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Essential mappings and transfinite dimension

by

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Abstract. We construct a compact metrizable space with inductive dimension $\omega+1$ that admits no essential mappings into Henderson's $(\omega+1)$-dimensional absolute retract $J^{\omega+1}$.

1. Introduction. A continuous mapping $f\colon X \to I^n = [0, 1]^n$ is called *essential* if there is no continuous extension $g\colon X \to \partial I^n$ of $f|f^{-1}(\partial I^n)$, where ∂I^n is the geometric boundary of I^n. The following characterization is well known (see e.g. Engelking [1], 3.2.10).

1.1 THEOREM. *A normal space has* $\dim \geqslant n$ *iff it admits an essential mapping into* I^n.

D. W. Henderson [2] has attempted to extend this result to transfinite inductive dimension.

1.2 DEFINITION. $\operatorname{Ind}(\emptyset) = -1$. Let α be an ordinal and X a normal space. $\operatorname{Ind}(X) \leqslant \alpha$ if every pair of disjoint closed subsets of X can be separated by a closed set with $\operatorname{Ind} < \alpha$ (S *separates* A and B in X if $X \setminus S$ is the union of disjoint open sets U and V with $A \subset U$ and $B \subset V$).

1.3 DEFINITION (Henderson). For each countable ordinal α we define a compact metric space J^α, its „boundary" T^α and a point $p^\alpha \in T^\alpha$.

(i) if α is finite then $J^\alpha = I^\alpha$, $T^\alpha = \partial I^\alpha$ and $p^\alpha = (0, 0, \ldots, 0)$.

(ii) If we have a successor $\alpha+1$ we define $J^{\alpha+1} = J^\alpha \times I$, $T^{\alpha+1} = (T^\alpha \times I) \cup \cup (J^\alpha \times \{0, 1\})$ and $p^{\alpha+1} = (p^\alpha, 0)$.

(iii) If α is a limit, put $K^\beta = J^\beta \cup L^\beta$ for every $\beta < \alpha$, where L^β is a half open arc such that $L^\beta \cap J^\beta = \{p^\beta\}$ = (the end-point of L^β). J^α is defined as the one-point compactification of the discrete sum $\bigoplus_{\beta<\alpha} K^\beta$; $T^\alpha = J^\alpha \setminus \bigcup_{\beta<\alpha} (J^\beta \setminus T^\beta)$ and p^α is the compactifying point.

A continuous mapping f from a space X into J^α is called *essential* if every continuous $g\colon X \to J^\alpha$ that satisfies $g|f^{-1}(T^\alpha) = f|f^{-1}(T^\alpha)$ is an onto mapping.

The following two theorems are due to Henderson [2].

1.4 THEOREM. *J^α is an absolute retract and* $\operatorname{Ind}(J^\alpha) = \alpha$.

1.5 THEOREM. *If there is an essential mapping from a normal space X into J^α then* $\operatorname{Ind}(X) \geqslant \alpha$ *or* $\operatorname{Ind}(X)$ *does not exist.*

Henderson asked the following question: if $\operatorname{Ind}(X)\geqslant\alpha$, is there an essential $f\colon X\to J^{\alpha}$? In view of Theorem 1.1 and the fact that there exist compact spaces with dim < Ind it seems reasonable to restrict ourselves to metric spaces. We show that the answer to the question is yes for $\alpha=\omega$ and no for $\alpha=\omega+1$. This also solves two questions that were raised by R. Pol ([3], p. 238) who independently showed the following. There exists a countable ordinal λ such that for every countable $\alpha>\lambda$ there is an α-dimensional compactum without essential mappings into J^{λ} ([3], Thm. 5.2). Note that Pol's result, which was obtained by a method completely different from our's, also gives different information concerning Henderson's question.

2. Two theorems. In this section we prove the converse of Theorem 1.5 for $\alpha=\omega$ and we give an addition theorem that will be used in the next section.

2.1 THEOREM. *If X is a normal space with infinite covering dimension then there is an essential $f\colon X\to J^{\omega}$.*

Proof. Let $\dim(X)=\infty$. We construct sequences $H_1\supset H_2\supset H_3\supset\ldots$ and $(A_i)_{i\in N}$ of closed subsets of X such that for every $i\in N$, $\dim(H_i)=\infty$, $\dim(A_i)\geqslant i$ and $A_i\subset H_i\setminus H_{i+1}$. Put $H_1=X$ and assume that H_i has been constructed. Since $\dim(H_i)=\infty$ there exist disjoint closed sets A and B in H_i such that every closed set M that separates A and B has $\dim\geqslant i$ (see Engelking [1], 3.1.27). Select a closed covering $\{F,G\}$ of H_i with $F\cap B=G\cap A=\emptyset$. The union of F and G is infinite-dimensional and hence (see Engelking [1], 3.1.8) one of them, say F, has infinite dim. Put $H_{i+1}=F$ and let A_{i+1} be a closed set that separates F and B in H_i. Then $A_{i+1}\subset H_i\setminus H_{i+1}$ separates A and B in H_i and hence $\dim(A_i)\geqslant i$. This completes the induction.

Consider now $J^{\omega}=\{p^{\omega}\}\cup\bigcup_{i=0}^{\infty}(J^i\cup L^i)$. Since $\dim(A_i)\geqslant i$ we may select for every $i\in N\cup\{0\}$ an essential $f_i\colon A_{2i+1}\to J^i$ and a continuous g_i from A_{2i+2} onto the closed interval $L^i\cup\{p^{\omega}\}$. Moreover, let h_{ω} be the constant function from $H=\bigcap_{i=1}^{\infty}H_i$ into $\{p^{\omega}\}$. Put $A=\bigcup_{i=1}^{\infty}A_i\cup H$, which is a closed subset of X. Since $\{A_i|i\in N\}$ is a pairwise disjoint collection of clopen subsets of A one easily verifies that

$$h=\bigcup_{i=0}^{\infty}f_i\cup\bigcup_{i=0}^{\infty}g_i\cup h_{\omega}$$

is a continuous mapping from A onto J^{ω}. The fact that the f_i's are essential guarantees in view of Henderson ([2], Prop. 3) that h is essential. Noting that J^{ω} is an absolute retract we can find an extension $\bar{h}\colon X\to J^{\omega}$ of h which is of course also essential.

2.2. COROLLARY. *If X is a metric space such that $\operatorname{Ind}(X)\geqslant\omega$, or such that $\operatorname{Ind}(X)$ does not exist, then there is an essential $f\colon X\to J^{\omega}$.*

2.3 THEOREM. *Let X be a hereditarily normal space and let α and β be two ordinals. If Y is a subset of X such that $\operatorname{Ind}(Y)<\beta$ and for every open neighbourhood U of Y, $\operatorname{Ind}(X\setminus U)<\alpha$ then $\operatorname{Ind}(X)<\alpha+\beta$.*

Proof. (By transfinite induction w.r.t. β.) If $\beta=0$ then $Y=\emptyset$. Since $\emptyset$ is a neighbourhood of Y we have that $\operatorname{Ind}(X)<\alpha=\alpha+\beta$.

Let β be a limit ordinal such that the theorem is valid for every $\beta'<\beta$. If $\operatorname{Ind}(Y)<\beta$ then there is a $\gamma<\beta$ such that $\operatorname{Ind}(Y)<\gamma$. By induction we have that $\operatorname{Ind}(X)<\alpha+\gamma<\alpha+\beta$.

Now assume that the induction hypothesis is valid for all ordinals $<\beta+1$. Let A and B be two disjoint closed subsets of X. Since X is normal there are closed, disjoint neighbourhoods A' and B' of A and B, respectively. Assume that $\operatorname{Ind}(Y)<\beta+1$. Then there are open, disjoint subsets O_1 and O_2 of Y such that $A'\cap Y\subset O_1$, $B'\cap Y\subset O_2$ and $\operatorname{Ind}(Y\setminus(O_1\cup O_2))<\beta$. It is easily seen that $\operatorname{Cl}_X(A\cup O_1)\cap\cap(B\cup O_2)=\emptyset=(A\cup O_1)\cap\operatorname{Cl}_X(B\cup O_2)$. Since X is hereditarily normal this implies (Engelking [1], 2.2.1) that there exist disjoint open sets U_1 and U_2 in X such that $A\cup O_1\subset U_1$ and $B\cup O_2\subset U_2$. Define $\tilde{X}=X\setminus(U_1\cup U_2)$ and $\tilde{Y}=Y\setminus(U_1\cup U_2)$. Then we have that $\operatorname{Ind}(\tilde{Y})\leqslant\operatorname{Ind}(Y\setminus(O_1\cup O_2))<\beta$. If V is an open neighbourhood of $\tilde{Y}$ in $\tilde{X}$ then $V\cup U_1\cup U_2$ is an open neighbourhood of Y in X and hence $\operatorname{Ind}(\tilde{X}\setminus V)=\operatorname{Ind}(X\setminus(V\cup U_1\cup U_2))<\alpha$. Applying the induction hypothesis we obtain that $\operatorname{Ind}(\tilde{X})<\alpha+\beta$. Since $\tilde{X}$ separates A and B in X we have proved that $\operatorname{Ind}(X)<\alpha+\beta+1$.

3. The counterexample. We construct a compact metric space $\tilde{X}$ that admits no essential mapping into $J^{\omega+1}$ and has $\operatorname{Ind}(\tilde{X})=\omega+1$.

Consider the Hilbert cube $Q=\prod_{i\in N}I$ and let $0=(0,0,0,\ldots)\in Q$. Define for $i\in N$ the i-cube B_i in Q by

$$B_i=\{(x_j)_j\in Q|x_j\in[0,1/i]\text{ for }j\leqslant i\text{ and }x_j=0\text{ for }j>i\}.$$

Let $A_i(i\in N)$ be the closed set $\bigcup_{j=i}^{\infty}B_j$. Consider now the Cantor set C, represented in the usual way by a subset of I. Let (a_i,b_i), $i\in N$, be an enumeration of the gaps of C. Select an order preserving quotient mapping $p\colon C\to I$ such that if $p(x)=p(y)$ then $x=y$ or $\{x,y\}=\{a_i,b_i\}$ for some i. Let $X=C\times A_1$ and construct a quotient space $\tilde{X}$ of X by identifying the points (a_i,x) and (b_i,x) for every $i\in N$ and $x\in A_i$. Let $q\colon X\to\tilde{X}$ be the natural mapping and define the "projections" $\pi_1\colon\tilde{X}\to I$ and $\pi_2\colon\tilde{X}\to A_1$ by

$$\pi_1\circ q(r,x)=p(r)$$

and

$$\pi_2\circ q(r,x)=x.$$

Since $\pi_1|q(C\times\{0\})$ is a homeomorphism, we identify $q(C\times\{0\})$ with I.

3.1 CLAIM. *$\tilde{X}$ is a compact metrizable space.*

Proof. Since $\tilde{X}$ is a quotient of a compact metrizable space, it suffices to show that $\tilde{X}$ is Hausdorff. Since π_1 and π_2 are continuous, we only have to separate the points (a_i, x) and (b_i, x) for $x \notin A_i$. It is easily verified that

$$q(([0, a_i] \cap C) \times (A_1 \setminus A_i)) \quad \text{and} \quad q(([b_i, 1] \cap C) \times (A_1 \setminus A_i))$$

are disjoint open neighbourhoods of (a_i, x) and (b_i, x), respectively.

3.2 CLAIM. $\operatorname{Ind}(\tilde{X}) \leqslant \omega + 1$.

Proof. This is a straightforward application of Theorem 2.3. We put $Y = I$, $a = \omega$ and $\beta = 2$. If K is the complement of a neighbourhood of I in $\tilde{X}$ then there is an $i \in N$ with $K \subset \pi_2^{-1}(A_1 \setminus A_i)$. It is left to the reader to verify that $\operatorname{Ind}(\pi_2^{-1}(A_1 \setminus A_i)) = i-1$.

3.3 CLAIM. $\operatorname{Ind}(\tilde{X}) \geqslant \omega + 1$.

Proof. Let $\{F, G\}$ be a closed covering of $\tilde{X}$ such that $F \cap \pi_1^{-1}(\{1\}) = G \cap \pi_1^{-1}(\{0\}) = \emptyset$. Assume that $\operatorname{Ind}(F \cap G) \leqslant n$ for some $n \in N$. We shall prove that for every $r \in C$, $q(\{r\} \times A_{n+2})$ is contained in either F or G.

Let $r \in C$ and consider $\{F \cap q(\{r\} \times B_k), G \cap q(\{r\} \times B_k)\}$ for $k \geqslant n+2$. Note that $q|\{r\} \times A_1$ is an embedding. Since the cube $q(\{r\} \times B_k)$ is a k-dimensional Cantor-manifold (Engelking [1], 1.8.13) we have that either $q(\{r\} \times B_k) \subset F$ or $q(\{r\} \times B_k) \subset G$. If $q(\{r\} \times B_k) \subset F$ then $q(\{r\} \times (B_k \cap B_{k+1}))$, which is a k-dimensional face of $q(\{r\} \times B_{k+1})$, is contained in F. Since also $q(\{r\} \times B_{k+1}) \subset F$ or $q(\{r\} \times B_{k+1}) \subset G$ we have that $q(\{r\} \times B_{k+1}) \subset F$. So we may conclude that $q(\{r\} \times A_{n+2})$ is contained in either F or G.

Having established this consider $s = \sup\{r \in C | \{r\} \times A_{n+2} \subset q^{-1}(F)\}$. Since $q^{-1}(F)$ is closed we have that $\{s\} \times A_{n+2} \subset q^{-1}(F)$. If $t = \inf\{r \in C | r \geqslant s \text{ and } \{r\} \times A_{n+2} \subset q^{-1}(G)\}$ then $\{t\} \times A_{n+2} \subset q^{-1}(G)$. Suppose that $s = t$. In this case $q(\{s\} \times A_{n+2})$, which is homeomorphic to A_{n+2}, is contained in $F \cap G$ and hence $\operatorname{Ind}(F \cap G) \geqslant \omega$. If $s \neq t$ then there is an $i \in N$ such that $s = a_i$ and $t = b_i$. Put $k = \max\{n+2, i\}$ and note that $q(\{s\} \times A_k) = q(\{t\} \times A_k)$. This means that $\operatorname{Ind}(F \cap G) \geqslant \operatorname{Ind}(q(\{s\} \times A_k)) = \omega$.

3.4 CLAIM. *There is no essential mapping from $\tilde{X}$ into $J^{\omega+1}$.*

Proof. Let f be an essential mapping from $\tilde{X}$ into $J^{\omega+1}$. Recall that $J^{\omega+1} = (\{p^\omega\} \cup \bigcup_{i=0}^{\infty} (J^i \cup L^i)) \times I$ and put $D_i = f^{-1}(J^i \times I)$. Observe that $f|D_i\colon D_i \to J^i \times I$ is essential for every $i \in N \cup \{0\}$. We shall prove that for every $n \in N$ there are x_n and y_n in $\pi_2^{-1}(A_n)$ such that $|\pi_1(x_n) - \pi_1(y_n)| < 1/n$, $f(x_n) \in J^\omega \times \{0\}$ and $f(y_n) \in J^\omega \times \{1\}$. If this is true then $(x_n)_n$ and $(y_n)_n$ have the same set L of cluster points. This implies that $f(L) \subset (J^\omega \times \{0\}) \cap (J^\omega \times \{1\}) = \emptyset$ which contradicts the compactness of $\tilde{X}$.

Let λ be Lebesgue measure on I and pick an arbitrary natural number n. Since $\{D_i | i \in N\}$ is a collection of pairwise disjoint, closed sets we can find an $i > n$ such that $\lambda(D_i \cap I) < 1/n$. This enables us to select points $0 = p_0 < p_1 < \ldots < p_k = 1$ in I such that $p_{i+1} < p_i + 1/n$ and $\{p_1, p_2, \ldots, p_{k-1}\} \cap D_i = \emptyset$. Since D_i is closed

there is a neighbourhood U of $\{p_1, p_2, \ldots, p_{k-1}\}$ in I and a $j > i$ such that $\pi_1^{-1}(U) \cap \pi_2^{-1}(A_j) \cap D_i = \emptyset$. Note that $\{p(a_m) | m > j\}$ is dense in I. Select $m(1), m(2), \ldots, m(k-1)$ greater than j such that $p(a_{m(l)}) \in U$ for $l = 1, 2, \ldots, k-1$,

$$p(a_{m(1)}) < 1/n, \quad 1 - p(a_{m(k-1)}) < 1/n \quad \text{and} \quad 0 < p(a_{m(l+1)}) - p(a_{m(l)}) < 1/n$$

for $l = 1, 2, \ldots, k-2$. Then

$$\mathscr{P} = \{[0, a_{m(1)}] \cap C, [b_{m(1)}, a_{m(2)}] \cap C, \ldots, [b_{m(k-2)}, a_{m(k-1)}] \cap C, [b_{m(k-1)}, 1] \cap C\}$$

is a finite partition of C with clopen sets. Since for every $l < k$, $m(l) > j$ we have that $\{q(K \times A_1) \cap D_i | K \in \mathscr{P}\}$ is a clopen partition of D_i. Note that

$$\operatorname{diam}(\pi_1 \circ q(K \times A_1)) < 1/n$$

for every $K \in \mathscr{P}$. Since $f|D_i$ is essential we have that $f|q(K \times A_1) \cap D_i$ is essential for some $K \in \mathscr{P}$. Then $f(q(K \times A_n) \cap D_i)$ is dense in $J^i \times I$ and hence $f(q(K \times A_n) \cap D_i) = J^i \times I$. This can be seen as follows. Let $x \in J^i \times I$ and let V be a canonical closed neighbourhood of x in $J^i \times I$, i.e. V is an $(i+1)$-cell. If $f^{-1}(V) \cap q(K \times A_1) \cap D_i$ is contained in the $(n-1)$-dimensional set $\pi_2^{-1}(A_1 \setminus A_n)$ then $f|f^{-1}(V) \cap q(K \times A_1) \cap D_i$ is not an essential mapping into V. This implies that

$$f|q(K \times A_1) \cap D_i\colon q(K \times A_1) \cap D_i \to J^i \times I$$

is not essential. So we may pick x_n and y_n in $q(K \times A_n)$ such that $f(x_n) \in J^i \times \{0\}$ and $f(y_n) \in J^i \times \{1\}$. This proves the claim.

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