# Finite-to-one restrictions of continuous functions (*)

by

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**Abstract.** It is shown that if $f\colon X\to Y$ is a map of locally compact metric spaces and $p=\dim Y<\infty$, then there exists a set $A\subset X$ such that $\dim A\leqslant \sup\{\dim f^{-1}(y)\colon y\in Y\}$ and no fiber $f^{-1}(y)$, $y\in Y$, contains more than $p$ points of $X\setminus A$. A connection between this result and the problem of characterization of $Q$-manifolds is indicated.

Let $f\colon X\to Y$ be a map of a locally compact metric space $X$. In this note we consider the question whether the structure of $f$ can be significantly simplified by passing to a restriction $f|X\setminus A$, where $A$ is an appropriately chosen subset of $X$ of dimension comparable to $\dim(f)=\sup\{\dim f^{-1}(y)\colon y\in Y\}$. In §§ 1–3 we show that if $\dim Y<\infty$ then this is in fact so. Specifically, we prove:

THEOREM 1. *Let $f\colon X\to Y$ be a $\sigma$-closed map of separable metric spaces such that $k=\dim(f)$ and $p=\dim Y$ are finite. Then, there is a set $A\subset X$ such that $\dim A\leqslant k$ and $f|X\setminus A$ is $p$-to-1.*

COROLLARY 2. *In notation of Theorem 1 there is a set $B\subset X$ such that $\dim B\leqslant k+E(p/2)$ and $f|X\setminus B$ is 1-to-1.*

Here, we denote by $E(x)$ the integer part of $x$ and we say that a map $f\colon X\to Y$ is $p$-to-1 if no fiber $f^{-1}(y)$, $y\in Y$, contains more than $p$ points.

The above results are applied in § 4 to give certain conditions under which maps $[0,1]^s\to R^n$ may be approximated by maps whose images are transverse, in a very vague sense, to all fibers of a given map $f\colon R^n\to Y$.

The statements of §§ 3, 4 may be viewed as selection type results, with Corollary 2 providing a certain lower bound for the maximal dimension of closed subsets of $Y$ over which $f$ admits a continuous selection. (In case $f$ is open, a classical result on sections of $f$ is given in [RC]).

The considerations of this note have been motivated by the problem of characterization of Hilbert cube manifolds. A dimension-theoretic approach to this problem naturally leads to questions concerning the possibility of improving properties of a map by neglecting a "small" subset of its domain; however in contrast

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to Theorem 1 the range of the map may be of infinite dimension. We formulate some of these questions in § 6 and we preceed this by showing in § 5 how Theorem 1 can be applied to derive a characterization of $Q$-manifolds. We note that, although formally new, this characterization can be derived from a result of R. D. Daverman and J. Walsh [DW] and yet another dimension-theoretic result, due to J. Walsh [W]. Nevertheless, we sketch a proof of it to indicate the approach alluded to above and to illustrate the connection between a problem concerning $Q$-manifolds and results, or questions, in dimension theory.

**Notation.** All spaces are assumed to be separable and topologized by a fixed metric which we denote by $\varrho$. We say that $f\colon X\to Y$ is *$\sigma$-closed* if $X$ is the union of countably many closed sets $X_i$ such that each restriction $f|X_i\colon X_i\to F(X_i)$ is a closed map. Closed maps with compact fibers are called proper.

Remark 1. By a lemma of I. A. Vainstein ([E], p. 139) if $f\colon X\to Y$ is closed then there is an open set $U\subset X$ such that $f|X\setminus U$ is proper and $f(U)$ is a countable set.

We say that a family of sets is of *size* $\varepsilon$ if the diameter of each of its members is $<\varepsilon$. The boundary of a set $A$ is denoted by $\partial A$. We write $N$ (resp. $R$) for the set of integers (resp. real numbers) and $I$ for the segment $[0,1]$. Undefined notions have the meaning of [E].

§ 1. **0-dimensional maps.** In this section we demonstrate Theorem 1 in the special case where $\dim(f)=0$. We need two lemmas.

LEMMA 1. *Let $f\colon X\to Y$ be a closed map with $\dim(f)=0$ and let $\varepsilon>0$. Then, there is an open cover $\mathscr{W}$ of $Y$ with the property that whenever $\{U\}$ refines $\mathscr{W}$ and $\overline{V}\subset\operatorname{int}U$ then there are discrete families $\mathscr{D}$ and $\mathscr{E}=\{E(D)\colon D\in\mathscr{D}\}$ of open subsets of $X$ such that:*

(a) *if $D\in\mathscr{D}$ then $\operatorname{diam}(D\cup E(D))<\varepsilon$ and $E(D)\subset f^{-1}(U\setminus\overline{V})$;*

(b) *$\mathscr{D}$ covers $f^{-1}(\overline{V})$ and $E(D)$ is a neighbourhood of $\partial D$, for each $D\in\mathscr{D}$;*

(c) *if $x_1,x_2\in\bigcup\mathscr{E}$ and $f(x_1)=f(x_2)$ then $\varrho(x_1,x_2)<\varepsilon$.*

Proof. For each $y\in Y$ there is a neighbourhood $G(y)$ of $f^{-1}(y)$ in $X$ which is the union of a discrete collection $\mathscr{G}(y)=\{G_1(y),G_2(y),\ldots\}$ of open sets of size $\varepsilon$. Since $f$ is closed we have $f^{-1}(W(y))\subset G(y)$ for some neighbourhood $W(y)$ of $y$. Let $\mathscr{W}=\{W(y)\colon y\in Y\}$. If $\overline{V}\subset\operatorname{int}U$ and $U\subset W(y)$ then let $F_0,F_1,\ldots$ be open neighbourhoods of $\overline{V}$ satisfying $\overline{F}_0\subset U$ and $\overline{F}_{i+1}\subset F_i$ for each $i$. Write

$$\mathscr{D}=\{D_i\colon i\in N\}\quad\text{where}\quad D_i=G_i(y)\cap f^{-1}(F_{4i}),$$

and

$$E(D_i)=G_i(y)\cap f^{-1}(F_{4i-1}\setminus\overline{F}_{4i+1}),\ i\in N.$$

Conditions (a) and (b) are clearly met. Also, $f(E(D_i))\cap f(E(D_j))=\emptyset$ which coupled with (a) yields (c).

DEFINITION. We say that $g\colon A\to B$ is a $(p,\varepsilon)$-map if each point-inverse $g^{-1}(b)$, $b\in B$, is a union of $p$ sets of size $\varepsilon$.



LEMMA 2. *Let $\varepsilon>0$, let $f\colon X\to Y$ be a closed map with $\dim(f)=0$ and $\dim Y=p<\infty$ and, for $i=1,2,\ldots,l$, let $K_i$ and $L_i$ be closed disjoint subsets of $X$. Then, there are open subsets $E_i$ of $X$ separating $X$ between $K_i$ and $L_i$ and such that $f|\overline{E}_1\cup\ldots\cup\overline{E}_l$ is a $(p,\varepsilon)$-map.*

Proof. Take closed sets $S_1,\ldots,S_l$ so that $S_i$ separates $X$ between $K_i$ and $L_i$; we may require that the metric $\varrho$ of $X$ is such that $\varrho(K_i\cup L_i,S_i)>\varepsilon$ for each $i\leqslant l$ (otherwise replace $\varrho(x_1,x_2)$ by $\varrho(x_1,x_2)+\sum_{i\leqslant 1}|\lambda_i(x_1)-\lambda_i(x_2)|$ for suitably chosen maps $\lambda_1,\ldots,\lambda_l$). Let $\mathscr{W}$ be a cover of $Y$ assured by Lemma 1. By a result of Morita there is a locally finite open cover $\mathscr{U}$ of $Y$ such that $\{\partial U\colon U\in\mathscr{U}\}$ is of order $p-1$ and $\mathscr{U}$ refines $\mathscr{W}$; see [E], p. 229. Let $\{V(U)\colon U\in\mathscr{U}\}$ be a closed shrinking of $\mathscr{U}$ such that $\{\overline{U}\setminus V(U)\colon U\in\mathscr{U}\}$ is of order $p-1$, and for $U\in\mathscr{U}$ let $\mathscr{D}(U)$ and $\{E(D)\colon D\in\mathscr{D}(U)\}$ be families provided by Lemma 1 for the pair $(U,V(U))$. We write

$$\mathscr{D}_i=\{D\in\bigcup\{\mathscr{D}(U)\colon U\in\mathscr{U}\}\colon D\cap S_i\neq\emptyset\},$$

$$T_i=\bigcup\{\partial D\colon D\in\mathscr{D}_i\}\quad\text{and}\quad\tilde{E}_i=\bigcup\{E(D)\colon D\in\mathscr{D}_i\}.$$

Then, $T_i$ contains the boundary of a neighbourhood of $S_i$ in $X\setminus(K_i\cup L_i)$ and hence separates $X$ between $K_i$ and $L_i$. Let $B=\tilde{E}_1\cup\ldots\cup\tilde{E}_l$. To show that $f|B$ is a $(p,\varepsilon)$-map fix $x\in B$ and let $\mathscr{U}_0=\{U\in\mathscr{U}\colon f(x)\in U\setminus V(U)\}$; then $\operatorname{card}\mathscr{U}_0\leqslant p$. We have

$$(f|B)^{-1}f(x)\subset f^{-1}f(x)\cap\bigcup\{E(D)\colon D\in\mathscr{D}(U)\text{ and }U\in\mathscr{U}_0\},$$

and for each $U\in\mathscr{U}_0$ the set $f^{-1}f(x)\cap\bigcup\{E(D)\colon D\in\mathscr{D}(U)\}$ is of size $\varepsilon$, by property (c) of $\{E(D)\colon D\in\mathscr{D}(U)\}$. Thus $(f|B)^{-1}f(x)$ is a union of $p$ sets of size $\varepsilon$ and we may let $E_i$ to be any neighbourhood of $T_i$ whose closure is contained in $\tilde{E}_i$.

PROPOSITION 1. *Let $f\colon X\to Y$ be a $\sigma$-closed map with $\dim(f)=0$ and $\dim Y=p<\infty$. Then, there is a set $A\in F_\sigma(X)$ such that $f|A$ is $p$-to-1 and $\dim(X\setminus A)=0$.*

Proof. Let $X_1\subset X_2\subset\ldots$ be closed subsets of $X$ such that $\bigcup X_i=X$ and each $f|X_i$ is a closed map. Let $\{A_i,B_i\colon i\in N\}$ be closed subsets of $X$ such that $A_i\cap B_i=\emptyset$ and both $\{A_i\colon i\in N\}$ and $\{X\setminus B_i\colon i\in N\}$ are bases of neighbourhoods of $X$. With $p=\dim Y$ we shall construct relatively open subsets $U_i^n$ and $V_i^n$ of $X_n$, $i\leqslant n$, so that

(a) $\overline{U}_i^n\cap\overline{V}_i^n=\emptyset$ and $U_i^n\supset X_n\cap A_i$, $V_i^n\supset X_n\cap B_i$;

(b) $U_i^{n+1}\supset\overline{U}_i^n$ and $V_i^{n+1}\supset\overline{V}_i^n$;

(c) $f|\bigcup_{i=1}^{n}X_n\setminus(U_i^n\cup V_i^n)$ is a $(p,1/n)$-map

The inductive construction. Suppose $\{U_i^n,V_i^n\colon i=1,\ldots,n\}$ are known. We write $U_{n+1}^n=\emptyset=V_{n+1}^n$ and

$$K_i=A_i\cap X_{n+1}\cup\overline{U}_i^n,\ L_i=B_i\cap X_{n+1}\cup\overline{V}_i^n\quad\text{for}\quad i\leqslant n+1$$

Lemma 2 applied to $f|X_{n+1}$ readily implies the existence of the required sets $\{U_i^{n+1}, V_i^{n+1}: i \leqslant n+1\}$. (The first step is analogous). Write

$$T_i^n = X_n \setminus \bigcup \{U_{ij}^k \cup V_i^k: k \in N\};$$

then $T_i^n$ is a closed set separating $X_n$ between $A_i$ and $B_i$. Thus $A = \bigcup \{T_i^n: i, n \in N\}$ is an $F_\sigma$-set in $X$ such that $\dim(X_n \setminus A) = 0$ for each $n$, yielding $\dim(X\setminus A) = 0$ by the countable sum theorem. Finally, condition (c) shows that $f|\bigcup\{T_i^n: i, n \leqslant k\}$ is a $(p, \varepsilon)$-map for each $k \in N$ and $\varepsilon > 0$, whence no fiber of $f|A$ contains $p+1$ points.

## § 2. Restrictions that lower a map's dimension.

PROPOSITION 2. *Let $f: X \to Y$ be a $\sigma$-closed map with* $\dim(f) = k$ *and* $\dim Y < \infty$. *Then for each $l < k$ there exists a set $X_l \in F_\sigma(X)$ such that* $\dim X_l \leqslant l$ *and* $\dim(f|X\setminus X_l) \leqslant k-l-1$.

Proof. If $X_{k-1}$ is constructed then it suffices to require for $l < k-1$ that $X_l$ be an $F_\sigma$-set in $X_{k-1}$ with $\dim X_l \leqslant l$ and $\dim(X_{k-1}\setminus X_l) \leqslant k-l-2$. The existence of $X_{k-1}$ in turn follows routinely from Remark 1 and the following

LEMMA 3. *Let $f: X \to Y$ be a proper map with* $\dim(f) = k < \infty$ *and* $\dim Y < \infty$, *and let $A$ and $B$ be disjoint closed subsets of $X$. Then, there is a closed set $T$ in $X$ such that* $\dim T \leqslant k-1$ *and, for each $y \in Y$, $T$ separates $f^{-1}(y)$ between $A$ and $B$.*

Proof. Let $\mathscr{F} = \bigcup \{N^k: k \geqslant 0\}$, the set of all finite sequences of integers. For $i \in \mathscr{F}$ define the integer $|i|$ by the requirement that $i \in N^{|i|}$. We angree that $*$ is the element of $N^0$ and if $i \in \mathscr{F}$, $p \in N$, then $(i, p)$ is the naturally defined member of $N^{|i|+1}$. We shall construct sets $F(i)$, $U(i)$, $V(i)$ so that the following conditions are satisfied for each $i \in \mathscr{F}$:

(a) $F(i)$ is closed in $Y$ and $U(i)$ and $V(i)$ are open sets in $X$ with $\overline{U}(i) \cap \overline{V}(i) = \emptyset$;

(b) $F(*) = Y$ and $U(*) \supset A$, $V(*) \supset B$;

(c) for each $p \in N$ we have $U(i, p) \supset U(i) \cap f^{-1}(F(i, p))$ and $V(i, p) \supset V(i) \cap f^{-1}(F(i, p))$;

(d) $F(i) = \bigcup \{F(i, p): p \in N\}$ and $\operatorname{diam} F(i) < 1/|i|$;

(e) the set $E(i) = f^{-1}(F(i)) \setminus (U(i) \cup V(i))$ admits an open cover of size $1/|i|$ and order $k-1$;

(f) in notation of (e), the family $\{E(i, p): p \in N\}$ is discrete in $X$.

Assuming the above sets to be constructed write

$$T_n = \bigcup \{E(i): |i| = n\} \quad \text{and} \quad T = \bigcap \{T_n: n \geqslant 0\}.$$

If $y \in Y$ then, by (d), there are $i_1, i_2, \ldots \in N$ such that

$$y \in F(i_1) \cap F(i_1, i_2) \cap \ldots,$$

and we have $f^{-1}(y) \setminus T = U(y) \cup V(y)$, where the set

$$U(y) = f^{-1}(y) \cap \bigcup \{U(i_1, \ldots, i_p): p \in N\}$$



and the analogously defined set $V(y)$ form the necessary partition of $f^{-1}(y) \setminus T$. To show that $\dim(T) \leqslant k-1$ we notice that, by (e), each set $T_n$ is closed and admits an open cover of size $1/n$ and order $k-1$. Thus compacta in $T$ are of dimension $\leqslant k-1$ and $\dim T \leqslant k-1$ in case $X$ is compact. In the general case we infer that, at least, each fiber of $f|T$ is of dimension $\leqslant k-1$. Moreover, $f|T = \beta\alpha$, where $\alpha: T \to N^\infty$ and $\beta: \operatorname{im}(\alpha) \to Y$ are defined by the requirements that

$$t \in E(i_1) \cap E(i_1, i_2) \cap \ldots, \quad \text{where} \quad (i_1, i_2, \ldots) = \alpha(t),$$

and

$$\beta(i_1, i_2, \ldots) \in \bigcap_{p=1}^{\infty} F(i_1, \ldots, i_p).$$

Both $\alpha$ and $\beta$ are continuous, c.f. (d) and (f), and so $f|T = \beta\alpha$ implies that $\alpha$ is proper and $\dim(\alpha) \leqslant \dim(f|T) \leqslant k-1$. Therefore $\dim T \leqslant k-1$ by a classical theorem of Hurewicz ([E], p. 136).

It remains to construct the sets satisfying (a)–(e), which is done by induction on $|i|$. Assume that $F = F(i)$, $U = U(i)$ and $V = V(i)$ are constructed so that (a) and (b) hold. Write $q = \dim Y$ and let $W_1, \ldots, W_q$ be disjoint open subsets of $X$ such that each of them separates $X$ between $\overline{U}$ and $\overline{V}$. For $l \leqslant q$ and $y \in F$ let $P_l(y)$ be a closed set separating $f^{-1}(y)$ between $\overline{U}$ and $\overline{V}$ and such that $\dim P_l(y) < \dim f^{-1}(y) \leqslant k$ and $P_l(y) \subset W_l$. Let $Q_l(y)$ be a neighbourhood of $P_l(y)$ in $W_l$ which admits an open cover of size $(n+1)^{-1}$ and order $k-1$. Since $f$ is closed it follows that there is a neighbourhood $G(y)$ of $y$ in $F$ such that $f^{-1}(\overline{G}(y)) \cap Q_l(y)$ separates $f^{-1}(\overline{G}(y))$ between $\overline{U}$ and $\overline{V}$, for each $l \leqslant q$. By a result of Ostrand ([E], p. 228) there is an open cover $\mathscr{G}$ of $F$ which refines $\{G(y): y \in F\}$ and is the union of its discrete sub-families $\mathscr{G}_1, \ldots, \mathscr{G}_q$. Then, for each $G \in \mathscr{G}$ there are open subsets $U(G)$ and $V(G)$ of $X$, with disjoint closures, such that

$$U(G) \supset f^{-1}(G) \cap U, \quad V(G) \supset f^{-1}(G) \cap V,$$

and if $G \in \mathscr{G}_l$ then

$$f^{-1}(G) \setminus (U(G) \cup V(G)) \subset Q_l(y) \quad \text{for some} \quad y \in F.$$

Let $\{F(G): G \in G\}$ be a closed shrinking of $\mathscr{G}$ of size $1/|i|$ and let

$$E(G) = f^{-1}(F(G)) \setminus (U(G) \cup V(G))$$

for $G \in \mathscr{G}$. Each of the families $\{E(G): G \in \mathscr{G}_l\}$ is discrete and for $G \in \mathscr{G}_l$ and $H \in \mathscr{G}_m$ we have $E(G) \cap E(H) \subset W_l \cap W_m = \emptyset$. Thus $\{E(G): G \in \mathscr{G}\}$ is discrete. With $G_1, G_2, \ldots$ being an enumeration of $\mathscr{G}$ we may thus write

$$F(i, p) = F(G_p), \quad U(i, p) = U(G_p), \quad V(i, p) = V(G_p)$$

to get the desired sets satisfying conditions (a)–(e). This concludes the proof of the Lemma and of Proposition 1.

Remark 2. In case $X$ is compact Proposition 2 can also be proved using a theorem announced by B. A. Pasynkov in [P]. Conversely, Pasynkov's theorem can be extended to $\sigma$-closed maps and derived as a consequence of Proposition 2:

COROLLARY 1 (C. f. [P]). *Assume* $f\colon X\to Y$ *is $\sigma$-closed and* $\dim(f)=k<\infty$, $\dim Y<\infty$. *Then, there exists a map* $g\colon X\to I^k$ *such that* $\dim(f\times g)=0$.

Proof. By Proposition 2 there is a set $A\in F_\sigma(X)$ such that $\dim A=0$ and $\dim(f|X\setminus A)\leqslant k-1$. Take a map $u\colon X\to I$ such that $f|A$ is 1-to-1; then $\dim(f\times u)\leqslant k-1$ and the result follows by induction on $k$.

**§ 3. Proof of Theorem 1 and of Corollary 2.** Assume notation of Theorem 1. By Proposition 2 there is a set $P\subset X$ such that $\dim P\leqslant k-1$ and $\dim(f|X\setminus P)=0$. By a well-known theorem of Tumarkin ([E], p. 45) there is a $G_\delta$-set $\tilde P$ in $X$ such that $\tilde P\supset P$ and $\dim\tilde P\leqslant k-1$. Then, $f|X\setminus\tilde P$ is $\sigma$-closed any by Proposition 1 there is a set $Q\in F_\sigma(X\setminus\tilde P)$ such that $\dim(X\setminus\tilde P\setminus Q)=0$ and $f|Q$ is $p$-to-1. We let $A=X\setminus Q$ to get the desired set.

Proof of Corollary 2. We consider first the case where $\dim(f)=0$. Take sets $Y=Y_p\supset Y_{p-1}\supset\ldots$ so that $\dim Y_i=i$ and $\dim(Y_i\setminus Y_{i-1})=0$ for $i=0,1,\ldots,p$. (Simply require that $Y_{i-1}$ be the union of the boundaries of all members of an appropriate basis of the topology of $Y_i$). Let $Y_q=Y$ if $q>p$ and

$$X_j=f^{-1}(Y_{2j+1}\setminus Y_{2j-1}),\quad j=0,1,\ldots,r+1,$$

where $r$ is such that $2r+1\leqslant p$ and $2r+3>p$.

Applying Theorem 1 to $f|X_j\colon X_j\to Y_{2j+1}\setminus Y_{2j-1}$ we get sets $A_j\subset X_j$ such that $\dim A_j=0$ and $f|X_j\setminus A_j$ is 1-to-1. We let $A=X_{r+1}\cup A_0\cup\ldots\cup A_r$; then $f|X\setminus A$ is 1-to-1 and it follows easily that $\dim A\leqslant E(p/2)$. (Observe that $\dim X_{r+1}\leqslant\dim(f)+\dim(Y\setminus Y_{2r+1})\leqslant 0$). This proves the Corollary under the additional assumption $k=0$. The general case now follows easily from Proposition 2, c.f. the proof of Theorem 1.

**§ 4. Cells general positioned with respect to a map.** In this section we fix $f\colon R^n\to Y$ where $p=\dim Y<\infty$. Our goal is to approximate maps $I^s\to R^n$, $s<n-\dim(f)$, by maps with images intersecting each fiber of $f$ in finitely many points or, possibly, in sets of cardinality $\leqslant 1$. If $s\geqslant 2$ then the results are obtained under a global assumption on $f$ stating that the *singular set* of $f$,

$$S(f)=\{x\in I^s\colon f^{-1}f(x)\neq\{x\}\}$$

is tamely embedded, in sense of Štanko demension theory ([E1], [Š]). We write $C(I^s,R^n)$ for the space of all maps $I^s\to R^n$ (compact-open topology).

THEOREM 2. *If* $\dim(f)\leqslant n-2$ *then* $\{\alpha\in C(I,R^n)\colon f\alpha$ *is $p$-to-1*$\}$ *is dense in* $C(I,R^n)$.

Addendum. *If* $\dim(f)+E(p/2)\leqslant n-2$ *then* $\{\alpha\in C(I,R^n)\colon f\alpha$ *is 1-to-1*$\}$ *is dense in* $C(I,R^n)$.



Proof. Let $E=C(I,R^n)$ and

$$G(\varepsilon)=\{\alpha\in E\colon f\alpha \text{ is a } (p,\varepsilon)\text{-map}\},$$

$$G=\{\alpha\in E\colon f\alpha \text{ is } p\text{-to-1}\}=\bigcap\{G(\varepsilon)\colon \varepsilon>0\}$$

(In the case of the Addendum replace $p$ by 1 in these definitions). We omit the verification that each $G(\varepsilon)$ is open in $E$. By the Baire category theorem it remains to prove that whenever $\alpha\in E$ and $\varepsilon>0$ are given then $\alpha\in\bar G(\varepsilon)$.

Fix $r>0$ and take $\delta\in(0,r)$ so that if $|t-s|\geqslant\varepsilon/2$ then $\varrho(\alpha(s),\alpha(t))>3\delta$. Let $\mathcal{J}$ be a triangulation of $I$ with $\operatorname{diam}\alpha(J)<\varepsilon$ for $J\in\mathcal{J}$. By § 3, there is a set $A\subset R^n$ such that $\dim A\leqslant n-2$ and $f|R^n\setminus A$ is $p$-to-1. There is no loss of generality in assuming that $\alpha(\partial J)\subset R^n\setminus A$ for $J\in\mathcal{J}$ and

$$\text{if } \alpha^{-1}\alpha(x)\neq\{x\}\quad\text{then}\quad \operatorname{card}\alpha^{-1}\alpha(x)=2 \text{ and } x\in\bigcup\{\partial J\colon J\in\mathcal{J}\}.$$

Let $\{U(J)\colon J\in\mathcal{J}\}$ be a collection of pair-wise disjoint open connected sets in $R^n$ such that $U(J)\supset\alpha(J\setminus\partial J)$ and $\operatorname{diam}U(J)<\delta$. It follows from Mazurkiewicz theorem ([E], p. 80) that there are continua $C(J)$ such that $\alpha(\partial J)\supset C(J)$ and $C(J)\setminus\alpha(\partial J)\subset U(J)\setminus A$ for each $J\in\mathcal{J}$. Then, $C=\bigcup\{C(J)\colon J\in\mathcal{J}\}$ is a compact set in $R^n\setminus A$ and thus there is a neighborhood $V$ of $C$ such that $f|V$ is a $(p,\delta)$-map. With $S=\bigcup\{\alpha(\partial J)\colon J\in\mathcal{J}\}$ we may use the Hahn-Mazurkiewicz theorem to get for each $J$ an arc $\beta_J\subset V\cap U(J)\setminus f^{-1}f(S)\cup\alpha(\partial J)$ having $\alpha(\partial J)$ as end-points, and we define $\beta\in E$ so that $\beta(J)=\beta_J$ for all $J\in\mathcal{J}$. Then, $\operatorname{im}(\beta)\subset V$ and

(a) $\varrho(\beta,\alpha)<\delta<r$;
(b) if $y\in Y$ then $f^{-1}(y)\cap V$ is a union of $p$ sets size $\delta$;
(c) if $K\subset\operatorname{im}(\beta)$ is a set of size $\delta$ then $\operatorname{diam}\beta^{-1}(K)<\varepsilon$.

(In fact, (a) implies that $\beta^{-1}(K)\subset\alpha^{-1}(\tilde K)$ where $\operatorname{diam}\tilde K<3\delta$) .Thus $\beta\in G_\varepsilon$ and $\varrho(\beta,\alpha)<r$; by the arbitrareness of $r$ this shows that $\alpha\in\bar G(\varepsilon)$.

It is convenient to quote now the following

LEMMA 4. *Let $A$ be a subset of $R^n$ such that every map $(I^2,\partial I^2)\to(R^n,R^n\setminus A)$ is relative $\partial I^2$ approximable by mappings with images missing $A$. Let $s\in N$ satisfy $s+\dim A<n$.*

(i) *If $A$ is $\sigma$-compact then* $\{\beta\in C(I^s,R^n)\colon \operatorname{im}(\beta)\cap A=\emptyset\}$ *is dense in* $C(I^s,R^n)$;

(ii) *If $\alpha\colon\partial I^s\to R^n$ satisfies $\operatorname{im}(\alpha)\cap A=\emptyset$ then there is a compact set $C\subset R^n\setminus A$ such that $\alpha$ is null-homotopic in every neighbourhood $N$ of $C$ in $R^n$.*

Comment. Part (i) needs to be shown for compacta only and follows by induction on $s$, using Alexander duality and Hurewicz theorem. Se [Š], [B], [E1]. Part (ii) follows in a similar but more delicate manner, using Sitnikov duality [M] for non-closed subsets of $R^n$ and Hurewicz theorem in Borsuk's weak shape theory. For a proof see the recent paper [S] of S. Spież.

PROPOSITION 3. *If $s \leqslant n-\dim(f)$ and $n \geqslant 5$ then $\{\alpha \in C(I, R^n): \dim(f\alpha) = 0\}$ is dense in $C(I^s, R^n)$.*

Proof. If $s = 2$ then it follows using Theorem 2 that the set

$$H = \{\alpha \in C(I^2, R^n): f\alpha|W \text{ is } p\text{-to-}1\},$$

where

$$W = \{(s, t) \in I^2: \text{ either } s \text{ or } t \text{ are rational}\},$$

is dense in $C(I^2, R^n)$. If $\alpha \in H$ then each fiber of $f\alpha$ meets $W$ in finitely many points whence $\dim(f\alpha) = 0$.

If $s > 2$ we fix embeddings $\alpha_i: I^2 \to R^n$ such that $\dim(f\alpha_i) = 0$ for each $i \in N$ and $\{\alpha_i: i \in N\}$ is dense in $C(I^2, R^n)$.

CLAIM. *There is a set $A \in F_\sigma(R^n)$ such that*

(i) $\dim A \leqslant \dim(f)-1$ *and* $\dim(f|R^n \setminus A) = 0$, *and*

(ii) $A \cap \bigcup \{\operatorname{im}(\alpha_i): i \in N\} = \emptyset$.

Assuming the claim it follows from Lemma 4 that

$$G = \{\alpha \in C(I^s, R^n): \operatorname{im}(\alpha) \cap A = \emptyset\}$$

is dense in $C = C(I^s, R^n)$. By Baire category theorem $G \cap H$ is dense in $C$, where $H = \{\alpha \in C: \dim(\alpha) = 0\}$. Evidently, $\dim(f\alpha) = 0$ for $\alpha \in G \cap H$.

The proof of the claim can be repeated after that of Proposition 2, except that the set $T$ of Lemma 3 has to be constructed so that $T \cap \operatorname{im}(\alpha_i) = \emptyset$, for each $i$. To assure this we require that the sets inductively constructed in the proof of that Lemma satisfy, in addition to conditions (a)–(f),

(g) $$E(i) \cap \operatorname{im}(\alpha_{|i|}) = \emptyset.$$

The previous construction works in this setting for the sets $K(y) = \operatorname{im}(\alpha_{|i|}) \cap f^{-1}(y)$ being 0-dimensional it follows that the sets $S_i(y)$ separating $A$ from $B$ in $f^{-1}(y)$ may be taken to be disjoint from $K(y)$.

THEOREM 3. *Assume that any map $I^2 \to R^n$ is approximable by maps with images missing $S(f)$, the singular set of $f$. If $s+\dim(f)+E(p/2) < n$ then the set $\{\alpha \in C(I^s, R^n): f\alpha$ is 1-to-1$\}$ is dense in $C(I^s, R^n)$.*

Proof. We consider two cases

(1) $\dim(f) = 0$. Fix $\beta: I^s \to R^n$ and $\varepsilon > 0$; we shall construct an $\alpha: I^s \to R^n$ such that $\operatorname{dist}(\alpha, \beta) < \varepsilon$ and $\alpha \in G(\varepsilon)$, where

$$G(\varepsilon) = \{\alpha \in C(I^s, R^n): \alpha \text{ and } f|\operatorname{im}(\alpha) \text{ are } (1, \varepsilon)\text{-maps}\}.$$

The assertion will then follow by Baire category theorem.

The construction of $\alpha$. Let $\mathcal{T}$ be any triangulation of $I^s$ which is so fine that any set $\beta(\sigma)$, $\sigma \in \mathcal{T}$, is contained in an open ball $U(\sigma)$ in $R^n$ of radius $\varepsilon/2$. Let $\{\varphi_i: I^2 \to R^n \setminus S(f)\}$ be a dense subset of $C(I^2, R^n)$ consisting of embeddings. By induction and Baire category theorem we may assume that $\beta$ is such that

(a) $f\beta||\mathcal{T}^{s-1}|$ is 1-to-1, and

(b) $\beta(|\mathcal{T}^{s-1}|) \cap \bigcup \{\operatorname{im}(\varphi_i): i \in N\} = \emptyset$,

By Corollary 1 there is a $\sigma$-compact set $A \subset S(f)$ such that $\dim A = E(p/2)$ and $f|R^n \setminus A$ is 1-to-1. Write

$$B = A \cup f^{-1}f\beta(|\mathcal{T}^{s-1}|) \setminus \beta(|\mathcal{T}^{s-1}|)$$

We have $\dim f^{-1}f\beta(|\mathcal{T}^{s-1}|) \leqslant \dim(f)+s-1$ and so $\dim B \leqslant \max(E(p/2), s-1) < n-s$ (we are using assumptions on $s$ and results in [E], pp. 136 and 43).

By Lemma 4 there are compacta $C(\sigma) \subset U(\sigma) \setminus B$, $\sigma \in \mathcal{T} \setminus \mathcal{T}^{s-1}$, such that $\beta|\partial\sigma$ is null-homotopic in every neighbourhood of $C(\sigma)$ in $R^n$. With $C = \bigcup \{C(\sigma): \sigma \in \mathcal{T} \setminus \mathcal{T}^{s-1}\}$ it is transparent that $f|C$ is 1-to-1. By compactness of $C$ there is a neighbourhood $N$ of $C$ in $R^n$ such that $f|N$ is a $(1, \varepsilon)$-map; the properties of $C(\sigma)$'s then allow us to construct a map $\alpha: I^s \to R^n$ such that $\alpha(\sigma) \subset U(\sigma) \cap N$ and $\alpha|\partial\sigma = \beta|\partial\sigma$, for each $\sigma \in \mathcal{T} \setminus \mathcal{T}^{s-1}$. By general position we may additionally require that $\alpha$ be an embedding and hence a desired member of $G(\varepsilon)$ with $\operatorname{dist}(\alpha, \beta) < \varepsilon$. (Here, property (b) and Lemma 2 are used to construct $\alpha$ so that $\alpha(|\sigma| \setminus |\partial\sigma|) \cap \beta(|\mathcal{T}^{s-1}|) = \emptyset$, for every $\sigma \in \mathcal{T} \setminus \mathcal{T}^{s-1}$).

(2) The general case. We embed $I^s$ standartly in $I^t$, where $t = n-\dim(f)$, and extend a given $\beta: I^s \to R^n$ to a map $u: I^t \to R^n$. Using Proposition 3 and assumptions on $S(f)$, along with Baire theorem, we get a map $v: I^t \to R^n$ which is as close to $u$ as we wish and satisfies the following conditions:

(i) $\dim(fv) = 0$;

(ii) $v(W) \cap S(f) = \emptyset$, where $W = \{x \in I^t$: all but 2 co-ordinates of $x$ are rational$\}$;

(iii) $S(v) \cap W = \emptyset$.

Then, $S(fv) \cap W = \emptyset$ and we may apply the special case (1) above, obtaining a map $j: I^s \to I^t$ such that

(iv) $j$ closely approximates the embedding $I^s \to I^t$, and

(v) $(fv)j$ is 1-to-1.

It is clear that $\alpha = vj$ closely approximates $\beta$ and $f\alpha$ is 1-to-1.

Remark 3. With assumptions on $s$ in Theorem 3 replaced by "$s+\dim(f) < n$" it follows by similar arguments that each map $I^s \to R^n$ is approximable by maps $\alpha: I^s \to R^n$ with $f\alpha$ being finite-to-1, e.g. $q$-to-1 where $q = p(s^{-1}-n^{-1})^{-1}$.

Remark 4. The results of this section remain valid with $I^s$ being replaced by any (separable metric) space $P$ of dimension $\leqslant s$. (In this setting, $C(P, R^n)$ is given the *limitation topology* generated by all balls in supremum-metric on $C(P, R^n)$ induced by bounded admissible metrics for $R^n$). A proof follows by considering first the case where $P$ is a locally finite polyhedron of dimension $\leqslant s$ and then approximating maps $P \to R^n$ by compositions $P \to K \to R^n$ with $K$ a polyhedron as above.

**§ 5. A relation to the problem of characterization of $Q$-manifolds.** In this section all spaces are locally compact and $Q = [-1, 1]^\infty$, the Hilbert cube. $Q$-manifolds may be characterized using the following properties

$(*)_n$ any map $I^n \times \{1, 2\} \to X$ is approximable by maps sending $I^n \times \{1\}$ and $I^n \times \{2\}$ to disjoint sets;

namely, $X$ is a $Q$-manifold iff $X \in$ ANR and $X \in (*)_n$ for each $n$. (See [T] and [E3]). On the other hand it has been conjectured by J. W. Cannon that the property $(*)_2$ distinguishes $n$-manifolds among ANR's having a finite dimension $n \geqslant 5$ and local homology groups that of $R^n$. This conjecture has been turned into a theorem by results of R. D. Edwards (see [E2]) and of F. Quinn [Q]. Returning to $Q$-manifolds, no characterization of $Q$-manifolds in terms of $(*)_2$ and local homology groups is known, although R. D. Daverman and J. Walsh [DW] have characterized them using $(*)_2$ and certain homology analogues of $(*)_n$, $n \geqslant 3$. (C.f. also [LW].)

Applying duality-type results (see [DW], p. 414, for more general statements) it is easy to see that the $Q$-manifold analogue of Cannon's conjecture can equivalently be formulated in the following way which exhibits directly its dimension-theoretic aspect:

(C) Let $X \in \mathrm{ANR} \cap (*)_2$ be such that $H_*(X, X \setminus \{x\}) = 0$ for each $x \in X$ and let $n \geqslant 3$. Then, any map $I^n \to X$ is approximable by maps sending $I^n$ to finite-dimensional sets.

In this section we provide a further illustration of this aspect of the characterization problem by using Theorem 1 in the proof of the following

PROPOSITION 4. *Suppose $X \in \mathrm{ANR} \cap (*)_2$. Then, $X$ is a $Q$-manifold iff, for every $n$, there is a map $g: X \to Y$ such that $\dim Y < \infty$ and every fiber $g^{-1}(y)$, $y \in Y$, is a $z_n$-set in $Y$.*

Here, we say that a closed set $K \subset X$ is a *$z_n$-set* (resp. a *$Z_n$-set*) in $X$ iff $H_i(U, U \setminus K) = 0$ (resp. $\pi_i(U, U \setminus K) = 0$) for every $i < n$ and every open set $U \subset X$.

Proof. We omit some details as our aim is mainly to present some of the motivation for the results of this paper and for the questions stated in § 6. Fix $f_0: I^n \to X$, embedd standartly $I^n$ into $R^{2n}$, and let $f: R^{2n} \to X$ be a map extending $f_0$. Let $g: X \to Y$ be a map such that $\dim Y < \infty$ and every fiber of $g$ is a $z_{2n}$-set in $X$. Replacing $Y$ by $Y \times I^5$ and using the property $(*)_2$ of $X$ it can be achieved that each fiber of $g$ be a $Z_{2n}$-set in $X$. (A closed set $K$ in an ANR-space $X$ has property $Z_p$ iff it has properties $z_p$ and $Z_2$). We need the following

LEMMA 5. *If $g$ is as above then given $u: I^n \to X$ and $v: I^{2n} \to X$, there is a map $w: I^{2n} \to X$ which is as close to $v$ as we wish and satisfies $gu(\{x\}) \cap gw(\{x\} \times I^n) = \emptyset$, for each $x \in I^n$.*



The proof of the Lemma reduces to constructing $I^n$-preserving maps $I^n \times I^n \to I^n \times X$ with images disjoint from $K = \bigcup \{\{x\} \times g^{-1}gu(x): x \in I^n\}$. Since $K$ intersects each fiber $\{x\} \times X$ along a $Z_{2n}$-set this can be done analogously as in [Wo].

Returning to the proof of the Proposition, let $\mathscr{A}$ denote the family of all $n$-element subsets of $\{1, \ldots, 2n\}$. For $A \in \mathscr{A}$ let $p_A: R^{2n} \to R^A$ be the projection, let $V(A)$ be a fixed dense countable subset of $R^A$ and let $\varphi_{i,A}: R^A \to R^{2n}$ be linear sections of $p_A$ with $\bigcup \{\mathrm{im}(\varphi_{i,A}): i \in N\} = V(\{1, \ldots, 2n\} \setminus A) \times R^A$. Using Lemma 5 and Baire category theorem, or a convergence procedure, we get a map $h: R^{2n} \to X$ which is in the compact open topology as close to $f$ as we wish and satisfies the following condition

(i) Given $A, B \in \mathscr{A}$ we have $gh(p_B^{-1}(x+v) \setminus \varphi_{i,A}(x)) \cap gh\varphi_{i,A}(x) = \emptyset$, for every $x \in R^A$ and $v \in V(B)$.

Employing $(*)_2$ we may also require that $h|W$ be 1-to-1, where

$$W = \{x \in R^{2n}: \text{at most 2 co-ordinates of } x \text{ are irrational}\}.$$

If $F = (gh)^{-1}(y)$ is any fiber of $gh$ then either $F$ is in the Menger–Nöbeling set

$$R^{2n} \setminus \bigcup \{\mathrm{im}(\varphi_{i,A}): i \in N,\ A \in \mathscr{A}\}$$

and hence $\dim F \leqslant n-1$, or else $F$ is of the form $(gh)^{-1}gh\varphi_{i,A}(x)$ for some $i \in N$, $A \in \mathscr{A}$ and $x \in R^A$. In the latter case it follows from (i) that $F$ is contained in another set

$$\{\varphi_{i,A}(x)\} \cup (R^{2n} \setminus \bigcup \{p_B^{-1}(x+v): v \in V(B),\ B \in \mathscr{A}\})$$

of the Menger–Nöbeling type. Hence in either case $\dim F \leqslant n-1$ and by Theorem 1 there is a set $K \in F_\sigma(R^{2n})$ such that $\dim(R^{2n} \setminus K) \leqslant n-1$ and $gh|K$ is $p$-to-1, where $p = \dim Y$. By Hurewicz theorem [E, p. 136] we infer that $\dim h(K) < \infty$ and so there is a $G_\sigma$-set $L$ in $R^{2n}$ such that $L \supset K \cup W$ and $\dim h(L) < \infty$. By Lemma 4, the inclusion $I^n \hookrightarrow R^{2n}$ is approximable by maps $j$ with $\mathrm{im}(j) \subset L$. It is clear that $hj$ may serve as the desired approximation to $f_0$ having a finite-dimensional image.

**§ 6. Some questions.** The requirement $\dim Y < \infty$ in Proposition 4 is caused directly by a similar assumption in Theorem 1. As without it Proposition 4 would provide a proof of (C), it is of interest to investigate the following "limit" statements of results of §§ 2, 3 (all spaces are compact and $Q = I^\infty$):

1. Let $f: Q \to Y$ be a map with $\dim(f) = 0$, let $n \in N$ and in case $n > 1$ assume that $S(f)$ is a countable union of $Z_2$-sets. Is then any map $I^n \to Q$ approximable by maps $\alpha: I^n \to Q$ for which $f\alpha(I^n)$ is a countable union of finite-dimensional compacta?

2. Is there an $n \in N$ such that whenever $f: X \to Y$ and $\dim(f) = 0$, $\dim X = n$, then there exists a set $A \subset X$ such that $\dim(X \setminus A) \leqslant n-1$ and $f(A)$ is a countable union of finite-dimensional compacta?

Incidentally, if $X = I^n \times$ (Cantor set) then a set $A$ as above exists, see [Bu]. In 1 and in 2 the words "countable u. of f.d. compacta" could be replaced by "weakly infinite-dimensional" to obtain alternative version of potential interest; the latter property is assured if $f$ is $\aleph_0$-to-1 on $A$ or on $\alpha(I^n)$.

The problem of exhibiting infinite-dimensional versions of Theorems 1–3 seems to be interesting also because if there are no such then these theorems could serve as guidelines how to assure infinite-dimensionality of certain decomposition spaces. E.g. the question whether dimension is preserved by 1-dimensional CE-maps (c.f. [KW]) is equivalent to:

3. Is there a CE-map $f\colon I^n \to Y$ such that $\dim(f) = 1$ and any $\sigma$-compact set with $\dim(I^n \setminus A) = 1$ intersects a fiber of $f$ in more than $n$ points?

Conceivably, $f$ as above may be constructed so that each arc connecting a given pair of points of $I^n$ intersects a fiber of $f$ in at least 2 points ($n \geqslant 5$; compare Theorem 2).

Finally, in connection with Theorem 2 and the problem of detecting property $(*)_2$ in product spaces let us ask the following:

4. Let $f\colon I^n \to Y \in \mathrm{ANR}$ be a map with $\dim(f) = 0$ and $n = \dim Y \geqslant 4$. Is the set $\{\alpha \in C(I, I^n)\colon f\alpha \text{ is 1-to-1}\}$ connected and locally connected?

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