Essential mappings and transfinite dimension

by

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Abstract. We construct a compact metrizable space with inductive dimension $\omega + 1$ that admits no essential mappings into Henderson’s $(\omega + 1)$-dimensional absolute retract $J^{\omega +1}$.

1. Introduction. A continuous mapping $f: X \to I^* = [0, 1]^*$ is called essential if there is no continuous extension $g: X \to \partial I^*$ of $f|f^{-1}(\partial I^*)$, where $\partial I^*$ is the geometric boundary of $I^*$. The following characterization is well known (see e.g. Engelking [1], 3.2.10).

1.1 Theorem. A normal space has dim $\geq n$ if and only if it admits an essential mapping into $I^n$.

D. W. Henderson [2] has attempted to extend this result to transfinite inductive dimension.

1.2 Definition. Ind$(\emptyset) = -1$. Let $\alpha$ be an ordinal and $X$ a normal space. Ind$(X)$ is the smallest ordinal such that whenever every pair of disjoint closed subsets of $X$ can be separated by a closed set with Ind $< \alpha$. Ind$(X)$ is the union of disjoint open sets $U$ and $V$ with $A \subset U$ and $B \subset V$.

1.3 Definition (Henderson). For each countable ordinal $\alpha$ we define a compact metric space $J^\alpha$, its "boundary" $T^\alpha$ and a point $p^\alpha \in T^\alpha$.

(i) If $\alpha$ is finite then $J^\alpha = I^\alpha$, $T^\alpha = \partial I^\alpha$ and $p^\alpha = (0, 0, ..., 0)$.

(ii) If $\alpha$ has a successor $\alpha + 1$ then $J^{\alpha + 1} = J^\alpha \times I$, $T^{\alpha + 1} = (T^\alpha \times I) \cup ((J^\alpha \times \{0, 1\})$ and $p^{\alpha + 1} = (p^\alpha, 0)$.

(iii) If $\alpha$ is a limit, put $K^\alpha = J^\alpha \cup J^\beta$ for every $\beta < \alpha$, where $L^\beta$ is a half open arc such that $L^\beta \cap J^\beta = \{p^\beta\}$ is the end-point of $L^\beta$. $J^\alpha$ is defined as the one-point compactification of the discrete sum $\bigoplus_{\beta < \alpha} K^\beta$; $T^\alpha = J^\alpha \cup (\bigoplus_{\beta < \alpha} T^\beta)$ and $p^\alpha$ is the compactifying point.

A continuous mapping $f$ from a space $X$ into $J^\alpha$ is called essential if every continuous $g: X \to J^\alpha$ that satisfies $g|f^{-1}(T^\beta) = f|f^{-1}(T^\beta)$ is an onto mapping.

The following two theorems are due to Henderson [2].

1.4 Theorem. $J^\alpha$ is an absolute retract and Ind$(J^\alpha) = \alpha$.

1.5 Theorem. If there is an essential mapping from a normal space $X$ into $J^\alpha$ then Ind$(X) \geq \alpha$ or Ind$(X)$ does not exist.
Henderson asked the following question: if \( \text{Ind}(X) > \omega \), is there an essential \( f: X \to J^* \)? In view of Theorem 1.1 and the fact that there exist compact spaces with \( \dim < \text{Ind} \) it seems reasonable to restrict ourselves to metric spaces. We show that the answer to the question is yes for \( \alpha = \omega \) and no for \( \alpha = \omega + 1 \). This also solves two questions that were raised by R. Pol ([3], p. 238) who independently showed the following. There exists a countable ordinal \( \lambda \) such that for every countable \( \alpha > \lambda \) there is an \( \alpha \)-dimensional compactum without essential mappings into \( J^* \) ([3], Thm. 5.2). Note that Pol’s result, which was obtained by a method completely different from our’s, also gives different information concerning Henderson’s question.

2. Two theorems. In this section we prove the converse of Theorem 1.5 for \( \alpha = \omega \) and we give an addition theorem that will be used in the next section.

2.1 Theorem. If \( X \) is a normal space with infinite covering dimension then there is an essential \( f: X \to J^* \).

Proof. Let \( \dim(X) = \omega \). We construct sequences \( H_1 \supset H_2 \supset H_3 \supset \ldots \) and \( (A_\alpha)_{\alpha \in \omega} \) of closed subsets of \( X \) such that for every \( \alpha \in \omega \), \( \dim(H_\alpha) = \alpha \), \( \dim(A_\alpha) \geq \alpha \) and \( A_\alpha \subseteq H_\alpha \setminus H_{\alpha+1} \). Let \( H = X \) and assume that \( H_1 \) has been constructed. Since \( \dim(H) = \omega \) there exist disjoint closed sets \( A \) and \( B \) in \( H_1 \) such that every closed set \( M \) that separates \( A \) and \( B \) has \( \dim(M) \) (see Engling ([1], 3.1.27)). Select a closed covering \( \{F, G\} \) of \( H_1 \) with \( F \cap G = \emptyset \) and \( \dim(F) = \alpha > \omega \). The union of \( F \) and \( G \) is infinite-dimensional and hence (see Engling ([1], 3.1.8)) one of them, say \( F \), has infinite dim. Put \( H_{\alpha+1} = F \) and let \( A_{\alpha+1} \) be a closed set that separates \( F \) and \( B \) in \( H_{\alpha+1} \). Then \( A_{\alpha+1} \subset H_{\alpha+1} \setminus H_{\alpha+2} \) separates \( A \) and \( B \) in \( H_{\alpha+1} \) and hence \( \dim(A_{\alpha+1}) \geq \alpha + 1 \). This completes the induction.

Consider now \( J^* = \{\alpha^*\} \cup \bigcup_{i=0}^{\omega}(J^* \cup J^*) \). Since \( \dim(A_\alpha) \geq \alpha \) we may select for every \( i \in \omega \cup \{0\} \) an essential \( f_i: A_{2i+1} \to J^* \) and a continuous \( g_i \) from \( A_{2i,2} \) onto the closed interval \( \{0\} \cup \{\alpha^*\} \). Moreover, let \( h \) be the constant function from \( H = \bigcup_{i=0}^{\omega} H_i \) into \( \{0\} \cup \{\alpha^*\} \). Put \( A = \bigcup_{i=0}^{\omega} A_i \cup H \), which is a closed subset of \( X \). Since \( \{A_\alpha\}_{\alpha \in \omega} \) is a pairwise disjoint collection of clopen subsets of \( A \) one easily verifies that

\[
\hat{h} = \bigcup_{i=0}^{\omega} f_i \cup \bigcup_{i=0}^{\omega} g_i \cup h
\]

is a continuous mapping from \( A \) onto \( J^* \). The fact that the \( f_i \)'s are essential guarantees in view of Henderson ([2], Prop. 3) that \( h \) is essential. Noting that \( J^* \) is an absolute retract we can find an extension \( \hat{h}: X \to J^* \) of \( h \) which is of course also essential.

2.2 Corollary. If \( X \) is a metric space such that \( \text{Ind}(X) \geq \omega_\alpha \), or such that \( \text{Ind}(X) \) does not exist, then there is an essential \( f: X \to J^* \).

2.3 Theorem. Let \( X \) be a hereditarily normal space and let \( \alpha \) and \( \beta \) be two ordinals. If \( Y \) is a subset of \( X \) such that \( \text{Ind}(Y) < \beta \) and for every open neighbourhood \( U \) of \( Y \), \( \text{Ind}(X \setminus U) < \alpha \) then \( \text{Ind}(X) < \alpha + \beta \).

Proof. (By transfinite induction w.r.t. \( \beta \)) If \( \beta = 0 \) then \( Y = \emptyset \). Since \( \emptyset \) is a neighbourhood of \( Y \) we have that \( \text{Ind}(X) < \alpha + \beta \).

Let \( \beta \) be a limit ordinal such that the theorem is valid for every \( \beta' < \beta \). If \( \text{Ind}(Y) < \beta \) then there is a \( \gamma < \beta \) such that \( \text{Ind}(Y) < \gamma \). By induction we have that \( \text{Ind}(X) < \alpha + \gamma < \alpha + \beta \).

Now assume that the induction hypothesis is valid for all ordinals \( \beta' < \beta + 1 \). Let \( A \) and \( B \) be two disjoint closed subsets of \( X \). Since \( X \) is normal there are closed, disjoint neighbourhoods \( A' \) and \( B' \) of \( A \) and \( B \), respectively. Assume that \( \text{Ind}(Y) < \beta + 1 \). Then there are open, disjoint subsets \( O_1 \) and \( O_2 \) of \( X \) such that \( A' \cap \text{int} O_1 \) and \( B' \cap \text{int} O_2 \) are disjoint and \( \text{Ind}(Y \setminus (O_1 \cup O_2)) < \beta \). It is easy to see that \( \text{Cl}(A \cup O_1) \cap \text{Cl}(B \cup O_2) = \emptyset \) and \( \text{Cl}(O_1 \cup O_2) = \emptyset \). Since \( X \) is hereditarily normal this implies (Engling ([1], 2.2.1)) that there exist disjoint open sets \( U_1 \) and \( U_2 \) in \( X \) such that \( A \cup O_1 \subseteq U_1 \) and \( B \cup O_2 \subseteq U_2 \). Define \( X = X \setminus (U_1 \cup U_2) \) and \( Y = Y \setminus (U_1 \cup U_2) \). Then we have that \( \text{Ind}(Y) < \text{Ind}(X \setminus (O_1 \cup O_2)) < \beta \). If \( Y \) is an open neighbourhood of \( Y \) in \( X \) then \( V \cup U_1 \cup U_2 \) is an open neighbourhood of \( Y \) in \( X \) and hence \( \text{Ind}(X \setminus (U_1 \cup U_2)) < \beta \). Applying the induction hypothesis we obtain that \( \text{Ind}(X) < \alpha + \beta \). Since \( X \) separates \( A \) and \( B \) in \( X \) we have proved that \( \text{Ind}(X) < \alpha + \beta + 1 \).

3. The counterexample. We construct a compact metric space \( \bar{X} \) that admits no essential mapping into \( J^{**} \) and has \( \text{Ind}(\bar{X}) = \omega + 1 \).

Consider the Hilbert cube \( Q = \prod_{i \in N} I_i \) and let \( 0 = (0, 0, 0, \ldots) \in Q \). Define for \( i \in N \) the \( i \)-cube \( B_i \) in \( Q \) by

\[
B_i = \{ (y_i) \in Q | y_i \in [0, 1] \} \text{ for } i \in N \text{ and } x_i = 0 \text{ for } j \neq i.
\]

Let \( A_i \subseteq B_i \) be the closed set \( \bigcup_{j \neq i} B_j \). Consider now the Cantor set \( C \), represented in the usual way by a subset of \( I \). Let \( \{a_i, b_i\}, i \in N \) be an enumeration of the gaps of \( C \). Select an order preserving quotient mapping \( p: C \to I \) such that \( p(a_i) = p(y_i) \) then \( x = y \) or \( x = y \).

Let \( X = X \setminus \bigcup_{i=0}^{\omega} A_i \). Let \( Q: \bar{X} \to X \) be the natural mapping and define the “projections” \( p_1: \bar{X} \to I \) and \( p_2: \bar{X} \to A_i \) by

\[
p_1: q(f(\bar{x})) = p(t)
\]

and

\[
p_2: q(f(\bar{x})) = x
\]

Since \( p_1(q(C \times \{0\})) \) is a homeomorphism, we identify \( q(C \times \{0\}) \) with \( I \).

3.1 Claim. \( X \) is a compact metrizable space.
Proof. Since $\tilde{X}$ is a quotient of a compact metrizable space, it suffices to show that $X$ is Hausdorff. Since $\pi_1$ and $\pi_2$ are continuous, we only have to separate the points $(a_1, x)$ and $(a_2, x)$ for $a_1 \neq a_2$. It is easily verified that
\[ q(\{(a, u) \cap C\} \times (A_1 \setminus A_2)) \] and
\[ q(\{(a, 1) \cap C\} \times (A_1 \setminus A_2)) \] are disjoint open neighbourhoods of $(a_1, x)$ and $(a_2, x)$, respectively.

3.2. Claim. \( \text{Ind}(\tilde{X}) \leq \omega + 1 \).

Proof. This is a straightforward application of Theorem 2.3. We put $Y = I$, $a = 0$, and $b = 1$. If $K$ is the complement of a neighbourhood of $J$ in $X$, then there is an $i \in N$ with $K \subseteq \pi_i^{-1}(A_i \setminus A_i)$. It is left to the reader to verify that $\text{Ind}(\pi_i^{-1}(A_i \setminus A_i)) = i - 1$.

3.3. Claim. \( \text{Ind}(\tilde{X}) \geq \omega + 1 \).

Proof. Let $\{F, G\}$ be a closed covering of $\tilde{X}$ such that $F \cap \pi_i^{-1}(\{1\}) = G \cap \pi_i^{-1}(\{0\}) = \emptyset$. Assume that $\text{Ind}(F \cap G) \leq n$ for some $n \in N$. We shall prove that for every $r \in C$, $q(\{r\} \times A_n)$ is contained in either $F$ or $G$.

Let $r \in C$ and consider $F \cap q(\{r\} \times B_0), G \cap q(\{r\} \times B_0)$ for $k \geq n + 2$. Note that $q(\{r\} \times A_1)$ is an embedding. Since the cube $q(\{r\} \times B_0)$ is a k-dimensional Cantor-manifold (Engelking [1], 1.8.13) we have that either $q(\{r\} \times B_0) \subseteq F$ or $q(\{r\} \times B_0) \subseteq G$. If $q(\{r\} \times B_0) \subseteq F$ then $q(\{r\} \times B_0) \subseteq G$, which is a k-dimensional face of $q(\{r\} \times B_0)$, is contained in $F$. Since also $q(\{r\} \times B_0) \subseteq F$, we have that $q(\{r\} \times B_0) \subseteq F$. So we may conclude that $q(\{r\} \times A_n)$ is contained in either $F$ or $G$.

Having established this consider $s = \sup \{r \in C \mid \{r\} \times A_{n+2} \subseteq q^{-1}(F)\}$. Since $q^{-1}(F)$ is closed we have that $\{t \times A_{n+2} \subseteq q^{-1}(F)\}$. If $t = \infty \cap C \neq \emptyset$ and $t \times A_{n+2} \subseteq q^{-1}(G)$ then $\{t \times A_{n+2} \subseteq q^{-1}(G)\}$. Suppose that $s = t$. In this case $q(\{t\} \times A_{n+2})$, which is homeomorphic to $A_{n+2}$, is contained in $F$ and $G$, and hence $\text{Ind}(F \cap G) \geq \omega$. If $s \neq t$ then there is an $i \in N$ such that $s_1 = s$ and $t = b_i$. Put $k = \max \{n+2, i\}$ and note that $q(\{t\} \times A_k) = q(\{t\} \times A_k)$. This means that $\text{Ind}(F \cap G) \geq \text{Ind}(q(\{t\} \times A_k)) = \omega$.

3.4. Claim. There is no essential mapping from $\tilde{X}$ into $J^{n+1}$.

Proof. Let $f$ be an essential mapping from $\tilde{X}$ into $J^{n+1}$. Recall that $J^{n+1} = (J^n \cup \bigcup_{i=0}^{n} \{J^i \times L^i\}) \times I$ and put $D_i = f^{-1}(J^i \times I)$. Observe that $f(D_0) \cap D_1 = J \cap I$ is essential for every $i \in N \cup \{0\}$. We shall prove that for every $n \in N$ there are $x_n$ and $y_n$ in $\pi_n^{-1}(A_n)$ such that $|x_n - y_n| < 1/n$, $f(x_n) \in J^n \times \{0\}$, and $f(y_n) \in J^n \times \{1\}$. This is then $x_n$, and $y_n$ have the same set $L$ of cluster points. This implies that $f(L) \subseteq (J^n \times \{0\}) \cup (J^n \times \{1\}) = \emptyset$, which contradicts the compactness of $\tilde{X}$.

Let $\lambda$ be Lebesgue measure on $I$ and pick an arbitrary natural number $n$. Since $\{D_i \mid i \in N\}$ is a collection of pairwise disjoint, closed sets we can find an $i > n$ such that $\lambda(D_i \cap I) < 1/n$. This enables us to select $0 < p_0 < p_1 < \ldots < p_{n-1} = 1$ in $I$ such that $p_{i+1} < p_i < \ldots < p_{n-1} = 1$ in $I$ such that $p_{i+1} < p_i < \ldots < p_{n-1} = 1$ in $I$ and a $\lambda > 1$ such that $\frac{1}{\lambda} < \lambda(U) \cap \pi_n^{-1}(A_n) \cap D_i = \emptyset$. Note that $\{p(\alpha) \mid \alpha > n\}$ is dense in $I$. Select $m(1), m(2), \ldots, m(k-1)$ greater than $\lambda$ such that $p(\alpha_i) \in \gamma$ for $i = 1, 2, \ldots, k-1$.

\[ \begin{align*}
\lambda(U) & < 1/n, \\
1 - \lambda(U) & > 1/n, \quad 0 < \lambda(U) < 1/n.
\end{align*} \]

for $i = 1, 2, \ldots, k-2$. Then $\mathcal{P} = \{0, \alpha_i \mid \alpha \in \gamma \cap \pi_n^{-1}(A_n) \cap D_i = \emptyset\}$. Since for every $i < k$, $m(i) > j$ we have that $\{q(\pi_i \times A_1) \cap D_i \mid i \in \mathcal{P}\}$ is a clopen partition of $D_i$. Note that $\text{diam}(\pi_i \times A_1) < 1/n$ for every $i \in \mathcal{P}$. Since $f(D_i)$ is essential we have that $f(q(\pi_i \times A_1) \cap D_i)$ is essential for some $i \in \mathcal{P}$. Then $f(q(\pi_i \times A_1) \cap D_i)$ is dense in $f(J \times I)$ and hence $f(q(\pi_i \times A_1) \cap D_i) = f(J \times I)$. This can be seen as follows. Let $x \in f(J \times I)$ and let $V$ be a canonical closed neighbourhood of $x$ in $J^{n+1}$, i.e., $V$ is an $(l+1)$-collar. If $f^{-1}(V) \cap q(\pi_i \times A_1) \cap D_i$ is contained in the $(n+1)$-dimensional set $\pi_n^{-1}(A_n) \cap D_i$ then $f^{-1}(V) \cap q(\pi_i \times A_1) \cap D_i$ is not an essential mapping into $V$. This implies that $f(q(\pi_i \times A_1) \cap D_i) = f(J \times I)$ is not essential. So we may pick $x_n$ and $y_n$ in $q(\pi_i \times A_1)$ such that $f(x_n) \in J^n \times \{0\}$ and $f(y_n) \in J^n \times \{1\}$. This proves the claim.

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