Y.

In the sequel we shall denote the closure, the interior and the boundary of a set in the above topologies on X or Y by cl, int and Fr respectively, while the closure, the interior and the boundary of a set in the usual Euclidean topology shall be denoted by cl^m, int^m and Fr^m, respectively.

First we shall show that X is $\pi_{\mathcal{M}}$-embedded in Y. Let $M\in\mathcal{M}$ and let $f\colon X\times M\to I$ be a continuous function. A function $F\colon Y\times M\to I$ such that $f^{-1}(0)\subset F^{-1}(0)$ and $f^{-1}(1)\subset F^{-1}(1)$ will be constructed as in the proof of Urysohn's lemma. For every rational $q\in I$ we shall define an open set $U_q\subset Y\times M$ and a closed set $F_q\subset Y\times M$ such that $f^{-1}(0)\subset F_0$, $f^{-1}(1)\subset Y\times M\setminus U_1$, $\mathrm{cl}\,U_q\subset F_q$ and if $q<q'$ then $F_q\subset U_{q'}$. Then we shall define the function F by the formula

$$F(y,m)=\begin{cases}\inf\{q: \langle y,m\rangle\in U_q\} & \text{for } \langle y,m\rangle\in U_1,\\ 1 & \text{for } \langle y,m\rangle\notin U_1.\end{cases}$$

Let us arrange all rational numbers in the interval (0,1) into an infinite sequence $q_3,q_4,\ldots$ and let $q_1=0$, $q_2=1$. Let us choose the numbers $r_n\in I$ and $s_n\in I$ such that for $n\in N$ $r_n<s_n$ and if $q_n<q_m'$ then $s_n<r_m$.

Let $\mathcal{U}=\bigcup_{n\in N}\mathcal{U}_n$ be a σ-discrete base in M and let $\mathcal{G}$ be a countable base in I. For $G\in\mathcal{G}$ and $a\in B\cup C$ define

$$\mathcal{D}_{a,G}=\begin{cases}\{[G_1\cap(B\cup C)]\times\{0\}\times(G\cap B)\subset X: G_1\in\mathcal{G},\ a\in G_1\} & \text{for } a\in C\cup(B\setminus G),\\ \{[\{a\}\times\{0\}\times(G\cap B)]\cup W_a: W_a \text{ is a neighbourhood} & \\ \quad\text{of the set } \{a\}\times\{0\}\times(G\cap B) \text{ in } X \text{ and } W_a & \\ \quad\subset\{\langle x,0,y\rangle: (y\geqslant 2x-a,\ y\geqslant -2x+3a) \text{ or} & \\ \quad(y\leqslant 2x-a,\ y\leqslant -2x-3a)\}\} & \text{for } a\in B\cap G.\end{cases}$$

For $U\in\mathcal{U}$, $G\in\mathcal{G}$ and $n\in N$ define

$\tilde{A}^0_{U,G,n}=\{a\in B\cup C$: there exists $D_a\in\mathcal{D}_{a,G}$ such that for every $x\in D_a$ and for every $m\in U$ the inequality $f(x,m)<s_n$ holds$\}$,

$\tilde{A}^1_{U,G,n}=\{a\in B\cup C$: there exists $D_a\in\mathcal{D}_{a,G}$ such that for every $x\in D_a$ and for every $m\in U$ the inequality $f(x,m)>r_n$ holds$\}$,

$$A^0_{U,G,n}=\bigcup\{D_a: a\in\tilde{A}^0_{U,G,n}\},$$
$$A^1_{U,G,n}=\bigcup\{D_a: a\in\tilde{A}^1_{U,G,n}\}.$$

Let us observe that if $x \in \mathrm{cl}^m A^0_{U,G,n} \setminus \bigcup\{T_{q,n}: q \in Q, n \in N\}$ then every $m \in \mathrm{cl}\, U$ satisfies $f(x, m) \leqslant s_n$ and if $x \in \mathrm{cl}^m A^1_{U,G,n} \setminus \bigcup\{T_{q,n}: q \in Q, n \in N\}$ then every $m \in \mathrm{cl}\, U$ satisfies $f(x, m) \geqslant r_n$. It follows that for $G_1, G_2 \in \mathcal{G}$ and $U_1, U_2 \in \mathcal{U}$ such that $\mathrm{cl}\, U_1 \cap \mathrm{cl}\, U_2 \neq \emptyset$ we have

$$(*) \qquad \mathrm{cl}^m A^0_{U_1,G_1,n} \cap \mathrm{cl}^m A^1_{U_2,G_2,n} \subset \bigcup\{T_{q,n}: q \in Q, n \in N\}.$$

Let us arrange the elements of the set $\{\langle G, n\rangle: G \in \mathcal{G}, n \in N\}$ into a sequence $\langle G_1, n_1\rangle, \langle G_2, n_2\rangle, \ldots$ We shall construct inductively the sets $K^0_{i,n}$ and $K^1_{i,n}$ such that $\bigcup\{A^j_{U,G_i,n} \times U: U \in \mathcal{U}_{n_i}\} \subset \mathrm{int}^m K^j_{i,n}$ for $j \in \{0, 1\}$ and $\mathrm{cl}^m K^0_{i,n} \cap \mathrm{cl}^m K^1_{l,n} = \emptyset$ for $i, l \in N$.

We shall omit the first step in our inductive construction since it is similar to the following ones.

Assume that for $l < i$ the sets $K^j_{l,n}$ have been already defined. For $U \in \mathcal{U}_{n_i}$ and $j \in \{0, 1\}$ put

$$\tilde{K}^{j,1}_{U,i,n} = \{\langle a, t, b\rangle \in Y: \langle a, 0, b\rangle \in \mathrm{int}^m_X A^j_{U,G_i,n} \text{ and } t < 1/i\},$$

$$\tilde{K}^{j,2}_{U,i,n} = \bigcup\{[B\langle b, 1/i\rangle, 1/i\} \cup \{\langle b, 0\rangle\}] \times (G_i \cap B): \langle b, 0, b\rangle \in A^j_{U,G_i,n} \setminus \mathrm{int}^m_X A^j_{U,G_i,n}\},$$

$$\tilde{K}^j_{U,i,n} = \tilde{K}^{j,1}_{U,i,n} \cup \tilde{K}^{j,2}_{U,i,n},$$

$$\tilde{K}^j_{i,n} = \bigcup\{\tilde{K}^j_{U,i,n} \times U: U \in \mathcal{U}_{n_i}\},$$

$$A^j_{i,n} = \bigcup\{A^j_{U,G_i,n} \times U: U \in \mathcal{U}_{n_i}\}.$$

Let us observe that the sets $\tilde{K}^j_{U,i,n}$ are open in Y and that $\mathrm{cl}^m_Y \tilde{K}^j_{U,i,n} \cap X = \mathrm{cl}^m_Y A^j_{U,G_i,n}$. At the same time from $(*)$ and from the fact that the families $\mathcal{U}_{n_i}$ are discrete it follows that

$$\mathrm{cl}^m A^0_{i,n} \cap \mathrm{cl}^m A^1_{j,n} \subset [(\bigcup\{T_{q,n}: q \in Q, n \in N\}) \times M] \setminus (A^0_{i,n} \cup A^1_{j,n}).$$

Consequently,

$$\mathrm{cl}^m_{Y\times M} K^0_{i,n} \cap \mathrm{cl}^m_{Y\times M} K^1_{j,n} \cap (X \times M) \subset [(\bigcup\{T_{q,n}: q \in Q, n \in N\}) \times M] \setminus (\tilde{K}^0_{i,n} \cup \tilde{K}^1_{j,n}).$$

It follows that

$$A^0_{i,n} \subset K^0_{i,n} = \tilde{K}^0_{i,n} \setminus \bigcup_{j=1}^{i} \mathrm{cl}^m \tilde{K}^1_{j,n} \quad \text{and} \quad A^1_{i,n} \subset K^1_{i,n} = \tilde{K}^1_{i,n} \setminus \bigcup_{j=1}^{i} \mathrm{cl}^m \tilde{K}^0_{j,n}.$$

Moreover, the sets $K^0_{i,n}$ and $K^1_{i,n}$ are open. Putting

$$K^0_n = \bigcup_{i=1}^{\infty} K^0_{i,n} \cup [f^{-1}([0, s_n)) \cap \bigcup\{T_{q,n}: q \in Q, n \in N\}]$$

and

$$K^1_n = \bigcup_{i=1}^{\infty} K^1_{i,n} \cup [f^{-1}((r_n, 1]) \cap \bigcup\{T_{q,n}: q \in Q, n \in N\}]$$

we obtain

$$f^{-1}([0, s_n)) \subset \bigcup_{i=1}^{\infty} A^0_{i,n} \cup [f^{-1}([0, s_n)) \cap \bigcup\{T_{q,n}: q \in Q, n \in N\}] \subset K^0_n = \mathrm{int} K^0_n,$$

$$f^{-1}((r_n, 1]) \subset \bigcup_{i=1}^{\infty} A^1_{i,n} \cup [f^{-1}((r_n, 1]) \cap \bigcup\{T_{q,n}: q \in Q, n \in N\}] \subset K^1_n = \mathrm{int} K^1_n$$

and $K^0_n \cap K^1_n = \emptyset$.

Now, we shall construct inductively the sets $U_{q_n} = U_n$ and $F_{q_n} = F_n$. Put $U_1 = \emptyset$, $F_1 = f^{-1}(0)$, $U_2 = Y \times M \setminus f^{-1}(1)$, $F_2 = Y \times M$.

Assume that the sets U_i and F_i have been already defined for $i < n$ and that they satisfy:

(a) $f^{-1}(0) \subset F_1$, $f^{-1}(1) \subset (Y \times M) \setminus U_2$;

(b) $\mathrm{cl}\, U_i \subset F_i$;

(c) if $q_i < q_j$ then $F_i \subset U_j$;

(d) $f^{-1}([0, s_i)) \subset U_i \cap (X \times M) \subset F_i \cap (X \times M) \subset f^{-1}([0, r_i])$.

Let $n' < n$ and $n'' < n$ be such that $q_{n'} = \max\{q_i: i < n \text{ and } q_i < q_n\}$ and $q_{n''} = \min\{q_i: i < n \text{ and } q_i > q_n\}$. The sets $F_{n'} \setminus (X \times M)$ and $Y \times M \setminus [U_{n''} \cup (X \times M)]$ are disjoint and closed in $(Y \setminus X) \times M$. The topology on $Y \setminus X$ coincides with the usual Euclidean topology. Thus the product $(Y \setminus X) \times M$ is metrizable and there exist disjoint sets W_1' and W_2', containing $F_{n'} \setminus (X \times M)$ and $(Y \times M) \setminus [U_{n''} \setminus (X \times M)]$ respectively, which are open in $(Y \setminus X) \times M$ and therefore open in $Y \times M$. We shall show that there exist disjoint sets W_1'' and W_3 open in $Y \times M$ such that $F_{n'} \setminus (X \times M) \subset W_1''$ and $(X \times M) \setminus F_{n'} \subset W_3$. Let us generate a new topology on $Y' = (Y \times M) \setminus [F_{n'} \cap (X \times M)]$. The neighbourhoods of the point $\langle x, m\rangle \in (X \times M) \setminus F_{n'}$ are of the form $U \setminus F_{n'}$, where U is a neighbourhood of $\langle x, m\rangle$ in the metric topology on $Y \times M$. All other points have neighbourhoods as in the product topology on $Y \times M$. Let us observe that for the latter points the topology on $Y \times M$ coincides with the metric topology. The space Y' is metrizable and the sets $F_{n'} \setminus (X \times M)$ and $(X \times M) \setminus F_{n'}$ are closed in Y'. Thus there exist required sets W_1' and W_3 that are open in Y'. The topology of Y' is coarser than the subspace topology induced by the topology of $Y \times M$. Thus the sets W_1' and W_3 are open in $Y \times M$.

Similarly, one can verify that there exist disjoint sets W_2'' and W_4, both open in $Y \times M$, such that $(Y \times M) \setminus U_{n''} \setminus (X \times M) \subset W_2''$ and $(X \times M) \cap U_{n''} \subset W_4$. Define

$$W_1 = W_1' \cap W_1'', \quad W_2 = W_2' \cap W_2'', \quad U_n = (W_4 \cap K^0_n) \cup W_1,$$

$$F_n = (Y \times M) \setminus [(W_3 \cap \mathrm{int} K^1_n) \cup W_2].$$

One can easily check that the sets U_n and F_n satisfy conditions (a)–(d). The construction of the sets U_{q_i} and F_{q_i} and the proof of the fact that X is $\pi_{\mathcal{M}}$-embedded in Y are complete.

Now we are going to show that X is $\pi_{\mathscr{C}}$-embedded in Y. Let $\mathscr{U} = \{U_s\}_{s\in S}$ be a locally finite cover of X consisting of functionally open sets and let $S' = \{s \in S\colon U_s \cap \Delta \neq \emptyset\}$. We shall show that $|S'| \leqslant \aleph_0$. For $s \in S'$ put $B_s = \{b \in B\colon \langle b, 0, b\rangle \in U_s\}$ and choose $b_s \in B_s$ and $n_s \in N$ such that $\{b_s\} \times \{0\} \times (b_s - 1/n_s, b_s + 1/n_s) \subset U_s$.

Define $S'_n = \{s \in S'\colon n_s \geqslant n\}$. If the inequality $|S'| > \aleph_0$ held, there would exist an $n \in N$ such that $|S'_n| > \aleph_0$. Thus, there would exist an $a \in B \cup C$ such that $a \in \mathrm{cl}(\{b_s \in B\colon s \in S'_n\} \setminus \{a\})$ and each neighbourhood of $\langle a, 0, b\rangle$, where b is a point from $(a - 1/n, a) \cap B$, would intersect infinitely many sets U_s, which contradicts the fact that the cover $\mathscr{U}$ is locally finite.

As we have shown, the family $\mathscr{U}' = \{U_s\colon s \in S'\} \cup \{\bigcup_{s\in S\setminus S'} U_s\}$ is a countable, locally finite, functionally open cover of the space X. The set X is $\pi_{\mathscr{M}}$-embedded in Y, so X is also C-embedded in Y. Thus there exists a functionally open, locally finite cover $\mathscr{V}' = \{V_s\colon s \in S'\} \cup \{V\}$ of Y such that $V_s \cap X \subset U_s$ and $V \cap X \subset \bigcup\{U_s\colon s \in S \setminus S'\}$. The set $Y \setminus V$ is functionally closed and the set $\bigcup_{s\in S'} V_s$ is functionally open in Y. Moreover, $Y \setminus V \subset \bigcup_{s\in S'} V_s$. Thus there exists a functionally open set $W \subset Y$ such that $Y \setminus \bigcup_{s\in S'} V_s \subset W \subset \mathrm{cl}\, W \subset V$. Since $V \subset Y \setminus \Delta$ is metrizable and $V \cap X$ is functionally closed in V, there exists a locally finite, functionally open cover $\mathscr{V}'' = \{V_s\colon s \in S \setminus S'\}$ of V such that $V_s \cap X \subset U_s \cap V$. The family $\mathscr{V} = \{V_s\colon s \in S'\} \cup \{V_s \cap W\colon s \in S \setminus S'\}$ is a locally finite functionally open cover of the space Y such that $\mathscr{V} \restriction X \prec \mathscr{U}$. The proof that X is $\pi_{\mathscr{C}}$-embedded in Y is complete.

Now, we shall prove that X is not $\pi_{\mathscr{P}}$-embedded in Y. Let

$$\Lambda = \{\langle p_{q,n}, q, n\rangle\colon q \in Q,\ n \in N\} \subset B \times (Q \times N) \subset B \times \beta(Q \times N),$$

where Q is equipped with the discrete topology, and let $Z = \mathrm{cl}_{B\times\beta(Q\times N)} \Lambda \subset B \times \beta(Q \times N)$. We shall define a function $f\colon X \times \Lambda \to \{0, 1\}$ which is extendable to $\tilde{f}\colon X \times Z \to \{0, 1\}$, but not to $F\colon Y \times Z \to I$. Let

$$f(x, p_{q,n}, q, n) = \begin{cases} 0, & \text{if } x \notin T_{q,n}, \\ 1, & \text{if } x \in T_{q,n}. \end{cases}$$

First we shall show that f is extendable to $\tilde{f}\colon X \times Z \to \{0, 1\}$ namely that $\mathrm{cl}_{X\times B\times\beta(Q\times N)} f^{-1}(0) \cap \mathrm{cl}_{X\times B\times\beta(Q\times N)} f^{-1}(1) = \emptyset$. Let $x_0 \in X$ and let $\langle b, z_0\rangle \in Z \setminus \Lambda$. Then $z_0 \in \beta(Q \times N) \setminus (Q \times N)$. We shall consider the following two cases: $x_0 \in \Delta$ and $x_0 \notin \Delta$.

If $x_0 = \langle b_0, 0, b_0\rangle \in \Delta$, put $C_0 = \{\langle q, n\rangle\colon \langle b_0, 0, p_{q,n}\rangle \notin T_{q,n}\}$, and $C_1 = \{\langle q, n\rangle\colon \langle b_0, 0, p_{q,n}\rangle \in T_{q,n}\}$. Then the fact that $C_0 \cap C_1 = \emptyset$ implies the existence of a neighbourhood U of the point z_0 in $\beta(Q \times N)$ such that either $U \cap C_0 = \emptyset$ or $U \cap C_1 = \emptyset$. Assume that $U \cap C_0 = \emptyset$. Let V' be such a neighbourhood of the set $\{b_0\} \times \{0\} \times (B \setminus \{b_0\})$ in X that if $\langle b_0, 0, p_{q,n}\rangle \in T_{q,n}$ then

$$[(B \cup C) \times \{0\} \times \{p_{q,n}\}] \cap V' \subset T_{q,n}$$

and if $\langle b_0, 0, p_{q,n}\rangle \notin T_{q,n}$ then $V' \cap T_{q,n} = \emptyset$ (such a neighbourhood exists, because the family $\{T_{q,n}\colon q \in Q,\ n \in N\}$ is discrete in $I^2 \setminus \Delta$). Then $V = V' \cup \{\langle b_0, 0, b_0\rangle\}$ is a neighbourhood of x_0 in X and $V \times B \times U$ is a neighbourhood of $\langle x_0, z_0, b\rangle$ in $X \times B \times \beta(Q \times N)$ disjoint from $f^{-1}(0)$.

If $x_0 \notin \Delta$ there exists a neighbourhood U of the point x_0 in X which intersects at most one interval T_{q_0,n_0}. Then $U \times B \times [\beta(Q \times N) \setminus \{\langle q_0, n_0\rangle\}]$ is a neighbourhood of $\langle x_0, z_0, b\rangle$ disjoint from $f^{-1}(1)$.

We have shown that $\mathrm{cl}_{X\times B\times\beta(Q\times N)} f^{-1}(0) \cap \mathrm{cl}_{X\times B\times\beta(Q\times N)} f^{-1}(1) = \emptyset$. We have to verify that there does not exist any extension $F\colon Y \times Z \to I$ of f. We shall prove that there exist no disjoint sets U_0, U_1, open in $Y \times Z$ and containing $f^{-1}(0)$ and $f^{-1}(1)$ respectively. Assume the contrary. Then for each $b \in B$ and $z \in \beta(Q \times N)$ such that $\langle b, z\rangle \in Z$, there exist basic neighbourhoods V, U and W of b in B, of z in $\beta(Q \times N)$ and of $\langle b, 0, b\rangle$ in Y respectively, satisfying $W \times [(V \times U) \cap Z] \subset U_0$ or $W \times [(V \times U) \cap Z] \subset U_1$.

Since for every $b \in B$ the set $Z \cap [\{b\} \times \beta(Q \times N)]$ is compact, there exists a neighbourhood V_b of b in B and a neighbourhood W_b of $\langle b, 0, b\rangle$ in Y such that for every $z \in \beta(Q \times N)$ either $W_b \times [(V_b \times \{z\}) \cap Z] \subset U_0$ or $W_b \times [(V_b \times \{z\}) \cap Z] \subset U_1$. The set B is not a first category set. Thus there exist: a set $B_0 \subset B$ dense in some set U open in I, a set V open in B, and an integer n, such that for each $b \in B_0$ we have $V_b = V$ and $W_{b,n} \times V \subset W_b$, where $W_{b,n}$ is a disc of radius $1/n$ tangent to $I \times \{0\}$ at $\langle b, 0\rangle$.

Take $q \in Q \cap U$ and $m \in N$ such that $p_{q,m} \in V$ and $U \not\subset I_{q,n}$, and take $b_0, b_1 \in B$ such that $b_0 \in (U \cap B_0) \setminus I_{q,n}$, $b_1 \in U \cap B_0 \cap I_{q,n}$ and $|b_0 - b_1| < 1/n$. Then $W_{b_0,n} \cap W_{b_1,n} \neq \emptyset$, and thus there exist $y \in (W_{b_0,n} \cap W_{b_1,n}) \times \{p_{q,n}\}$ and

$$w = \langle y, p_{q,m}, q, m\rangle \in (W_{b_0,n} \times V \times V \times \{\langle q, m\rangle\}) \cap (W_{b_1,n} \times V \times V \times \{\langle q, m\rangle\}).$$

At the same time $w_0 = \langle b_0, 0, p_{q,m}, p_{q,m}, q, m\rangle \in W_0 = W_{b_0,n} \times V \times V \times \{\langle q, m\rangle\}$, $w_1 = \langle b_1, 0, p_{q,m}, p_{q,m}, q, m\rangle \in W_1 = W_{b_1,n} \times V \times V \times \{\langle q, m\rangle\}$, $\tilde{f}(w_0) = f(w_0) = 0$ and $\tilde{f}(w_1) = f(w_1) = 1$. Since the sets $W_0 \cap (Y \times Z)$ and $W_1 \cap (Y \times Z)$ are contained in U_0 or in U_1, we have $W_0 \cap (Y \times Z) \subset U_0$ and $W_1 \cap (Y \times Z) \subset U_1$. Thus $w \in W_0 \cap (Y \times Z) \cap W_1 \subset U_0 \cap U_1 = \emptyset$ which is a contradiction. ■

Remark 1. This example together with the theorem implies that not every closed subset of a paracompact p-space is π-embedded (see [3], Example 1.7).

Remark 2. The above example actually proves that $\pi_{\mathscr{M}\times\mathscr{C}}$-embedding does not imply $\pi_{\mathscr{K}}$-embedding, where $\mathscr{K}$ denotes the class of all Čech-complete Lindelöf spaces. Thus, the example together with the theorem implies that Čech-complete Lindelöf space need not be absolutely P-embeddable and consequently it answers in the negative problem raised by K. Morita in [1].

References

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A correction of two papers concerning Hilbert manifolds

by

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In [8] the author has claimed the following result: if X is a complete separable ANR and there is a Z-set K in X such that $X\setminus K$ is a manifold modelled on the Hilbert space l_2, then X is a Hilbert manifold itself. This result is false as will be seen from an example due to P. Bowers and J. Walsh and presented in §A. The author would like to express his sincere thanks to his colleagues from the University of Tennessee, and particularly to M. Bestvina, P. Bowers and J. Walsh, for referring this example to him.

The purpose of this corrigendum is to prove in §B a modified version of the result claimed in [8] and to introduce in §C necessary changes in another author's paper [9] where that result has been used as a technical tool in establishing a characterization of l_2-manifolds. In the appendix we include a version of Bing's shrinking criterion which allows to consider in §B certain manifolds modelled on spaces different from l_2, as is done in [8].

All the unexplained notions have the meaning of [8] or of [9]. Function spaces are considered in the limitation topology and we say that f is approximable by functions in a set S if $f\in \operatorname{cl} S$.

§A. Z-sets versus $\bar{Z}$-sets. Let us say that K is a Z-set in X (resp. a $\bar{Z}$-set in X) if it is closed in X and id_X is approximable by maps $f\colon X\to X$ satisfying $f(X)\cap K=\emptyset$ (resp. $\operatorname{cl} f(X)\cap K=\emptyset$, cl denoting closure). In [6] D. W. Henderson has given characterizations of Z-sets in ANR's (cf. also [4], [1]) and, in connection with an application in [2] and [10] and a question posed on a conference in Oberwolfach in 1970, has established the following fact:

A1. Let X be a manifold modelled on a locally convex metric linear space E such that $E\cong E^\infty$ or $E\cong E^\infty_f=\{(x_i)\in E^\infty\colon x_i=0 \text{ for almost all } i\text{'s}\}$. Then the family of $\bar{Z}$-sets in X coincides with that of Z-sets in X.

It was also known to Henderson that these families may differ if X is merely an ANR (oral communication to the author from 1972). The following example illustrating this is due to P. Bowers and J. Walsh:

A2. Example. Let $X=[0,1]\times\{0\}\cup\{2^{-n}\colon n\in N\}\times[0,1]\subset I^2$ and $p=(0,0)$. Then $K=\{p\}$ is a Z-set in X but is not a $\bar{Z}$-set in X.