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On the transformers of the Zahorski classes of functions

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Abstract. Let $\mathscr{M}_i$ $(i = 1, \dots, 5)$ denote the Zahorski classes of functions defined on interval $I = (0, 1)$. By $H_{\mathscr{M}_i}$ we denote the set of all increasing homeomorphism h of interval I onto itself which leaves $\mathscr{M}_i$ invariant, i.e., such that for every $f \in \mathscr{M}_i$ $f \circ h \in \mathscr{M}_i$. In this paper it is shown that there are proper inclusion $H_{\mathscr{M}_i} \supset H_{\mathscr{M}_{i+1}}$, for $i = 1, 2, 3$. We give a characterization of the class $H_{\mathscr{M}_4}$ and the examples of homeomorphisms showing that classes $H_{\mathscr{M}_5} \setminus H_{\mathscr{M}_4}$ and $H_{\mathscr{M}_4} \setminus H_{\mathscr{M}_5}$ are nonempty. We also establish the inclusion $H_{\mathscr{M}_4} \supset H_{b\Delta}$ where $H_{b\Delta}$ is the class of homeomorphisms preserving bounded derivatives.

Let g be a homeomorphism of $(0, 1)$ onto itself, $g(0) = 0$, $g(1) = 1$. If $\mathscr{F}$ is an arbitrary family of real functions defined on $(0, 1)$ then g is said to be a transformer on $\mathscr{F}$ if $f \circ g \in \mathscr{F}$ holds for every $f \in \mathscr{F}$. Let $H_{\mathscr{F}}$ denote the class of transformers of $\mathscr{F}$.

M. Laczkovich and G. Petruska in [7] have given a necessary and sufficient condition for g to be a transformer on a class of derivatives.

In 1950, Z. Zahorski [9] considered a hierarchy of classes of functions, $\mathscr{M}_k$, $k = 0, \dots, 5$. The largest class, $\mathscr{M}_0$, turned out to be Darboux and first class of functions: the smallest, $\mathscr{M}_5$, turned out to be class of approximately continuous functions. He showed how the classes of derivatives and bounded derivatives fit into the scheme.

In the present paper we study a hierarchy of classes $H_{\mathscr{M}_k}$, $k = 0, \dots, 5$ (for the sake of simplicity we use the notation H_k, $k = 0, \dots, 5$) and consider a number of related properties.

Throughout this paper the word "set" means a Lebesgue measurable subset of open interval $(0, 1)$, the word "homeomorphism" means an increasing homeomorphism of $(0, 1)$ onto itself. By a "function" we means a real function on $(0, 1)$. $A' = (0, 1) \setminus A$, $|A|$ being the Lebesgue measure of the set A and $d(A, a)$ being the density of the set A in the point a. As usual, $d^+(A, a)$ and $d^-(A, a)$ denote the righthand side and lefthand side densities and $\bar{d}(A, a)$ the upper density.

We begin with the definition of the Zahorski classes of sets.

Definition 1. Let E be a nonempty set of type F_σ. We say E *belongs to class*:

M_0 if every point of E is a point of bilateral accumulation of E;

M_1 if every point of E is a point of bilateral condensation of E;

M_2 if each one-side neighbourhood of each $x \in E$ intersects E in a set of positive measure;

M_3 if for each $x \in E$ and each sequence $\{I_n\}_{n=1}^{\infty}$ of closed intervals converging to x and not containing x such that $|I_n \cap E| = 0$ for each n, we have

$$\lim_{n\to\infty} \frac{|I_n|}{\operatorname{dist}(x, I_n)} = 0;$$

M_4 if there exist a sequence of closed sets $\{F_n\}_{n=1}^{\infty}$ and a sequence of positive numbers $\{\eta_n\}_{n=1}^{\infty}$ such that $E = \bigcup_{n=1}^{\infty} F_n$ and for $x \in F_n$ and each $c>0$ there exists a number $\varepsilon(x, c) = 0$ such that if h and h_1 satisfy $hh_1>0$, $h/h_1<c$, $|h+h_1|<\varepsilon(x, c)$, then

$$\frac{|E \cap (x+h, x+h_1)|}{|h_1|} > \eta_n;$$

M_5 if every point of E is a point of density of E.

DEFINITION 2. Let f be real function defined on an interval $I = (0, 1)$, $k = 1, 2, \ldots, 5$. We say f is *in class* $\mathscr{M}_k$ if every associated set of f is in M_k. The associated sets of f are all sets of the from $E_\alpha(f) = \{x: f(x)<\alpha\}$ or $E^\alpha(f) = \{x: f(x)>\alpha\}$, for $\alpha \in R$.

DEFINITION 3. Let H_i, $i = 1, \ldots, 5$, denote the class of all homeomorphisms h such that for all $f \in \mathscr{M}_i$, $f \circ h \in \mathscr{M}_i$.

LEMMA 1. *$h \in H_i$ if and only if for each $A \in M_i$ $h^{-1}(A) \in M_i$* ($i = 1, \ldots, 5$).

Proof. Let h be a homeomorphism such that for every $A \in M_i$, $h^{-1}(A) \in M_i$. Let $f \in \mathscr{M}_i$. Observe that $E_\alpha(f \circ h) = h^{-1}(E_\alpha(f))$. Since $E_\alpha(f) \in M_i$, $h^{-1}(E_\alpha(f)) \in M_i$. Thus, $E_\alpha(f \circ h) \in M_i$. In the similar way we may prove that $F_\alpha(f \circ h) \in M_i$. Thus, we see that $h \in H_i$. Let $h \in H_i$ and $A \in M_i$. Then there exists a function $f \in \mathscr{M}_i$ such that $0<f(x)<1$ for all $x \in A$, and $f(x) = 0$ for all $x \notin A$ (see [1]) .Therefore $E^0(f \circ h) = h^{-1}(A)$. Then $h^{-1}(A) \in M_i$.

We are now ready to establish inclusion relationships for H_i, $i = 1, \ldots, 5$.

THEOREM 1. *The following proper inclusions holds*: $H_1 \supset H_2 \supset H_3 \supset H_4$.

Proof. Theorems 1.1 ([3], p. 9) and 3.5 ([3], p. 16) show that $\mathscr{M}_0 = \mathscr{M}_1$ and that H_1 is the set of all homeomorphisms. Let $h \notin H_2$, and $g = h^{-1}$. By Lemma 1 there exists a set $A \in M_2$, such that $g(A) \notin M_2$. Then, there exist $y \in g(A)$ and $\eta>0$ such that $|(y, y+\eta) \cap g(A)| = 0$ or $|(y-\eta, y) \cap g(A)| = 0$. Suppose that $|(y, y+\eta) \cap g(A)| = 0$ (in some way, we obtain the result for the second condition). Let $x = g^{-1}(y)$ and $x+\varepsilon = g^{-1}(y+\varepsilon)$. Since $A \in M_2$, $|(x, x+\varepsilon) \cap A|>0$. Let $B = \{t\colon t \in (x, x+\varepsilon) \cap A$ and $d(A, t) = 1\}$. Let B^* denote the set of type F_σ, such that $B^* \subset B$ and $|B^*| = |B|$. Thus $B^* \in M_5$ and also $B^* \in M_3$. Since $|g(B^*)| = 0$, $h \notin H_3$. This completes the proof of the inclusion $H_2 \supset H_3$.

Let us now focus our attention on the inclusion $H_3 \supset H_4$. The proof is similar to that of the inclusion $H_2 \supset H_3$.

Suppose that $h \notin H_3$. By Lemma 1 there exists a set $A \in M_3$, such that $g(A) \notin M_3$ ($g = h^{-1}$). Then, there exists $y \in g(A)$ and a sequence of closed intervals $\{I_n\}_{n=1}^{\infty}$ converging to y, such that $|I_n \cap g(A)| = 0$ and $|I_n|>c\operatorname{dist}(y, I_n)$, where $c>0$, ($n = 1, 2, \ldots$). Suppose that $x = g^{-1}(y)$, $J_n = g^{-1}(I_n)$, ($n = 1, 2, \ldots$). We may assume that for all $n \in N$ $|A \cap J_n| = 0$ (if there exists $n_0 \in N$ such that $|A \cap J_{n_0}|>0$, then there exists a set A^* such that $A^* \in M_5$ and $|g(A^*)| = 0$, then by Lemma 1 $g^{-1} \notin H_4$). Since $A \in M_3$, we have

$$\lim_{n\to\infty} \frac{|J_n|}{\operatorname{dist}(x, J_n)} = 0.$$

Let $B = (0, 1) \setminus \bigcup_{n=1}^{\infty} J_n$. We have $B \in M_3$. In the same way we obtain the result for a set $E = (0, 1) \setminus \bigcup_{k=1}^{\infty} J_{n_k}$, where $\{J_{n_k}\}_{k=1}^{\infty}$ is a subsequence of the sequence $\{J_n\}_{n=1}^{\infty}$. Choose a subsequence $\{J_{n_k}\}_{k=1}^{\infty}$ of sequence $\{J_n\}_{n=1}^{\infty}$ such that $d(\bigcup_{k=1}^{\infty} J_{n_k}, x) = 0$. Let $E = (0, 1) \setminus \bigcup_{k=1}^{\infty} J_{n_k}$. Since $d(E, x) = 1$, the set $E \in M_5$. It follows that $E \in M_4$ and $g(E) \notin M_3$. Thus $h \notin H_4$. This completes the proof of all the inclusions.

We now prove that the inclusions are proper. We construct three examples. Let the homeomorphism g_1 map some set of positive measure onto a set of measure zero. Thus $h_1 = g_1^{-1} \notin H_2$.

Let $A = (0, x] \cup \bigcup_{n=1}^{\infty} J_n$, where $J_n = (a_n, b_n)$, $d(A, x) = 1$, $x \in (0, 1)$, $x<a_{n+1} < b_{n+1} < a_n < b_1 = 1$. Let $B = (0, x] \cup \bigcup_{n=1}^{\infty} I_n$, where $I_n = (a_n', b_n')$, $x<a_{n+1}'<b_{n+1}' < a_n' < b_1' = 1$ and

$$\lim_{n\to\infty} \frac{|(b_{n+1}', a_n')|}{b_{n+1}'} = \alpha > 0.$$

Define the homeomorphism g_2 by the formula $g_2(t) = t$ for all $t \in (0, x]$, $g_2(a_n) = a_n'$, $g_2(b_n) = b_n'$ ($n = 1, 2, \ldots$), and g_2 is linear on each interval (a_n, b_n) or (b_{n+1}, a_n) ($n = 1, 2, \ldots$). The proof that $h_2 = g_2^{-1} \in H_2 \setminus H_3$ follows immediately from Lemma 1.

We now show that $H_3 \neq H_4$. Let

$$E = (0, x] \cup \bigcup_{n=n_0}^{\infty} \left(x+\frac{1}{n}, x+\frac{1}{n}+r_n\right),$$

where

$$x \in (0, 1), \quad 1/n(n-1)>r_n>0, \quad d(E, x) = 1.$$

Let

$$A = (0, x] \cap \bigcup_{n=n_0}^{\infty} \left(x+\frac{1}{n}, x+\frac{1}{n}+r_n'\right), \quad \frac{1}{n(n-1)}>r_n'>0 \quad \text{and} \quad d^+(A, x) = 0.$$

Consequently $A \notin M_4$. Define the homeomorphism $g_3(t) = t$ for all $t \in (0, x]$, $g_3(x+1/n) = x+1/n$, $g_3(x+1/n+r_n) = x+1/n+r_n'$ $(n = n_0, n_0+1, \ldots)$ and g_3 is linear on each interval $(x+1/n, x+1/n+r_n)$ or $(x+1/(n+1)+r_{n+1}, x+1/n)$, $n = n_0, n_0+1, \ldots$ and $(x+1/n_0+r_{n_0}, 1)$. Thus $d^+(A, x) = 0$ and $g_3(E) = A$, so $h_3 = g_3^{-1} \notin H_4$. We now show that $h_3 \in H_3$. Let $D \in M_3$, and $\{P_n\}_{n=1}^{\infty}$ be a sequence of closed intervals not containing x and converging to x such that $|g_3(D) \cap P_n| = 0$, for each $n \in N$. It follows that $|D \cap Q_n| = 0$, where $Q_n = g_3^{-1}(P_n)$, $n = 1, 2, \ldots$ Since $D \in M_3$, we have

$$\lim_{n\to\infty} \frac{|Q_n|}{\operatorname{dist}(x, Q_n)} = 0 .$$

There exists $s = s(n)$, $k = k(n)$ such that

$$\inf(Q_n) \in [a_{s(n)}, a_{s(n)-1}) \quad \text{and} \quad \sup(Q_n) \in (a_{s(n)-k(n)+1}, a_{s(n)-k(n)}) ,$$

where

$$a_n = x+1/n, \quad b_n = x+1/n+r_n .$$

We have

$$\frac{|Q_n|}{\operatorname{dist}(x, Q_n)} = \frac{|Q_n \cap (a_s, a_{s-1})|}{\operatorname{dist}(x, Q_n)} + \frac{|(a_{s-1}, a_{s-k+1})|}{\operatorname{dist}(x, Q_n)} + \frac{|(a_{s-k+1}, a_{s-k}) \cap Q_n|}{\operatorname{dist}(x, Q_n)} .$$

Since

$$\lim_{n\to\infty} \frac{|Q_n|}{\operatorname{dist}(x, Q_n)} = 0, \quad \lim_{n\to\infty} \frac{|(a_{s-1}, a_{s-k+1})|}{\operatorname{dist}(x, Q_n)} = 0,$$

we obtain

$$\lim_{n\to\infty} \frac{|(a_{s(n)-1}, a_{s(n)-k(n)+1})|}{a_{s(n)-1}-x} = 0,$$

that is, $\lim\limits_{n\to\infty} k(n)/(s(n)-k(n)) = 0$. Observe that

$$\frac{|P_n|}{\operatorname{dist}(x, P_n)} \leqslant \frac{|(a_s, a_{s-k})|}{a_s - x} = \frac{x+\dfrac{1}{s-k}-x+\dfrac{1}{s}}{1/s} = \frac{k}{s-k} .$$

It follows that $\lim\limits_{n\to\infty} |P_n|/\operatorname{dist}(x, P_n) = 0$, which means that $g_3(D) \in M_3$ (at the other points of the set $g_3(D)$, the definition of the M_3-set holds, because g_3 is linear from each side), and the proof of the theorem is complete.

Definition 4. The *d_4-upper density of the set X at the point a* denoted by $d_4(X, a)$ is defined by

$$d_4(X, a) = \sup\left\{\overline{\lim_{n\to\infty}} \frac{|X \cap I_n|}{|I_n|}\right\}$$

where supremum is taken for all the sequences of intervals $\{I_n\}_{n=1}^{\infty}$ for with $I_n \to a$ (that is the endpoints of I_n converge to a) and there exists a positive number c such that $|I_n| > c \cdot \operatorname{dist}(a, I_n)$ $(n = 1, 2, \ldots)$.

Lemma 2. *For every $X \subset (0, 1)$, X is M_4-set if and only if there exists a sequence of closed sets $\{F_n\}_{n=1}^{\infty}$ and a sequence of positive numbers $\{\eta_n\}_{n=1}^{\infty}$ such that $X = \bigcup_{n=1}^{\infty} F_n$ and for every n and $x \in F_n$ we have $d_4(X', x) < 1-\eta_n$.*

Proof. This is only a reformulation of the definition of M_4-set (see [6]).

Lemma 3. *For every $X \subset (0, 1)$ and $a \in (0, 1)$ we have*

(i) $0 \leqslant \overline{d}(X, a) \leqslant d_4(X, a) \leqslant 1$

(ii) *If $d(X, a)$ exists then $d_4(X, a) = d(X, a)$.*

The proof of this lemma can be found in [6].

We shall use the following notation (p_4-property for the sets A, B): $A \subset_{p_4} B$ means that $A \subset B$ and there exists a sequence $\{F_n\}_{n=1}^{\infty}$ of closed sets and a sequence of positive numbers $\{\eta_n\}_{n=1}^{\infty}$ such that $A = \bigcup_{n=1}^{\infty} F_n$ and for every n and $x \in F_n$ we have $d_4(B', x) < 1-\eta_n$.

Definition 5. We say that the homeomorphism g *satisfies condition* (H) if for every pair of perfect sets A, B such that $A \subset_{p_4} B$, we have $g(A) \subset_{p_4} g(B)$.

Theorem 2. *A necessary and sufficient condition for $g^{-1} \in H_4$ is that g satisfies condition* (H).

Proof. Suppose g satisfies condition (H). Let $f \in \mathcal{M}_4$. Then there exists a p_4-system for f (see [2]), that is, a system of perfect sets $\{A_n^{y_r}, A_{y_r}^n\}$, $r = 1, 2, \ldots$, $n = r, r+1, \ldots$, $y_1, y_2, \ldots$ — is an enumeration of the rationals, satisfying

1. $$\bigcup_{n=r}^{\infty} A_n^{y_r} = E^{y_r}(f), \quad \bigcup_{n=r}^{\infty} A_{y_r}^n = E_{y_r}(f),$$

2. $$A_n^{y_r} \subset_{p_4} A_{n+1}^{y_r}, \quad A_{y_r}^n \subset_{p_4} A_{y_r}^{n+1},$$

and if $y_s < y_t$ and $n \geqslant \max(s, t)$, then

3. $$A_n^{y_s} \subset_{p_4} A_n^{y_t}, \quad A_{y_t}^n \subset_{p_4} A_{y_s}^n .$$

But since g preserves the p_4-property, the system of sets $\{g(A_n^{y_r}), g(A_{y_r}^n)\}$ is a p_4-system for $f \circ g^{-1}$. By Theorem 5 (see [2]), $f \circ g^{-1} \in \mathcal{M}_4$ if and only if there exists p_4-system for $f \circ g^{-1}$. Then $f \circ g^{-1} \in \mathcal{M}_4$ and $g^{-1} \in H_4$. Suppose that $g^{-1} \in H_4$. At first we show that if A is a closed set, B is a perfect set, $A \subset B$, and $\sup\limits_{x \in A} \{d_4(B', x)\} < 1-\eta < 1$ then $g(A) \subset_{p_4} g(B)$.

Let $E = B_1 \cup A$, where B_1 is a set of type F_σ, $|B_1| = |B|$, $B_1 \subset B$ and every point of B_1 is a point of density B_1. It follows from the inequality $d_4(E', x) < 1-\eta$ for every $x \in A$, that $E \in M_4$. From Lemma 1 we have $g(E) \in M_4$. By Lemma 2 there exists a sequence $\{F_n\}_{n=1}^{\infty}$ of closed sets and a sequence $\{\eta_n\}_{n=1}^{\infty}$ of positive numbers such that $g(E) = \bigcup_{n=1}^{\infty} F_n$ and for every n and $y \in F_n$ $d_4(g(E'), y) < 1-\eta_n$.

Thus $g(A) = \bigcup_{n=1}^{\infty} g(A) \cap F_n$ and for every n and $y \in g(A) \cap F_n$, we obtain $d_4(g(E'), y) < 1-\eta_n$. From the inclusion $E' \supset B'$ it follows that $d_4(g(B'), y) < 1-\eta_n$ and then $g(A) \subset_{p_4} g(B)$. Now let A, B be a pair of perfect sets such that $A \subset_{p_4} B$. By the definition of p_4-property we have $A = \bigcup_{n=1}^{\infty} F_n$ where $\{F_n\}_{n=1}^{\infty}$ is a sequence of closed sets and there exists a sequence $\{\eta_n\}_{n=1}^{\infty}$ of positive numbers such that for every n and $x \in F_n$, $d_4(B', x) < 1-\eta_n$. Since F_n is closed, we have $g(F_n) \subset_{p_4} g(B)$ which gives us that $g(A) = \bigcup_{n=1}^{\infty} g(F_n) \subset_{p_4} g(B)$. This completes the proof of theorem.

THEOREM 3. *The homeomorphism classes $H_5 \setminus H_4$ and $H_4 \setminus H_5$ are nonempty.*

Proof. Let C be the Cantor set in $(0, 1)$ and let $\{G_i\}_{i=1}^{\infty} = \{(a_i, b_i)\}_{i=1}^{\infty}$ denote the sequence of the interval contiguous to C. Define for $i = 1, 2, \ldots$ $I_{ik} = [a_i + 2^{-k} - 2^{-k}(p+2)^{-1}, a_i + 2^{-k}]$ where p is such that $|G_i| = 3^{-p}$, $k \geq k(i)$ and $k(i)$ is such that $2^{-k(i)} < (b_i - a_i)/4$. Let us denote for i, k and $j = 0, 1, \ldots, k$, $x_j = \inf(I_{ik}) + j/k|I_{ik}|$. Moreover, let for $i = 1, 2, \ldots$ and $k > k(i)$

$$D_{ik} = \bigcup_{s=1}^{k} \left(x_{s-1}, x_{s-1} + \frac{x_s - x_{s-1}}{p}\right),$$

where p is such that $b_i - a_i = 3^{-p}$,

$$H_{ik} = \bigcup_{s=1}^{k} \left(x_{s-1}, x_{s-1} + \frac{x_s - x_{s-1}}{2}\right) \quad \text{if} \quad b_i - a_i < 3^{-1}$$

and

$$H_{ik} = \bigcup_{s=1}^{k} (x_{s-1}, x_s) \quad \text{if} \quad b_i - a_i = \tfrac{1}{3},$$

$$D = C \cup \bigcup_{i=1}^{\infty} (G_i \setminus \bigcup_{k=k(i)}^{\infty} I_{ik}) \cup \bigcup_{i=1}^{\infty} \bigcup_{k=k(i)}^{\infty} D_{ik}$$

and

$$H = C \cup \bigcup_{i=1}^{\infty} (G_i \setminus \bigcup_{k=k(i)}^{\infty} I_{ik}) \cup \bigcup_{i=1}^{\infty} \bigcup_{k=k(i)}^{\infty} H_{ik}.$$

For the sake of simplicity of our notation we keep on writing

$$A_j = G_j \setminus \bigcup_{k=k(j)}^{\infty} I_{jk} \quad (j = 1, 2, \ldots).$$

J. Lipiński in [8] proved that $D \notin M_4$.

We show that for every $a \in H$, $d_4(H', a) < 1 - \frac{1}{8}$. In the proof we are using the method presented by J. Lipiński in [8]. He proved that for all $a \in C \setminus \bigcup_{i=1}^{\infty} \{a_i, b_i\}$,

$d(\bigcup_{i=1}^{\infty} A_i, a) = 1$. By inclusion $\bigcup_{i=1}^{\infty} A_i \subset H$, we get $d(H, a) = 1$ and by Lemma 3(ii) $d_4(H', a) = 0$. The set

$$\bigcup_{i=1}^{\infty} (G_i \setminus \bigcup_{k=k(i)}^{\infty} I_{ik}) \cup \bigcup_{i=1}^{\infty} \bigcup_{k=k(i)}^{\infty} H_{ik}$$

is open and it follows that for all a belonging to this set we have $d(H, a) = 1$ and, by Lemma 3(ii), also $d_4(H', a) = 0$. It remains to prove that $d_4(H', a) < 1 - 1/8$ for $a \in \bigcup_{i=1}^{\infty} \{a_i, b_i\}$. J. Lipiński proved also that for all $j = 1, 2, \ldots$

$$d^-(\bigcup_{i=1}^{\infty} A_i, a_j) = 1 \quad \text{and} \quad d^+(\bigcup_{i=1}^{\infty} A_i, b_j) = 1.$$

It follows that for all $i = 1, 2, \ldots$

(1) $$d^-(H, a_i) = 1 \quad \text{and} \quad d^+(H, b_i) = 1.$$

Observe that if $I \subset (a_i, b_i)$ and $I \cap A_i \neq \emptyset$ then

(2) $$|I \cap H|/|I| \geq 1/2.$$

We show that for all $i = 1, 2, \ldots$ $d_4(H', a_i) < 1 - 1/8$. Let $\{I_n\}_{n=1}^{\infty}$ be a sequence of closed intervals such that $\{I_n\}_{n=1}^{\infty}$ converging to a_i, $I_n \subset (a_i, b_i)$ and there exists $c > 0$ such that $|I_n| > c \cdot \operatorname{dist}(a_i, I_n)$. Thus, for estimation of the mean density $|I_n \cap H|/|I_n|$ we need only consider the case where for every n there exists $k = k(n)$ such that $I_n \subset H_{ik(n)}$.

Indeed, if it was not true then, by conditions (1), (2),

$$|I_n \cap H|/|I_n| \geq \min(1, 1/2) = 1/2.$$

We have inequality

$$\operatorname{dist}(a_i, I_{ik}) = 2^{-k} - 2^{-k}(p+2)^{-1} < \operatorname{dist}(a_i, I_n) < |I_n|/c$$

so

$$|I_{ik}|(2^{-k} - 2^{-k}(p+2)^{-1})/|I_{ik}| < |I_n|/c$$

and

$$|I_{ik}|/|I_n| < 1/(c(p+1)).$$

Let $m = m(n)$ be a number of points $x_s \in I_{ik(n)}$ included in interval I_n. Then

$$\frac{|I_n \cap H_{ik}|}{|I_n|} \geq \frac{(m-1)/2k|I_{ik}|}{|I_{ik}|(m+1)/k} = \frac{1}{2}\frac{(m-1)}{(m+1)}.$$

Observe that

$$(m-1)/k|I_{ik}| \leq |I_n| < (m+1)/k|I_{ik}|.$$

Then

$$\frac{m-1}{m+1} \geq \frac{k|I_n|/|I_{ik}| - 2}{k|I_n|/|I_{ik}| + 2}$$

and

$$\frac{|I_n \cap H_{ik}|}{|I_n|} \geqslant \frac{1}{2}\frac{k-2/(c(p+1))}{k+2/(c(p+1))}.$$

Since the sequence $\{I_n\}_{n=1}^{\infty}$ converges to a_i, $k(n) \to \infty$. Thus, there exists N such that for every $n \geqslant N$

$$|I_n \cap H_{ik(n)}|/|I_n| \geqslant 1/8.$$

Hence $\varliminf_{n\to\infty} |I_n \cap H|/|I_n| \geqslant 1/8$ so $\varlimsup_{n\to\infty} |I_n \cap H'|/|I_n| \leqslant 1-1/8$, and consequently $d_4(H', a_i) \leqslant 1-1/8$.

By the same estimation for points b_i ($i = 1, 2, \ldots$) and (1), (2) we have then for $a \in H$ $d_4(H', a) \leqslant 1-1/8$.

Define $g(x) = x$ for $x \in (1/3, 2/3) \cup C \cup \bigcup_{i=1}^{\infty} (G_i \setminus \bigcup_{k=k(i)}^{\infty} I_{ik})$,

$$g(x_s) = x_s \ (s = 0, 1, 2, \ldots, k), \quad g(x_{s-1}+(x_s-x_{s-1})/2) = x_{s-1}+(x_s-x_{s-1})/p$$

where $p \geqslant 2$, $x_s \in I_{ik} \subset G_i$ and $|G_i| = 3^{-p}$ and g is linear on the contiguous intervals. It is easy to verify that $g(H) = D$. By Lemma 1 $h = g^{-1} \notin H_4$. To show that $h \in H_5$ we now prove that g fulfils the hypothesis of Theorem 5 [4], that is g is absolutely continuous homeomorphism, g' is essentially bounded and for every $x_0 \in (0, 1)$ there exists $n \in N$ such that x_0 is the point of dispersion of the set $\{x\colon g'(x) < 1/n\}$.

If $x_0 \in G_i$, where $|G_i| = 3^{-p}$, we take n greater than p and we have

$$\{x\colon g'(x) < 1/n\} \cap G_i = \emptyset, \text{ so } d(\{x\colon g'(x) < 1/n\}, x_0) = 0.$$

Let $x_0 \in C \setminus \bigcup_{i=1}^{\infty} \bar{G}_i$. Since $\{x\colon g'(x) < 1\} \subset (0, 1) \setminus \bigcup_{i=1}^{\infty} A_i$, $d(\{x\colon g'(x) < 1\}, x_0) = 0$.

The case where x_0 equals a_i or b_i is obviously reduced to a combination of the preceding cases. By the definition of the homeomorphism g, $g' \leqslant 2$ on the set of its existence, that g' is essentially bounded. On the intervals contiguous to Cantor set homeomorphism g is intervals linear, so g is absolutely continuous, hence applying Theorem 5 [4], we get the desired result.

Now we construct an example of a homeomorphism which is in $H_4 \setminus H_5$. Let $x \in (0, 1)$. Let for $n \geqslant n_0$ where $1/n_0 + x < 1$, $p_n = 1/n + x$, $q_n = p_n + r_n$, where $r_n = (1/(n-1) - 1/n)n/(n+1)$. Let $A = (0, x] \cup \bigcup_{n=n_0}^{\infty} (p_n, q_n)$. Then $d(A, x) = 1$. Now, let $q_n' = (p_n + q_n)/2$ ($n \geqslant n_0$), and define the homeomorphism by the following formulae $g(t) = t$, for $0 < t \leqslant x$, $g(p_n) = p_n$, $g(q_n) = q_n'$ for all $n \geqslant n_0$, and g is linear on each interval (p_n, q_n) or (q_n, p_{n-1}). From Lemma 1 we conclude that $h = g^{-1} \notin H_5$. To prove that $h \in H_4$ we use Lemma 1. Let $E \in M_4$ and $x \notin E$. We have $g(E) \bigcup_{n=1}^{\infty} g(F_n)$, where $\{F_n\}_{n=1}^{\infty}$ is a sequence of closed sets and for every n and $x_0 \in F_n$, $d_4(E', x_0) < 1-\eta_n$, where η_n is positive for all $n \in N$.

If $\{I_n\}_{n=1}^{\infty}$ is a sequence of closed intervals converging to y_0 and $|I_n|/\text{dist}(y_0, I_n) > c$ ($c > 0$), then for $j \geqslant j_0$ $|g(E') \cap I_j|/|I_j| = |E' \cap J_j|/|J_j|$, where $J_j = g^{-1}(I_j)$, and $|J_j| > c \cdot \text{dist}(x_0, J_j)$, where $x_0 = g^{-1}(y_0)$. This follows from the linearity of the homeomorphism g on each side of the point x_0.

Hence

$$\varlimsup_{n\to\infty} \frac{|g(E') \cap I_j|}{|I_j|} = \varlimsup_{n\to\infty} \frac{|E' \cap J_j|}{|J_j|} < 1-\eta_n.$$

Thus for every $y_0 \in g(F_n)$, $d_4(g(E'), y_0) < 1-\eta_n$ and Lemma 2 implies that $g(E) \in M_4$.

Now, let $x \in E$. We need only to show that there exists $\eta > 0$ such that $d_4(g(E'), x) < 1-\eta$. Since $E \in M_4$, there exists $\eta_x > 0$ such that $d_4(E', x) < 1-\eta_x$. Suppose $\{I_n\}_{n=1}^{\infty}$ is a sequence of closed intervals converging to x and there exists $c > 0$ such that $|I_n| > c \cdot \text{dist}(x, I_n)$.

We will show that there exist n_0 and $c_1 > 0$ such that for all $n > n_0$ $|J_n| > c_1 \cdot \text{dist}(x, J_n)$, where $I_n = g^{-1}(J_n)$, ($n = 1, 2, \ldots$). Let $I_n = [a_n, b_n]$, $n \in N$. Then $|I_n| > c \cdot \text{dist}(x, I_n)$ if and only if $b_n - a_n > c(a_n - x)$ (we assume that $x < a_n$ for all n). Then there exist $k = k(n)$, $s = s(n)$ such that

$$\inf(J_n) \in [p_{s(n)+1}, p_{s(n)}), \quad \sup(J_n) \in (p_{s(n)-k(n)+1}, p_{s(n)-k(n)}].$$

By the estimation

$$1/(s+1) + x < a_n < 1/s + x, \quad 1/(s-k+1) + x < b_n < 1/(s-k) + x$$

we have

$$\frac{b_n - a_n}{a_n - x} < \frac{1/(s-k) - 1/(s+1)}{1/(s+1)} = \frac{k+1}{s+1}.$$

Then

$$\varliminf_{n\to\infty} (k(n)+1)/(s(n)-k(n)) \geqslant c.$$

The sequence $\{I_n\}_{n=1}^{\infty}$ converges to x, then this inequality implies that $k(n)$, $s(n)$ converge to ∞. Thus there exist $c_1 > 0$ and n_0 such that

$$\frac{|J_n|}{\text{dist}(x, J_n)} \geqslant \frac{|(p_s, p_{s-k+1})|}{|(x, p_s)|} = \frac{k-1}{s-k+1} > c_1.$$

Next, we estimate the expression $|I_n \cap g(E)|/|I_n|$. $g' \geqslant 1/2$ at every point which exists, so

$$|I_n \cap g(E)|/|I_n| = |g(J_n \cap E)|/|I_n| \geqslant \tfrac{1}{2}|E \cap J_n|/|J_n| \cdot |J_n|/|I_n|.$$

We estimate expression $|J_n|/|I_n|$.

First we can see that

$$\frac{d(g^{-1})}{du}(u) = 2 \quad \text{for} \quad u \in (p_n, q_n')$$

and

$$\frac{d(g^{-1})}{du}(u) = \frac{1}{n+3} \quad \text{for} \quad u \in (q'_{n+1}, p_n).$$

Then we have

$$\begin{aligned}|I_n|-|J_n| = {} & |I_n \cap (p_{s+1}, q'_{s+1})| - |g^{-1}(I_n \cap (p_{s+1}, q'_{s+1}))| + \\ & + |I_n \cap (q'_{s+1}, p_s)| - |g^{-1}(I_n \cap (q'_{s+1}, p_s))| + \\ & + |I_n \cap (P_{s-k+1}, q'_{s-k+1})| - |g^{-1}(I_n \cap (p_{s-k+1}, q'_{s-k+1}))| + \\ & + |I_n \cap (q'_{s-k+1}, p_{s-k})| - |g^{-1}(I_n \cap (q'_{s-k+1}, p_{s-k}))| \\ & \leqslant (1-2/(s+3))|I_n \cap (q'_{s+1}, p_s)| + (1-2/(s-k+3))|I_n \cap (q'_{s-k+1}, p_{s-k})| \\ & \leqslant r_{s+1}/2 + r_{s-k+1}/2.\end{aligned}$$

That is,

$$|J_n|/|I_n| \geqslant 1 - \tfrac{1}{2}(r_{s+1}/|I_n| + r_{s-k+1}/|I_n|).$$

By the estimation $|I_n|$ and the definition r_n we see that

$$\frac{r_{s+1}}{|I_n|} \leqslant \frac{s-k+1}{(k-1)(s+1)(s+2)} \quad \text{and} \quad \frac{r_{s-k+1}}{|I_n|} \leqslant \frac{s}{(s-k)(s-k+1)(k-1)}.$$

Observe that $s = s(n)$, $k = k(n)$ converge to ∞ and $s-k>1$. Hence there exists $0<\delta<2$ and n_0 such that for every $n>n_0$

$$\frac{r_{s+1}}{|I_n|} + \frac{r_{s-k+1}}{|I_n|} < \delta,$$

that is, there exists $\eta_1>0$ such that

$$\frac{|J_n|}{|I_n|} > 1 - \frac{1}{2}\delta > \eta_1, \quad \text{for} \quad n>n_0.$$

The set $E \in M_4$. By the definition of M_4-sets there exists $n_1>n_0$ such that for all $n>n_1$

$$\frac{|E \cap J_n|}{|J_n|} > \eta_x.$$

Consequently

$$\frac{|g(E) \cap I_n|}{|I_n|} > \frac{1}{2}\eta_x\eta_1, \quad \text{for} \quad n>n_1.$$

Finally, $\varlimsup_{n\to\infty} |g(E') \cap I_n|/|I_n| \leqslant 1 - \frac{1}{2}\eta_x\eta_1$ and we obtain the desired inequality with $\eta = \frac{1}{2}\eta_x\eta_1$.

THEOREM 4. *If the homeomorphisms g and g^{-1} satisfy a Lipschitz condition then both g and g^{-1} are in H_4.*

Proof. Let g and g^{-1} fulfil a Lipschitz condition with constans L and l respectively. We show that g fulfils the condition H of Definition 5. Let A and B be a perfect sets, and suppose $A \subset_{p_4} B$. We shall prove that $g(A) \subset_{p_4} g(B)$. We first prove that if $\{I_n\}_{n=1}^{\infty}$ is a sequence of closed intervals converging to y and there exists a constant $c>0$ such that $|I_n|>c\cdot\operatorname{dist}(y, I_n)$, for $n = 1, 2, \ldots$, then there exists a constant $c_1>0$ such that $|J_n|>c_1\cdot\operatorname{dist}(x, I_n)$, where $J_n = g^{-1}(I_n)$, $x = g^{-1}(y)$, $n = 1, 2, \ldots$ Denote $I_n = [a_n, b_n]$. Since $|I_n|>c\cdot\operatorname{dist}(y, I_n)$, we have $(b_n-a_n)/(a_n-y)>c$ for all $n \in N$ (we consider $a_n>y$).

Let $J_n = [a'_n, b'_n]$, $n \in N$. Then

$$b_n - a_n = g(b'_n) - g(a'_n) \leqslant L(b'_n - a'_n)$$

and

$$\frac{b'_n-a'_n}{a'_n-x} \geqslant \frac{1}{L}\frac{b_n-a_n}{a_n-y}\frac{a_n-y}{a'_n-y} \geqslant \frac{1}{L}c\frac{1}{l} = \frac{c}{Ll}.$$

Thus

$$|J_n|>c_1\cdot\operatorname{dist}(x, J_n), \quad \text{where} \quad c_1 = c/(L\cdot l).$$

Let $A = \bigcup_{n=1}^{\infty} F_n$ where $\{F_n\}_{n=1}^{\infty}$ is a sequence of closed sets and for every $x \in F_n$ $d_4(B', x)<1-\eta_n$, where $\eta_n>0$. Let $\{I_n\}_{n=1}^{\infty}$ be a sequence of closed intervals converging to y such that $|I_n|>c\cdot\operatorname{dist}(y, I_n)$ with $c>0$, then we have

$$\frac{|g(B) \cap I_n|}{|I_n|} = \frac{|g(B) \cap g(J_n)|}{|g(J_n)|} > \frac{1}{L}\frac{1}{l}\frac{|B \cap J_n|}{|J_n|}, \quad \text{where} \quad J_n = g^{-1}(I_n).$$

Since $|J_n|>c_1\cdot\operatorname{dist}(x, J_n)$, $\varlimsup_{n\to\infty} |B' \cap J_n|/|J_n|<1-\eta_n$ and thus $\varlimsup_{n\to\infty} |g(B') \cap I_n|/|I_n| < 1 - \frac{1}{L}\frac{1}{l}\eta_n$. This completes the proof of the inclusion $g(A) \subset_{p_4} g(B)$. By Theorem 3 we have $g^{-1} \in H_4$.

The proof that $g \in H_4$ is similar.

PROPOSITION. *Let $b\Delta$ denote the class of bounded derivatives on* $(0, 1)$. *Let $H_{b\Delta}$ be the class of homeomorphisms such that $f \circ h \in b\Delta$ for every bounded derivative f. Then $H_{b\Delta} \subset H_4$ and the inclusion is proper.*

Proof. Let $h \in H_{b\Delta}$ and $E \in M_4$. By Theorem 2.6 ([3], p. 96) there exist $f \in B\Delta$ such that $E = \{y: f(y)>0\}$. If $g = h^{-1}$ then $g(E) = \{x: f(h(x))>0\}$. Let $\alpha \in R$. By Theorem 2.5 [3] $E^{\alpha}(f \circ h) \in M_4$ thus $g(E) = E^0(f \circ h) \in M_4$ and so $g^{-1} = h \in H_4$.

A. M. Bruckner in [5] gave the example of a homeomorphism h such that and h^{-1} satisfy Lipschitz condition but $h \notin H_{b\Delta}$. Thus, by Theorem 4, h is the homeomorphism such that $h \in H_4$ and $h \notin H_{b\Delta}$, so the inclusion is proper.

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Products of Baire topological vector spaces

by

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Abstract. Let I be a set whose cardinal number is larger than $\aleph_0$. In this article it is proved that there are dense subspaces of $c_0(I)$ with additional properties which are Baire and whose product is not Baire. The same properties are obtained taking $l^p(I)$, $0<p<\infty$, instead of $c_0(I)$.

1. Introduction. All the linear spaces we shall use are defined over the field K of real or complex numbers. If E is a set, we denote by E^ω the countable infinite product of copies of E.

Oxtoby [5] proved that the continuum hypothesis implies that there is a completely regular Baire topological space whose square is not Baire. Actually, Oxtoby uses the hypothesis that the union of $<2^{\aleph_0}$ subsets of Lebesgue measure zero of real numbers has Lebesgue measure zero. P. E. Cohen [2], using forcing techniques, gave an absolute example of Baire spaces whose product is not Baire. Later Fleissner and Kunen [3] gave new examples of Baire spaces whose products is not Baire space without using additional hypothesis of the theory of sets. In this article we give examples of Baire topological vector spaces whose product is not Baire using in part techniques of Fleissner and Kunen [3].

Given a set I and an ordinal α we denote by $\operatorname{card} I$ and $\operatorname{card}\alpha$ the cardinal numbers of I and α respectively. If $\beta<\alpha$, $[\beta,\alpha[$ is the interval of ordinal numbers closed in β and open in α, i.e.,

$$[\beta,\alpha[\;=\{\delta\colon\ \beta\leqslant\delta<\alpha\}\,.$$

We represent by ω_1 the first ordinal such that $\operatorname{card}\omega_1>\aleph_0$. We suppose $[0,\alpha]$ endowed with the order topology. A subset of $[0,\alpha[$ is said to be stationary if it meets every unbounded closed subset of $[0,\alpha]$. Let γ be the first ordinal such that $\operatorname{card} I=\operatorname{card}\gamma$ and let T_n be a mapping from $[0,\gamma[$ in $[0,\gamma[$, $n=1,2,\ldots$ We shall need the following results:

(a) *If* $\operatorname{card}\gamma>\aleph_0$ *the set*

$$\{\alpha<\gamma\colon\ T_n([0,\alpha[)\subset[0,\alpha[,\ n=1,2,\ldots\}$$

is unbounded and closed in $[0,\gamma[$,