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Products of Baire topological vector spaces

by

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Abstract. Let I be a set whose cardinal number is larger than $\aleph_0$. In this article it is proved that there are dense subspaces of $c_0(I)$ with additional properties which are Baire and whose product is not Baire. The same properties are obtained taking $l^p(I)$, $0<p<\infty$, instead of $c_0(I)$.

1. Introduction. All the linear spaces we shall use are defined over the field K of real or complex numbers. If E is a set, we denote by E^ω the countable infinite product of copies of E.

Oxtoby [5] proved that the continuum hypothesis implies that there is a completely regular Baire topological space whose square is not Baire. Actually, Oxtoby uses the hypothesis that the union of $<2^{\aleph_0}$ subsets of Lebesgue measure zero of real numbers has Lebesgue measure zero. P. E. Cohen [2], using forcing techniques, gave an absolute example of Baire spaces whose product is not Baire. Later Fleissner and Kunen [3] gave new examples of Baire spaces whose products is not Baire space without using additional hypothesis of the theory of sets. In this article we give examples of Baire topological vector spaces whose product is not Baire using in part techniques of Fleissner and Kunen [3].

Given a set I and an ordinal α we denote by $\operatorname{card} I$ and $\operatorname{card}\alpha$ the cardinal numbers of I and α respectively. If $\beta<\alpha$, $[\beta,\alpha[$ is the interval of ordinal numbers closed in β and open in α, i.e.,

$$[\beta,\alpha[\,=\{\delta\colon \beta\leqslant\delta<\alpha\}\,.$$

We represent by ω_1 the first ordinal such that $\operatorname{card}\omega_1>\aleph_0$. We suppose $[0,\alpha]$ endowed with the order topology. A subset of $[0,\alpha[$ is said to be stationary if it meets every unbounded closed subset of $[0,\alpha]$. Let γ be the first ordinal such that $\operatorname{card} I=\operatorname{card}\gamma$ and let T_n be a mapping from $[0,\gamma[$ in $[0,\gamma[$, $n=1,2,\dots$ We shall need the following results:

(a) *If* $\operatorname{card}\gamma>\aleph_0$ *the set*

$$\{\alpha<\gamma\colon T_n([0,\alpha[)\subset[0,\alpha[,\ n=1,2,\dots\}$$

is unbounded and closed in $[0,\gamma[$,

(b) *If* card $\gamma > \aleph_0$ *then any stationary subset of* $[0, \gamma[$ *can be split into* card γ *disjoint stationary subset of* $[0, \gamma[$ (see [4] and [8]).

2. Topological properties. Let I be an uncountable set. Let γ be the first ordinal such that card I = card γ. Let (M_n) be a sequence of stationary subsets of $[0, \gamma[$ pairwise disjoint. We set $M = \bigcup \{M_n: n = 1, 2, \ldots\}$. Let E be a Hausdorff topological space whose points are families $\{x_\beta: \beta < \gamma\}$ with

$$x_\beta \in K, \quad \operatorname{card}\{\beta: x_\beta \neq 0\} < \operatorname{card}\gamma .$$

We set O to denote the point $\{x_\beta: \beta<\gamma\}$ with $x_\beta = 0$, $\beta<\gamma$, and we suppose that $O \in E$.

For every α in M we set

$$E_\alpha = \{\{x_\beta: \beta<\gamma\} \in E: x_\beta = 0 \text{ if } \beta \geqslant \alpha\}$$

and we suppose E_α endowed with topology induced by E. Let $\mathscr{A}_\alpha$ be a family of open subsets of E such that $A \cap E_\alpha \neq \emptyset$ for every A in $\mathscr{A}_\alpha$ and

$$\{A \cap E_\alpha: A \in \mathscr{A}_\alpha\}$$

is a base for the topology of E_α. We suppose that

$$\operatorname{card}\mathscr{A}_\alpha < \operatorname{card} I$$

and if α is an ordinal limit of elements of M then

$$\mathscr{A}_\alpha = \bigcup \{\mathscr{A}_\beta: \beta<\alpha, \beta \in M\} .$$

Moreover we suppose that

$$\mathscr{A} = \bigcup \{\mathscr{A}_\alpha: \alpha \in M\}$$

is a base of the topology of E and if $\alpha_1 < \alpha_2$, α_1 and α_2 in M, then $\mathscr{A}_{\alpha_1} \subset \mathscr{A}_{\alpha_2}$.

PROPOSITION 1. *If for every* α *in* M_m, E_α^m *is a Baire space*, $m = 1, 2, \ldots$, *then every product of a finite number of copies of* E *is a Baire space.*

Proof. Given a positive number p, let $\{D_n: n = 1, 2, \ldots\}$ be a family of dense open subsets of E^p. We fix a positive integer n and β in M. If $A \in \mathscr{A}_\beta$ we find $D_n(A)$ in $\mathscr{A}$ such that

$$D_n(A)^p \subset D_n \cap A^p. \tag{1}$$

Since $\operatorname{card}\mathscr{A}_\beta < \operatorname{card} I$ there is an ordinal number $S_n(\beta)<\gamma$ such that $D_n(A)$ belongs to $\mathscr{A}_{S_n(\beta)}$ for every A in $\mathscr{A}_\beta$. Given any α in $[0, \gamma[$ let $\beta(\alpha)$ be the first ordinal in $[\alpha, \gamma[\cap M$. We set $T_n(\alpha) = S_n(\beta(\alpha))$. We apply result (a) to the family $\{T_n: n = 1, 2, \ldots\}$ of mappings from $[0, \gamma[$ in $[0, \gamma[$ to obtain that the set

$$V = \{\alpha<\gamma: T_n([0, \alpha[) \subset [0, \alpha[, \ n = 1, 2, \ldots\}$$

is unbounded and closed in $[0, \gamma[$.

Let us see now that $D = \bigcap \{D_n: n = 1, 2, \ldots\}$ is dense in E^p. We take $A \in \mathscr{A}_\alpha$, $\alpha \in M$. If X is the set of points of $[0, \gamma[$ which are limits of points of $[\alpha+1, \gamma[\cap M$ it is obvious that X is unbounded and closed in $[0, \gamma[$ and since M_p is stationary in $[0, \gamma[$ we can take an ordinal β in $M_p \cap X \cap V$. We fix now a positive integer n and we take any element B in $\mathscr{A}_\beta$. Since β is an ordinal number which is limit of elements of M, there is $\delta<\beta$, $\delta \in M$, such that B belongs to $\mathscr{A}_\delta$ and, therefore,

$$S_n(\delta) = T_n(\delta) < \beta \quad \text{and} \quad D_n(B) \in \mathscr{A}_{T_n(\delta)}$$

and thus $D_n(B)$ belongs to $\mathscr{A}_\beta$. From (1) it follows that

$$B^p \cap E_\beta^p \cap D_n \supset E_\beta^p \cap D_n(B)^p \neq \emptyset .$$

Consequently, $D_n \cap E_\beta^p$ is dense in E_β^p, $n = 1, 2, \ldots$, and since E_β^p is a Baire space, $D \cap E_\beta^p$ is dense in E_β^p. One has that $A \in \mathscr{A}_\beta$ and, therefore, $A^p \cap E_\beta^p$ is a nonvoid open subset of E_β^p from where it follows that

$$A^p \cap D \supset A^p \cap E_\beta^p \cap D \neq \emptyset .$$

Consequently, E^p is a Baire space. ∎

For every α in M we set

$$\mathscr{B}_\alpha = \{A_1 \times A_2 \times \ldots \times A_p \times E^\omega: A_1, A_2, \ldots, A_p \in \mathscr{A}_\alpha, \ p = 1, 2, \ldots\} .$$

One has that $B \cap E_\alpha^\omega \neq \emptyset$, $B \in \mathscr{B}_\alpha$, and

$$\{B \cap E_\alpha^\omega: B \in \mathscr{B}_\alpha\}$$

is a base of the topology of E_α^ω, $\operatorname{card}\mathscr{B}_\alpha < \operatorname{card} I$, and if α is an ordinal limit of elements of M, then

$$\mathscr{B}_\alpha = \bigcup \{\mathscr{B}_\beta: \beta<\alpha, \beta \in M\} .$$

Moreover one has that

$$\mathscr{B} = \bigcup \{\mathscr{B}_\alpha: \alpha \in M\}$$

is a base of the topology of E^ω and if $\alpha_1 < \alpha_2$, $\alpha_1, \alpha_2 \in M$, then $\mathscr{B}_{\alpha_1}$ is contained in $\mathscr{B}_{\alpha_2}$.

PROPOSITION 2. *If for every* α *in* M, E_α^ω *is a Baire space, then* E^ω *is a Baire space.*

Proof. Let $\{D_n: n = 1, 2, \ldots\}$ be a family of dense open subsets of E^ω. We fix a positive integer n and we take β in M. If B belongs to $\mathscr{B}_\beta$ we find $D_n(B)$ in $\mathscr{B}$ such that $D_n(B)$ is contained in $B \cap D_n$. There is an ordinal $S_n(\beta)<\gamma$ such that $D_n(B)$ belongs to $\mathscr{B}_{S_n(\beta)}$ for every B in $\mathscr{B}_\beta$. We proceed as we did in the proof of the previous proposition to obtain $\{T_n: n = 1, 2, \ldots\}$ and V.

Let us see now that $D = \bigcap \{D_n: n = 1, 2, \ldots\}$ is dense in E^ω. We take α in M and B in $\mathscr{B}_\alpha$, $\alpha<\gamma$. We obtain X as before and we take β in $M \cap X \cap V$. Proceeding again as in the previous proof we obtain that $D \cap E_\beta^\omega$ is dense in E_β^ω from where it follows easily that $D \cap B \neq \emptyset$ and thus E^ω is a Baire space. ∎

If $x = \{x_\beta\colon \beta<\gamma\}$ is an element of E and Q is a subset of $[0,\gamma[$ we set

$$x(Q) = \{y_\beta\colon \beta<\gamma\} \quad \text{with } y_\beta = x_\beta \text{ if } \beta\in Q \quad \text{and} \quad y_\beta = 0 \text{ if } \beta\notin Q\,.$$

We set

$$F = \{x(Q)\colon x\in E,\ Q\subset[0,\gamma[\}\,.$$

We suppose F endowed with a topology defined by a metric d such that

$$d\big(x(Q), y(Q)\big)\leqslant d(x,y)\,, \qquad x, y\in E\,, \qquad Q\subset[0,\gamma[$$

and E is a topological subspace of F.

For every x in E and every positive integer n we write

$$g(x) = \sup\{\beta<\gamma\colon x_\beta\neq 0\}\,, \qquad g_n(x) = \min\{\beta<\gamma\colon d\big(x, x([0,\beta[)\big)<1/n\}$$

and we suppose, when $x_{g(x)} = 0$, $g(x)>0$, that

$$\lim_{\beta<g(x)} \{x([0,\beta[)\colon <\} = x\,. \tag{2}$$

Proposition 3. *If $x = \{x_\beta\colon \beta<\gamma\}$ belongs to E and $\alpha<g(x)$, there is a neighbourhood U of x in E such that $\alpha<g(z)$ for every z in U.*

Proof. We find an ordinal δ such that $\alpha<\delta\leqslant g(x)$ and $x_\delta\neq 0$. Since the topology of F is finer than the topology of the pointwise convergence, there is a neighbourhood U of x in E such that if $z = \{z_\beta\colon \beta<\gamma\}$ is in U, then $z_\delta\neq 0$ and consequently $\alpha<g(z)$. ■

Proposition 4. *Given a positive integer n and $x = \{x_\beta\colon \beta<\gamma\}$ in E there is a neighbourhood U of x in E such that $g_n(z)\leqslant g_n(x)$ for every z in U.*

Proof. We take $\varepsilon>0$ such that

$$d\big(x([0, g_n(x)[), x\big)+2\varepsilon<1/n\,.$$

We set

$$U = \{y\in E\colon d(x,y)<\varepsilon\}\,.$$

Then, if z belongs to U, one has that

$$d\big(z([0, g_n(x)[), z\big)\leqslant d\big(z([0, g_n(x)[), x([0, g_n(x)[)\big)+ \\ + d\big(x([0, g_n(x)[), x\big)+d(x,z)\leqslant 2\varepsilon+d\big(x([0, g_n(x)[), x\big)<1/n$$

and thus $g_n(z)\leqslant g_n(x)$. ■

For every pair of positive integers p and q we write

$$F_{p,q} = \{x\in E\colon x_{g(x)} = 0,\ g(x)\in \bigcup\{M_j\colon j\leqslant q,\ j\neq p\}\}\cup\{0\}\,,$$
$$F_q = \prod\{F_{j,q}\colon j = 1,2,\ldots,q\}\,,$$
$$G_q = \{x\in E\colon x_{g(x)} = 0,\ g(x)\in\bigcup\{M_j\colon j = 1,2,\ldots,q-1,q+1,\ldots\}\}\cup\{0\}\,,$$
$$G = \prod\{G_q\colon q = 1,2,\ldots\}\,.$$

We suppose that given x in $F_{p,q}$, $\varepsilon>0$ and $\alpha<\gamma$ there is an element z in $F_{p,q}$ such that $d(x,z)<\varepsilon$ and $\alpha<g(z)$.

Proposition 5. *Given an integer $q>1$, F_q is not a Baire space.*

Proof. For every positive integer n, we set

$$D_n = \{x = (x_1, x_2, \ldots, x_q)\in F_q\colon \min\{g(x_j)\colon j = 1,2,\ldots,q\} \\ > \max\{g_n(x_j)\colon j = 1,2,\ldots,q\}\}\,.$$

We apply Propositions 3 and 4 to obtain a neighbourhood V_j of x_j such that

$$g(z_j)>\max\{g_n(x_i)\colon i = 1,2,\ldots,q\}\,,$$
$$g_n(z_j)\leqslant g_n(x_j)\,, \qquad z_j\in V_j,\ j = 1,2,\ldots,q\,.$$

Consequently,

$$\min\{g(z_j)\colon j = 1,2,\ldots,q\}>\max\{g_n(z_j)\colon j = 1,2,\ldots,q\}\,,$$
$$z_j\in V_j,\ j = 1,2,\ldots,q\,,$$

and thus D_n is open.

Given a neighbourhood V of a point $y = (y_1, y_2, \ldots, y_q)$ of F_q we find $\varepsilon>0$ and a neighbourhood W of y, $W\subset V$, such that

$$\{x = (x_1, x_2, \ldots, x_q)\in F_q\colon d(x_i, y_i)<\varepsilon\,,\ i = 1,2,\ldots,q\}\subset W\,,$$
$$g_n(u_j)\leqslant g_n(y_j)\,, \qquad (u_1, u_2, \ldots, u_q)\in W\,, \qquad j = 1,2,\ldots,q\,.$$

We can find $z = (z_1, z_2, \ldots, z_q)$ in F_q such that

$$\max\{g(y_i)\colon i = 1,2,\ldots,q\}<g(z_j)\,, \qquad d(y_j, z_j)<\varepsilon\,, \qquad j = 1,2,\ldots,q\,.$$

Then $z\in W$ and

$$\min\{g(z_i)\colon i = 1,2,\ldots,q\}>\max\{g(y_i)\colon i = 1,2,\ldots,q\} \\ \geqslant\max\{g_n(y_i)\colon i = 1,2,\ldots,q\}\geqslant\max\{g_n(z_i)\colon i = 1,2,\ldots,q\}\,.$$

Therefore $z\in W\cap D_n$ from where it follows that $V\cap D_n\neq\emptyset$ and thus D_n is dense in F_q.

Finally, let $x = (x_1, x_2, \ldots, x_q)$ be any point of F_q with $x_j\neq 0$, $j = 1,2,\ldots,q$. We can take two positive integers $h\neq k$, $h\leqslant q$, $k\leqslant q$ with $g(x_h)\neq g(x_k)$. Suppose that $g(x_h)<g(x_k)$. From (2) it follows that there is a positive integer n such that $g(x_h)<g_n(x_k)$. Then obviously x is not in D_n. We set

$$D_0 = \{x = (x_1, x_2, \ldots, x_q)\in F_q,\ x_j\neq 0,\ j = 1,2,\ldots,q\}\,.$$

Then D_0 is a dense open subset of F_q. We obtain now

$$\bigcap\{D_n\colon n = 0,1,2,\ldots\} = \emptyset\,.$$

Proposition 6. *The space G is not Baire.*

Proof. For every positive integer n we set

$$D_n = \{x = (x_1, x_2, \ldots) \in G:\ \min\{g(x_j): j = 1, 2, \ldots, n\} > \max\{g_n(x_j): j = 1, 2, \ldots, n\}\}.$$

As in Proposition 5, changing q by n, D_n is an open set. Given a neighbourhood V of a point $y = (y_1, y_2, \ldots)$ of G we can find an integer $m > n$ and a neighbourhood W of y, $W \subset V$, such that

$$\{x = (x_1, x_2, \ldots) \in G:\ d(x_i, y_i) < \varepsilon,\ i = 1, 2, \ldots, m\} \subset W,$$

$$g_n(u_j) \leqslant g_n(y_i), \quad (u_1, u_2, \ldots) \in W, \quad j = 1, 2, \ldots, n.$$

We can find $z = (z_1, z_2, \ldots)$ in G such that

$$\max\{g(y_i): i = 1, 2, \ldots, n\} < g(z_j), \quad d(y_j, z_j) < \varepsilon, \quad j = 1, 2, \ldots, m.$$

Then $z \in W$ and

$$\min\{g(z_i): i = 1, 2, \ldots, n\} > \max\{g(y_i): i = 1, 2, \ldots, n\} \geqslant \max\{g_n(y_i): i = 1, 2, \ldots, n\} \geqslant \max\{g_n(z_i): i = 1, 2, \ldots, n\}.$$

Therefore z belongs to $W \cap D_n$ from where it follows that $V \cap D_n \neq \emptyset$ and thus D_n is dense in G.

Finally, let $x = (x_1, x_2, \ldots)$ be any point of G with $x_j \neq 0$, $j = 1, 2, \ldots$ We can take two positive integers $h \neq k$ with $g(x_h) \neq g(x_k)$. Suppose $g(x_h) < g(x_k)$. From (2) it follows that there is a positive integer $n > \max(h, k)$ such that $g(x_h) < g_n(x_k)$. Then obviously x is not in D_n. We set $D_0 = \{x = (x_1, x_2, \ldots, x_j, \ldots) \in G: x_j \neq 0, j = 1, 2, \ldots\}$. Then D_0 is a countable intersection of dense open subsets of G. We obtain

$$\bigcap \{D_n: n = 0, 1, 2, \ldots\} = \emptyset. \ ■$$

3. Topological vector properties. Let I be a noncountable set. We set γ to denote the first ordinal such that $\operatorname{card}\gamma = \operatorname{card} I$. We write $c_0(\gamma)$ instead of $c_0([0, \gamma[)$. Obviously $c_0(\gamma)$ is a Banach space isomorphic to $c_0(I)$. If $\alpha < \gamma$ we set

$$L(\alpha) = \{x = \{x_\beta: \beta < \gamma\} \in c_0(\gamma):\ x_\beta = 0,\ \beta \geqslant \alpha\}.$$

We fix an ordinal number α such that there is a sequence of ordinal numbers

$$0 = \alpha_0 < \alpha_1 < \ldots < \alpha_n < \ldots$$

converging to α with

$$\operatorname{card}[0, \alpha_n[= \operatorname{card}[\alpha_{n-1}, \alpha_n[, \quad n = 1, 2, \ldots$$

Let Y be an algebraic complement of

$$\bigcup \{L(\alpha_n): n = 1, 2, \ldots\}$$

in $L(\alpha)$. We fix a positive integer m. One has that

$$\bigcup \{(Y + L(\alpha_n))^m: n = 1, 2, \ldots\} = L(\alpha)^m$$

and therefore there is a positive integer p such that $(Y + L(\alpha_p))^m$ is a dense Baire subspace of the Banach space $L(\alpha)^m$. For every $x = \{x_\beta: \beta < \gamma\}$ of $L(\alpha)$ we set

$$Tx = \{y_\beta: \alpha_p \leqslant \beta < \alpha\}, \quad y_\beta = x_\beta, \quad \alpha_p \leqslant \beta < \alpha.$$

It is immediate that T is a topological homomorphism from $L(\alpha)$ onto $c_0([\alpha_p, \alpha[)$. Let φ be a bijective mapping from $[\alpha_p, \alpha_{p+1}[$ onto $[0, \alpha_{p+1}[$. If $x = \{x_\beta: \alpha_p \leqslant \beta < \alpha\}$ belongs to $c_0([\alpha_p, \alpha[)$ we write

$$Sx = \{y_\beta: \beta < \gamma\}, \quad y_{\varphi(\beta)} = x_\beta \text{ if } \alpha_p \leqslant \beta < \alpha_{p+1},$$

$$y_\beta = x_\beta \text{ if } \alpha_{p+1} \leqslant \beta < \alpha, \quad \text{and} \quad y_\beta = 0 \text{ if } \alpha \leqslant \beta < \gamma.$$

It is obvious that S is a topological isomorphism from $c_0([\alpha_p, \alpha[)$ onto $L(\alpha)$. Since $(Y + L(\alpha_p))^m$ is a dense Baire subspace of $L(\alpha)^m$ it follows that

$$S \circ T(Y + L(\alpha_p)) = H_\alpha$$

is dense in $L(\alpha)$ and H_α^m is a Baire space. On the other hand, if $x = \{x_\beta: \beta < \gamma\}$ is a nonzero vector of H_α it is immediate that

$$g(x) = \sup\{\beta < \gamma: x_\beta \neq 0\} = \alpha.$$

Now we represent by Z the subset of $[0, \gamma[$ such that α belongs to Z, if and only if there is a sequence

$$0 = \alpha_0 < \alpha_1 < \ldots < \alpha_n < \ldots$$

converging to α and

$$\operatorname{card}[0, \alpha_n[= \operatorname{card}[\alpha_{n-1}, \alpha_n[, \quad n = 1, 2, \ldots$$

PROPOSITION 7. *Z is a stationary subset of $[0, \gamma[$.*

Proof. Let H be an unbounded closed subset of $[0, \gamma[$. We can take $\beta_1 < \beta_2 < \ldots < \beta_n < \ldots$ in H such that

$$\operatorname{card}[0, \beta_n[= \operatorname{card}[\beta_{n-1}, \beta_n[, \quad n = 1, 2, \ldots$$

If

$$\beta = \sup\{\beta_n: n = 1, 2, \ldots\}$$

then β belongs to H and also to Z. Thus $Z \cap H \neq \emptyset$. ■

We apply result (b) to split Z into a disjoint family

$$\{Z_{\beta, p}: \beta < \gamma, p = 1, 2, \ldots\}$$

of stationary subsets of $[0, \gamma[$. We set

$$Z_\beta = \bigcup \{Z_{\beta, p}: p = 1, 2, \ldots\}.$$

We fix $\beta<\gamma$. If α belongs to $Z_{\beta,p}$ we take H_α such that H_α^p is a Baire space and we set E_α to denote the linear hull of

$$\{H_\delta:\ \delta\leqslant\alpha,\ \delta\in Z_\beta\}.$$

Since H_α is dense in $L(\alpha)$ it follows that E_α^p is a Baire space. If X_β is the linear hull of

$$\{E_\alpha:\ \alpha\in Z_\beta\}$$

it is obvious that X_β is dense in $c_0(\gamma)$.

We take $\beta_1<\beta_2<\ldots<\beta_n<\ldots$ in $[0,\gamma[$. We set $F_{p,q}$ to denote the linear hull of

$$\bigcup\{X_{\beta_j}:\ j\leqslant q,\ j\neq p\}$$

and G_q for the linear hull of

$$\bigcup\{X_{\beta_j}:\ j=1,2,\ldots,q-1,\ q+1,\ldots\}.$$

We set

$$F_q=\prod\{F_{j,q}:\ j=1,2,\ldots,q\}\quad\text{and}\quad G=\prod\{G_q:\ q=1,2,\ldots\}.$$

If E is the linear hull of

$$\bigcup\{X_{\beta_j}:\ j=1,2,\ldots\}$$

it is immediate that the distance deduced from the norm $\|\cdot\|$ of $c_0(\gamma)$ satisfies the properties required for d in F in the former section. On the other hand, given $x=\{x_\delta:\ \delta<\gamma\}$ in $F_{p,q}$, $\varepsilon>0$, $\alpha<\gamma$, let r be a positive integer $r\neq p$, $r\leqslant q$. We find $\delta_1>\delta_2>\alpha$, $\delta_2>g(x)$, $\delta_1,\delta_2\in Z_{\beta_r}$ and we take

$$y=\{y_\delta:\ \delta<\gamma\},\quad y_\delta=0,\quad \delta\neq\delta_2,\quad y_{\delta_2}=\varepsilon/2.$$

Since H_{δ_1} is dense in $L(\delta_1)$ there is an element $z=\{z_\delta:\ \delta<\gamma\}$ in H_{δ_1} such that

$$\|z-(x+y)\|<\varepsilon/2.$$

Then z belongs to $F_{p,q}$ and

$$\|x-z\|\leqslant\|z-(x+y)\|+\|y\|<\varepsilon.$$

Consequently, Propositions 5 and 6 can be applied to F_q and G.

THEOREM 1. *There is a family $\{X_i:\ i\in I\}$ of dense subspaces of $c_0(I)$ satisfying the following properties*:

1. *For every positive integer m, H_i^m is a Baire space, $i\in I$.*
2. *$X_i\times X_j$ is not a Baire space, $i,j\in I$, $i\neq j$.*
3. *X_i^m is not homeomorphic to X_j^m for every positive integer m, $i,j\in I$, $i\neq j$.*

Proof. Since $c_0(I)$ is topologically isomorphic to $c_0(\gamma)$, it is enough to prove properties 1, 2, and 3 for the family $\{X_\beta:\ \beta<\gamma\}$ of dense subspaces of $c_0(\gamma)$.

We fix $\beta<\gamma$ and we take α in Z_β. One has that E_α coincides with

$$\{x=\{x_\delta:\ \delta<\gamma\}\in X_\beta:\ x_\delta=0,\ \delta\geqslant\alpha\}.$$

There is a dense subset U_α of E_α whose cardinal number coincides with cardα such that if $\alpha_1<\alpha_2$, $\alpha_1,\alpha_2\in Z_\beta$ then $U_{\alpha_1}\subset U_{\alpha_2}$ and if α is an ordinal number which is limit of elements of Z_β then

$$U_\alpha=\bigcup\{U_\delta:\ \delta<\alpha,\ \delta\in Z_\beta\}.$$

If x belongs to U_α and if n is a positive integer we set

$$A(x,n)=\{z\in F_\beta:\ \|x-z\|<1/n\}.$$

We set

$$\mathscr{A}_\alpha=\{A(x,n):\ x\in U_\alpha,\ n=1,2,\ldots\}.$$

The family $\{A_\alpha:\ \alpha\in Z_\beta\}$ satisfies the conditions required in the former section with X_β instead of E and $Z_{\beta,m}$ instead of M_m, $m=1,2,\ldots$ Consequently, Proposition 1 can be applied to obtain that X_β^m is a Baire space.

Taking $\beta_1<\beta_2$ in $[0,\gamma[$ one has that $F_{1,2}$ and $F_{2,2}$ coincide with X_{β_1} and X_{β_2} respectively. Consequently, $X_{\beta_1}\times X_{\beta_2}$ is not a Baire space. Finally, if X_{β_1} were homeomorphic to $X_{\beta_2}^m$ one would have that $X_{\beta_1}^m\times X_{\beta_2}^m$ which is not a Baire space would be homeomorphic to the Baire space $X_{\beta_1}^{2m}$, $m=1,2,\ldots$ This is a contradiction. ■

THEOREM 2. *Given an integer $q>1$ there are q dense subspaces $L_1,L_2,\ldots,L_q$ in $c_0(I)$ satisfying the following properties*:

1. *L_j^ω is a Baire space, $j=1,2,\ldots,q$.*
2. *$\prod\{L_j:\ j=1,2,\ldots,p-1,p+1,\ldots,q\}$ is a Baire space, $p=1,2,\ldots,q$.*
3. *$\prod\{L_j:\ j\leqslant q\}$ is not a Baire space.*

Proof. Since $c_0(I)$ is isomorphic to $c_0(\omega_1)\times c_0(I)$, if we determine q dense subspaces $L_1,L_2,\ldots,L_q$ of $c_0(\omega_1)$ satisfying conditions 1, 2 and 3, then $L_1\times c_0(I)$, $L_2\times c_0(I),\ldots,L_q\times c_0(I)$ are spaces isomorphic to dense subspaces of $c_0(I)$ satisfying conditions 1, 2 and 3 (observe that the product of a metrizable Baire space and a metrizable complete space is Baire [9]). So we can suppose card$I=\omega_1$. We take

$$L_j=F_{j,q},\quad j=1,2,\ldots,q,$$

from where it follows that $\prod\{L_j:\ j=1,2,\ldots,q\}$ is not a Baire space. For every $\alpha\in Z$, H_α is separable and consequently H_α^ω is a Baire space [5]. Therefore Proposition 2 can be applied to obtain that X_β^ω is a Baire space $\beta<\omega_1$. Since X_{β_p}, $p\leqslant q$, is a dense subspace of L_j, $j\neq p$, $j\leqslant q$, it follows that L_j^ω and $\prod\{L_j:\ j=1,2,\ldots,p-1,p+1,\ldots,q\}$ are Baire spaces. ■

THEOREM 3. *There is a countable family $L_1,L_2,\ldots,L_n,\ldots$ of dense subspaces of $c_0(I)$ satisfying the following properties*:

1. *L_j^ω is a Baire space, $j=1,2,\ldots$*
2. *$\prod\{L_j:\ j=1,2,\ldots,p-1,p+1,\ldots\}$ is a Baire space, $p=1,2,\ldots$*
3. *$\prod\{L_j:\ j=1,2,\ldots\}$ is not a Baire space.*

Proof. It is analogous to the previous one taking $L_j=G_j$, $j=1,2,\ldots$ ■

Note 1. Theorems 1, 2 and 3 are also valid changing $c_0(I)$ by $l^p(I)$, $0<p<\infty$, with the same proofs.

Note 2. In [1] the construction of Fleissner and Kunen [3] and the method of "coordinates" are used also to obtain dense Baire subspaces of $l^2(\omega_1)$ whose product is not a Baire space, but unfortunately the example given there does not satisfy the required properties.

Note 3. Independently of us R. Pol and J. van Mill have obtained normed Baire spaces with product non-Baire. They have proven in [7] that in a F-space of weight $\aleph_1$ there are dense Baire subspaces such that their product is not Baire. The reasoning used in [7] is based on a general version of the construction of Fleissner and Kunen [3] given in [6].

Open question. Let E be a nonseparable Fréchet space. Does there exists a dense Baire subspace F of E such that $F\times F$ is not Baire?

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Extensions of functions from products with compact or metric factors

by

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Abstract. This paper is a continuation of an earlier paper [3]. A subspace A of a space X is $\pi_{\mathscr{Z}}$-embedded in X, where $\mathscr{Z}$ denotes a non-empty class of spaces, if for every $Z\in\mathscr{Z}$ and for every continuous function $f\colon A\times Z\to I$ there exists an extension of f over $X\times Z$. We shall prove that $\pi_{\mathscr{M}\cup\mathscr{C}}$-embedding is equivalent to $\pi_{\mathscr{M}\times\mathscr{C}}$-embedding, where $\mathscr{M}\cup\mathscr{C}$ denotes the class consisting of all metric and all compact spaces, and $\mathscr{M}\times\mathscr{C}$ — the class of products of a metric and a compact space. We shall also give an example which shows that $\pi_{\mathscr{M}\times\mathscr{C}}$-embedding does not imply $\pi_{\mathscr{P}}$-embedding where $\mathscr{P}$ denotes the class of all paracompact p-spaces.

Throughout this paper by a topological space we mean a completely regular space, by a function or an extension — a continuous function or a continuous extension. Symbols $\mathscr{M}$, $\mathscr{C}$, $\mathscr{M}\times\mathscr{C}$, and $\mathscr{P}$ denote the classes of metric spaces, compact spaces, products of a metric and a compact space, and paracompact p-spaces, respectively. By P we shall denote the set of all irrational numbers and by Q — the set of all rational numbers in the unit interval I.

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Let us recall (cf. [2]) that a subspace A of a space X is $\pi_{\mathscr{Z}}$-embedded in X, where $\mathscr{Z}$ denotes a non-empty class of spaces, if for every $Z\in\mathscr{Z}$ and for every function $f\colon A\times Z\to I$ there exists an extension of f over $X\times Z$. Obviously, if $\mathscr{Z}\subset\mathscr{Z}'$ then $\pi_{\mathscr{Z}'}$-embedding implies $\pi_{\mathscr{Z}}$-embedding. Thus $\pi_{\mathscr{P}}$-embedding implies $\pi_{\mathscr{M}\times\mathscr{C}}$-embedding, which implies $\pi_{\mathscr{M}\cup\mathscr{C}}$-embedding. For dense subsets of topological spaces the inverse implications are also true: T. C. Przymusiński showed in [2] that for a dense subset of a topological space $\pi_{\mathscr{P}}$-embedding is equivalent to $\pi_{\mathscr{M}\cup\mathscr{C}}$-embedding. It turns out that in the case of closed subsets this theorem does not hold. We shall give an example of a space Y and its closed subspace X which is $\pi_{\mathscr{M}\cup\mathscr{C}}$-embedded but is not $\pi_{\mathscr{P}}$-embedded in Y. We shall also prove that $\pi_{\mathscr{M}\cup\mathscr{C}}$-embedding is equivalent to $\pi_{\mathscr{M}\times\mathscr{C}}$-embedding; thus our example will show that $\pi_{\mathscr{M}\times\mathscr{C}}$-embedding does not imply $\pi_{\mathscr{P}}$-embedding. These results will also provide a negative answer to the problem raised by K. Morita in [1], whether every Čech-complete, paracompact space is absolutely P-embeddable (a space X is absolutely