Finite-to-one restrictions of continuous functions (*)

by

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Abstract. It is shown that if \( f: X \to Y \) is a map of locally compact metric spaces and \( p = \dim Y < \infty \), then there exists a set \( A \subseteq X \) such that \( \dim f^{-1}(y), y \in Y \) and no fiber \( f^{-1}(y), y \in Y \), contains more than \( p \) points of \( X \setminus A \). A connection between this result and the problem of characterization of \( Q \)-manifolds is indicated.

Let \( f: X \to Y \) be a map of a locally compact metric space \( X \). In this note we consider the question whether the structure of \( f \) can be significantly simplified by passing to a restriction \( f|X \setminus A \), where \( A \) is an appropriately chosen subset of \( X \) of dimension comparable to \( \dim(f) = \sup \{ \dim f^{-1}(y) : y \in Y \} \). In §§ 1-3 we show that if \( \dim Y < \infty \) then this is in fact so. Specifically, we prove:

THEOREM 1. Let \( f: X \to Y \) be a \( \sigma \)-closed map of separable metric spaces such that \( k = \dim(f) \) and \( p = \dim Y \) are finite. Then, there is a set \( A \subseteq X \) such that \( \dim A \leq k \) and \( f|X \setminus A \) is \( p \)-to-1.

COROLLARY 2. In notation of Theorem 1 there is a set \( B \subseteq X \) such that \( \dim B \leq k + E(p/2) \) and \( f|X \setminus B \) is 1-to-1.

Here, we denote by \( E(x) \) the integer part of \( x \) and we say that a map \( f: X \to Y \) is \( p \)-to-1 if no fiber \( f^{-1}(y), y \in Y \), contains more than \( p \) points.

The above results are applied in § 4 to give certain conditions under which maps \( [0, 1]^n \to \mathbb{R}^n \) may be approximated by maps whose images are transverse, in a very vague sense, to all fibers of a given map \( f: \mathbb{R}^n \to Y \).

The statements of §§ 3, 4 may be viewed as selection type results; with Corollary 2 providing a certain lower bound for the maximal dimension of closed subsets of \( Y \) over which \( f \) admits a continuous selection. (In case \( f \) is open, a classical result on sections of \( f \) is given in [RC].)

The considerations of this note have been motivated by the problem of characterization of Hilbert cube manifolds. A dimension-theoretic approach to this problem naturally leads to questions concerning the possibility of improving properties of a map by neglecting a "small" subset of its domain; however, in contrast

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to Theorem 1 the range of the map may be of infinite dimension. We formulate some of these questions in § 6 and we proceed with showing in § 5 how Theorem 1 can be applied to derive a characterization of $Q$-manifolds. We note that, although formally new, this characterization can be derived from a result of R. D. Daverman and J. Walsh [DW] and yet another dimension-theoretic result, due to I. Walsh [W]. Nevertheless, we sketch a proof of it to indicate the approach alluded to above and to illustrate the connection between a problem concerning $Q$-manifolds and results, or questions, in dimension theory.

**Notation.** All spaces are assumed to be separable and topologized by a fixed metric which we denote by $d$. We say that $f: X \to Y$ is $\alpha$-closed if $X$ is the union of countably many closed sets $X_i$ such that each restriction $f(X_i): X_i \to f(X_i)$ is a closed map. Closed maps with compact fibers are called proper.

**Remark 1.** By a lemma of I. A. Vainstein ([EI], p. 139) if $f: X \to Y$ is closed then there is an open set $U \subseteq X$ such that $f(U) \neq \varnothing$ is proper and $f(U)$ is a countable set.

We say that a family of sets is of size $\leq s$ if the diameter of each of its members is $< s$. The boundary of a set $A$ is denoted by $\partial A$. We write $N$ (resp. $K$) for the set of integers (resp. real numbers) and $I$ for the segment $[0, 1]$. Undefined notions have the meaning of [EI].

**§ 1. 0-dimensional maps.** In this section we deduce Theorem 1 in the special case where dim($f$) = 0. We need two lemmas.

**Lemma 1.** Let $f: X \to Y$ be a closed map with dim($f$) = 0 and let $\varepsilon > 0$. Then, there is an open cover $\mathcal{U}$ of $Y$ with the property that whenever $\{U_i\}$ refines $\mathcal{U}$ and $\mathcal{V} \subseteq \mathcal{U}$ then there are discrete families $\mathcal{D}$ and $\mathcal{E} = \{E(D): D \in \mathcal{D}\}$ of open subsets of $X$ such that:

(a) if $D \in \mathcal{D}$ then $diam(D \cap E(D)) < \varepsilon$ and $E(D) \subseteq f^{-1}(U_i(V))$;

(b) $\mathcal{D}$ covers $f^{-1}(\mathcal{V})$ and $E(D)$ is a $\varepsilon$-neighbourhood of $\partial D$, for each $D \in \mathcal{D}$;

(c) if $x, x_i \in \partial D$ and $f(x_i) \neq f(x)$ then $d(x, x_i) < \varepsilon$.

**Proof.** For each $y \in Y$ there is a neighbourhood $G(y)$ of $f^{-1}(y)$ in $X$ which is the union of a discrete collection $\mathcal{G}(y) = \{G_1(y), G_2(y), \ldots\}$ of open sets of size $\varepsilon$. Since $f$ is closed we have $f^{-1}(W(y)) \subseteq G(y)$ for some neighbourhood $W(y)$ of $y$. Let $\mathcal{G} = \{W(y); y \in Y\}$. If $\mathcal{V} \subseteq \mathcal{U}$ then let $F_0, F_1, \ldots$ be open neighbourhoods of $\mathcal{V}$ satisfying $F_0 \subseteq U_0$ and $F_{i+1} \subseteq F_i$ for each $i$. We write

$$\mathcal{D} = \{D_i; i \in N\} \quad \text{where} \quad D_i = G(y) \cap f^{-1}(F_{i+1}),$$

and

$$E(D_i) = G(y) \cap f^{-1}(F_{i+1}), \quad i \in N.$$  

Conditions (a) and (b) are clearly met. Also, $f(E(D)) \cap f(E(D_i)) = \emptyset$ which coupled with (a) yields (c).

**Definition.** We say that $g: A \to B$ is a $(\alpha, \varepsilon)$-map if each point-inverse $g^{-1}(b), b \in B$, is a union of $\varepsilon$ sets of size $\alpha$.

**Proposition 1.** Let $f: X \to Y$ be a closed map with dim($f$) = 0 and dim $X = p < \infty$. Then, there is a set $A \subseteq f(X)$ such that $f|A$ is $p$-to-1 and $f(X) = X$.

**Proof.** Let $X_1 \subseteq \ldots$ be closed subsets of $X$ such that $\bigcup X_i = X = \partial X$ and for each $X_i$ is a closed map. Let $A_1, B_1; i \in N$ be closed subsets of $X$ such that $A_i \cap B_i = \emptyset$ and both $(A_i; i \in N)$ and $(X, X_i; i \in N)$ are bases of neighbourhoods of $X$. With $p = \dim Y$ we shall construct relatively open subsets $U_i, V_i$ of $X_i, i \in N$ so that:

(a) $U_i \cap V_i = \emptyset$ and $U_i \supseteq X_i \cap A_i, V_i \supseteq X_i \cap B_i$;

(b) $U_i^{n+1} = U_i$ and $V_i^{n+1} = V_i$;

(c) $f(\bigcup X_i \cap (U_i \cup V_i))$ is a $(p, 1/n)$-map.

The inductive construction. Suppose $\{U_i, V_i; i = 1, \ldots, n\}$ are known. We write $U_{n+1} = \emptyset = V_{n+1}$ and

$$K_n = A_1 \cap X_{n+1} \cap U_n, \quad L_n = B_1 \cap X_{n+1} \cap V_n.$$  

**Lemma 2.** Let $\varepsilon > 0$, let $f: X \to Y$ be a closed map with dim($f$) = 0 and dim $Y = p < \infty$ and, for $i = 1, 2, \ldots, n$, let $K_i$ and $L_i$ be closed disjoint subsets of $X$. Then, there are open subsets $E_i$ of $X$ separating $X$ between $K_i$ and $L_i$, and such that $f(E_i \cup \cdots \cup E_n) = (p, \varepsilon)$-map.

**Proof.** Take closed sets $S_1, \ldots, S_n$ so that $S_i$ separates $X$ between $K_i$ and $L_i$. We may require that the metric $q$ of $X$ is such that $q_1(x, y) < \varepsilon$ for each $i \leq \varepsilon$ (otherwise replace $q(x, y)$ by $q(x, y) + \sum_{i \leq \varepsilon} |l_i(x_i) - l_i(y_i)|$ for suitably chosen maps $l_1, \ldots, l_n$). Let $\mathcal{W}$ be a cover of $Y$ assured by Lemma 1. By a result of Morita there is a locally finite open cover $\mathcal{U}$ of $Y$ such that $\{U \subseteq Y; U \in \mathcal{U}\}$ is of order $p - 1$ and $\mathcal{U}$ refines $\mathcal{W}$; see [EI], p. 229. Let $\{V(U); U \in \mathcal{U}\}$ be a closed shrinking of $\mathcal{U}$ such that $\{U \cap V(U) = U \in \mathcal{U}\}$ is of order $p - 1$, and for $U \in \mathcal{U}$ let $\mathcal{S}(U)$ and $\{E(D); D \subseteq \mathcal{U}(U)\}$ be families provided by Lemma 1 for the pair $(U, V(U))$. We write

$$\mathcal{S}_1 = \{D \subseteq \mathcal{U}(U); U \in \mathcal{U}\} \quad \text{and} \quad \mathcal{E}_1 = \{E(D); D \subseteq \mathcal{U}(U)\}.$$

Then, $T_i$ contains the boundary of a neighbourhood of $K_i$ in $X \setminus (K_i \cup L_i)$ and hence separates $X$ between $K_i$ and $L_i$. Let $B = E_1 \cup \cdots \cup E_n$. To show that $f|B$ is a $(p, \varepsilon)$-map fix $x \in B$ and let $\mathcal{W}_0 = \{U \in \mathcal{W}; f(x) \in U \cap V(U)\}$ then card $\mathcal{W}_0 < p$. We have

$$f(E(D)) = f(E(D)) \subseteq f^{-1}(1)(x) \cap \{E(D); D \subseteq \mathcal{U}(U)\} = \{E(D); D \subseteq \mathcal{U}(U)\}$$

and for each $U \in \mathcal{U}_0$ the set $f^{-1}(1)(x) \cap \{E(D); D \subseteq \mathcal{U}(U)\}$ is of size $\leq s$, by property (c) of $(E(D); D \subseteq \mathcal{U}(U))$. Thus $f(E(D))$ is a union of $p$ sets of size $\leq s$ and we may let $E_i$ be any neighbourhood of $T_i$ whose closure is contained in $E_i$.
Lemma 2 applied to \( f|X_{n+1} \) readily implies the existence of the required sets \( \{T^n_i, V^n_i: i \leq n+1 \} \). (The first step is analogous.) Write

\[ T^n_i = X^n_i \cup \bigcup \{ T^{n+1}_j: j, n+1 \}

then \( T^n_i \) is a closed set separating \( X^n_i \) between \( A_i \) and \( B_i \). Thus \( A = \bigcup \{ T^n_i: n \in N \} \) is an \( F^n_i \)-set in \( X \) such that \( \dim (X_i, A) = 0 \) for each \( n \), yielding \( \dim (X, A) = 0 \) by the countable sum theorem. Finally, condition (c) shows that \( f| \bigcup T^n_i: n < k \) is a \((p, e)\)-map for each \( k \in N \) and \( e > 0 \), whence no fiber of \( f|A \) contains \( p+1 \) points.

\section{2. Restrictions that lower a map's dimension}

\textbf{Proposition 2.} Let \( f: X \to Y \) be a \( \sigma \)-closed map with \( \dim (f) = k \) and \( \dim Y < \infty \). Then for each \( 1 < i < k \) there exists a set \( X_i \in F_i(X) \) such that \( \dim X_i \leq 1 \) and \( \dim (f|X_i \times X) = k-1 \).

\textbf{Proof.} If \( X_{k-1} \) is constructed then it suffices to require for \( k < i < k-1 \) that \( X_i \) be an \( F^n_i \)-set in \( X_{k-1} \) with \( \dim X_i \leq 1 \) and \( \dim (f|X_i \times X) = k-2 \). The existence of \( X_{k-1} \) in turn follows routinely from Remark 1 and the following.

\textbf{Lemma 3.} Let \( f: X \to Y \) be a proper map with \( \dim (f) = k < \infty \) and \( \dim Y < \infty \), and let \( A \) and \( B \) be disjoint closed subsets of \( X \). Then, there is a closed set \( T \) in \( X \) such that \( \dim T = k-1 \) and, for each \( x \in X \), \( T \) separates \( f^{-1}(F) \) between \( A \) and \( B \).

\textbf{Proof.} Let \( \mathcal{F} = \bigcup \{ N^n_k: k < 0 \} \), the set of all finite sequences of integers. For each \( i \in \mathcal{F} \) define the integer \( |i| \) by the requirement that \( i \leq |i| \). We arrange that \( \mathcal{F} \) is the element of \( N^0 \) if \( i \neq \mathcal{F} \), \( p \in N \), then \( (i, p) \) is the naturally defined member of \( N^0 \). We shall construct sets \( F(i), U(i), V(i) \) so that the following conditions are satisfied for each \( i \in \mathcal{F} \):

\begin{enumerate}
\item \( F(i) \) is closed in \( Y \) and \( U(i), V(i) \) are open sets in \( X \) with \( U(i) \cap V(i) = \varnothing \);
\item \( F(\mathcal{F}) = Y \) and \( U(\mathcal{F}) \supseteq A, V(\mathcal{F}) \supseteq B \);
\item for each \( p \in N \) we have \( U(i, p) = U(i) \cap f^{-1}(F(i, p)) \) and \( V(i, p) = V(i) \cap f^{-1}(F(i, p)) \);
\item \( F(\mathcal{F}) = \bigcup \{ F(i, p): p \in N \} \) and \( \dim (F(i, p)) = 1/|i| \);
\item the set \( E(i) = f^{-1}(F(i)) \cup V(i) \) admits an open cover of size \( 1/|i| \) and order \( k-1 \); and \( i \) in notation of (c), the family \( \{ E(i, p): p \in N \} \) is discrete in \( X \).
\end{enumerate}

Assuming the above sets to be constructed write

\[ T^n_i = \bigcup \{ E(i): |i| = n \} \quad \text{and} \quad T = \bigcap \{ T^n_i: n \geq 0 \}.

If \( y \in Y \), then, by (d), there are \( i_1, i_2, \ldots \in N \) such that \( y \in F(i_1) \cap f(i_1, i_2) \cap \ldots \)

and we have \( f^{-1}(F) \cap \bigcup T^n_i = U(y) \cup V(y) \), where the set

\[ U(y) = f^{-1}(F(i_1)) \cup \bigcup \{ U(i_1, i_2, \ldots): p \in N \} \]

and the analogously defined set \( V(y) \) form the necessary partition of \( f^{-1}(F) \cap \bigcup T^n_i \).

To show that \( \dim (T) \leq k-1 \), we notice that, by (d), each set \( T^n_i \) is closed and admits an open cover of size \( 1/|n| \) and order \( k-1 \). Thus \( \bigcap \) compact in \( T \) of dimension \( \leq k-1 \) and \( \dim T \leq k-1 \) in \( X \) is compact. In the general case we infer that, at least, each fiber of \( f|T \) is of dimension \( \leq k-1 \). Moreover, \( f|T = \beta _a \), where \( a : T \to N^\infty \) and \( \beta : im(a) \to Y \) are defined by the requirements that

\[ t \in E(i_1) \cap E(i_2) \cap \ldots \quad \text{where} \quad (i_1, i_2, \ldots) = a(t), \]

and

\[ \beta (i_1, i_2, \ldots) = \bigcap \{ F(i_1, i_2, \ldots) \} \]

Both \( a = \beta \) is continuous, c.f. (d) and (f), and so \( f|T = \beta _a \) implies that \( a \) is proper and \( \dim (a) \leq \dim (f|T) = k-1 \).

Therefore, \( \dim T \leq k-1 \) by a classical theorem of Hurewicz (IE, p. 136).

It remains to construct the sets satisfying (a)-(f), which is done by induction on \( |i| \). Assume that \( F = f(i), U = U(i) \) and \( V = V(i) \) are constructed so that (a) and (b) hold. Write \( g = \dim Y \) and let \( W_1, W_2, \ldots \) be disjoint open subsets of \( X \) such that each of them separates \( U \) and \( V \). For \( 1 \leq q \) and \( \mathcal{F} \) let \( P(\mathcal{F}) \) be a closed set separating \( f^{-1}(F) \) between \( U \) and \( V \) such that \( \dim P(\mathcal{F}) = k-1 \) and \( \dim P(\mathcal{F}) \leq k \). Let \( \mathcal{Q}(\mathcal{F}) \) be a neighbourhood of \( P(\mathcal{F}) \) in \( W_1 \) which admits an open cover of size \( (n+1)^{-1} \) and order \( k-1 \). Since \( f \) is closed it follows that there is a \( \mathcal{Q}(\mathcal{F}) \) separating \( f^{-1}(F) \) between \( U \) and \( V \) for each \( 1 \leq q \). By a result of Ostrand (IE, p. 228) there is an open cover \( \mathcal{U} \) of \( F \) which refines \( \mathcal{Q}(\mathcal{F}) \), and the union of its discrete sub-families \( \mathcal{F}, \ldots \), \( \mathcal{F} \), then, for each \( g \in \mathcal{F} \) there are open subsets \( U(\mathcal{G}) \) and \( V(\mathcal{G}) \) of \( X \), with disjoint closures, such that

\[ U(\mathcal{G}) = f^{-1}(F) \cap U, \quad V(\mathcal{G}) = f^{-1}(F) \cap V, \]

and if \( g \in \mathcal{F} \), then

\[ f^{-1}(F) = (U(\mathcal{G}) \cup V(\mathcal{G})) = \mathcal{Q}(\mathcal{F}) \]

for some \( \mathcal{F} \in \mathcal{U} \). Each of the families \( \mathcal{F}(\mathcal{G}) \) is discrete and for \( g \in \mathcal{G} \), and \( H \in \mathcal{F}_a \), we have \( U(\mathcal{G}) \cap U(H) = \varnothing \) and \( W_2 = \varnothing \). Thus \( \{ F(\mathcal{G}) : \mathcal{G} \in \mathcal{G} \} \) is discrete. With \( G_1, G_2, \ldots \) being an enumeration of \( \mathcal{G} \) we may thus write

\[ F(i, p) = F(G_1), \quad U(i, p) = U(G_1), \quad V(i, p) = V(G_1) \]

to get the desired sets satisfying conditions (a)-(f). This concludes the proof of the Lemma and of Proposition 1.
Remark 2. In case $X$ is compact Proposition 2 can also be proved using a theorem announced by B. A. Pasynkov in [P]. Consequently, Pasynkov's theorem can be extended to $\sigma$-closed maps and derived as a consequence of Proposition 2:

**Corollary 1 (C. f. [P]).** Assume $f : X \to Y$ is $\sigma$-closed and $\dim(f) = k < \infty$, $\dim Y < \infty$. Then, there exists a map $g : X \to Y$ such that $\dim(g \circ f) = 0$.

**Proof.** By Proposition 2 there is a set $A \in F_k(X)$ such that $\dim A = 0$ and $\dim(f|_{X \setminus A}) \leq k-1$. Take a map $u : X \to I$ such that $f|_{A} = 1$-to-$1$; then $\dim(f \circ u) \leq k-1$ and the result follows by induction on $k$.

§ 3. Proof of Theorem 1 and of Corollary 2. Assume notation of Theorem 1. By Proposition 2 there is a set $F \subset X$ such that $\dim F < k-1$ and $\dim(f|_{F \setminus Y}) = 0$. By a well-known theorem of Tamarkin ([E], p. 45) there is a $G_\sigma$-set $\tilde{F}$ in $X$ such that $F \subset \tilde{F}$ and $\dim \tilde{F} < k-1$. Then, $\dim f|_{\tilde{F} \setminus Y}$ is $\sigma$-closed any by Proposition 1 there is a set $Q \subset F \setminus Y$ such that $\dim(\tilde{X} \setminus \tilde{F}, \tilde{Q}) = 0$ and $Q|\tilde{Q}$ is $p$-to-$1$. We let $A = X \setminus Q$ to get the desired set.

**Proof of Corollary 2.** We consider first the case where $\dim(f) = 0$. Take sets $Y = Y_0 \supset Y_1 \supset \ldots$ so that $\dim Y_i = i$ and $\dim(Y_{i+1} \setminus Y_i) = 0$ for $i = 0, 1, \ldots, p$. (Simply require that $Y_{i+1}$ be the union of the boundaries of all members of an appropriate basis of the topology of $Y_i$.) Let $X_i = Y_i$ if $\dim Y_i < p$ and

$$X_i = f^{-1}(Y_{i+1} \setminus Y_i), \quad j = 0, 1, \ldots, r+1,$$

where $r$ is such that $2r+1 < p$ and $2r+2 > p$.

Applying Theorem 1 to $f|X_i : X_i \to Y_{i+1} \setminus Y_{i+2}$ we get sets $A_j \subset X_j$ such that $\dim A_j = 0$ and $f|X_j \setminus A_j$ is $1$-to-$1$. We let $A = \bigcup A_j \cup \Delta_0 \cup \ldots \cup A_{r+1}$; then $\dim(f|A \setminus A) = 1$-to-$1$ and it follows easily that $\dim A = \dim f|A \setminus A)$. (Observe that $\dim Y_0 + \dim(f|A \setminus A) \leq \dim f|A \setminus A)$.) This proves the Corollary under the additional assumption $k = 0$. The general case now follows easily from Proposition 2, c.f. the proof of Theorem 1.

§ 4. Cells general position with respect to a map. In this section we fix $f : \mathbb{R}^s \to \mathbb{R}^n$ where $s = \dim Y < \infty$. Our goal is to approximate maps $f : \mathbb{R}^s \to \mathbb{R}^n$ by maps with intersecting each fiber of $f$ in finitely many points or, possibly, in sets of cardinality $\leq 1$. If $s > 2$ then the results are obtained under a global assumption on $f$ stating that the singular set of $f$

$$S(f) = \{ x \in \mathbb{R}^s : f^{-1}(x) \neq \{ x \} \}$$

is tamely embedded, in sense of Stanko demension theory ([E], [5]). We write $C(I, \mathbb{R}^n)$ for the space of all maps $I \to \mathbb{R}^n$ (compact-open topology).

**Theorem 2.** If $\dim(f) \leq n-2$ then $\{ x \in C(I, \mathbb{R}^n) : f(x) = p \text{-to}-1 \} \subset C(I, \mathbb{R}^n)$ is dense in $C(I, \mathbb{R}^n)$.

**Addendum.** If $\dim(f)+E(p) \leq n-2$ then $\{ x \in C(I, \mathbb{R}^n) : f(x) = 1 \text{-to}-1 \} \subset C(I, \mathbb{R}^n)$.

**Proof.** Let $E = C(I, \mathbb{R}^n)$ and $G \subset E \subset \mathbb{R}^n$. We define $G(x) = \{ a \in E : f|_{x \times I} = (p, x) \}$.

$$G = \{ x \in E : f|_{x \times I} = p \text{-to}-1 \} \cap G(a) $$

(In the case of the Addendum replace $p$ by $1$ in these definitions.) We omit the verification that each $G(a)$ is open in $E$. By the Baire category theorem it remains to show that whenever $a \in E$ and $c > 0$ then there is a $G(a)$.

Fix $r > 0$ and take $\delta > 0$ so that if $|t| > \delta$ then $e^{-t} > \delta$. Let $\delta > 0$ be a triangulation of $I$ with $\delta > 0$ for each $J \in \mathcal{J}$. By § 3, there is a set $A \subset \mathbb{R}^n$ such that $\dim A < n-2$ and $f|A \setminus A = p \text{-to}-1$. There is no loss of generality in assuming that $a(b) \subset \mathbb{R}^n$. For $J \in \mathcal{J}^n$ and $x \in |\delta| \setminus J \in \mathcal{J}$ then $\dim a^{-1}(x) = 2$ and $x \in |\delta| \setminus J \in \mathcal{J}$.

**Lemma 4.** Let $A$ be a subset of $\mathbb{R}^n$ such that every map $(I^2, \partial I^2) \to (\mathbb{R}^n, \mathbb{R}^n)$ is relative $G^\mathbb{R}$ approximable by mappings with images missing $A$. Let $x \in \mathbb{R}^n$ satisfy $x + \delta A < \mathbb{R}^n$.

(i) If $A$ is $\sigma$-compact then $\{ a \in C(I, \mathbb{R}^n) : f|_{x \times I} = \varnothing \}$ is dense in $C(I, \mathbb{R}^n)$.

(ii) If $a : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $f|_{x \times I} = \varnothing$ then there is a compact set $C \subset \mathbb{R}^n$ such that $x$ is null-homotopic in every neighborhood $N$ of $C$ in $\mathbb{R}^n$.

**Comment.** Part (i) needs to be shown for compacta only and follows by induction on $s$, using Alexander duality and Hurewicz theorem. Se [[S], [B], [E]]. Part (ii) follows in a similar but more delicate manner, using Sitnikov duality [[M]] for non-closed subsets of $\mathbb{R}^n$ and Hurewicz theorem in Borsuk's weak shape theory. For a proof see the recent paper [S] of S. Spież.
PROPOSITION 3. If $\sigma \leqslant n - \dim(f)$ and $n \geqslant 3$ then $\{\sigma \in C(I', R^n) : \dim(f\sigma) = 0\}$ is dense in $C(I', R^n)$.

Proof. If $\sigma = 2$ then it follows using Theorem 2 that the set

$$W = \{\sigma \in C(I', R^n) : f\sigma(W) = p\text{-to-}1\},$$

where

$$W = \{(x, t) \in I^2 : \text{either } x \text{ or } t \text{ are rational}\},$$

is dense in $C(I', R^n)$. If $\sigma \in H$ then each fiber of $f\sigma$ meets $W$ in finitely many points whenever $\dim(f\sigma) = 0$.

If $\sigma > 2$ we fix embeddings $\alpha : I' \to R^n$ such that $\dim(f\alpha) = 0$ for each $i \in N$ and $\{x_i : i \in N\}$ is dense in $C(I', R^n)$.

CLAIM. There is a set $A \in F_2(R^n)$ such that

1. $\dim(A) \leqslant \dim(f) - 1$ and $\dim(f(R^n \setminus A)) = 0$, and
2. $A \cap \{\im(x_0) : i \in N\} = \emptyset$.

Assuming the claim it follows from Lemma 4 that

$$G = \{\sigma \in C(I', R^n) : \im(x_0) \cap A = \emptyset\}$$

is dense in $C = C(I', R^n)$. By Baire category theorem $G \cap H$ is dense in $C$, where $H = \{\sigma \in C : \dim(\sigma) = 0\}$. Evidently, $\dim(f\sigma) = 0$ for $\sigma \in G \cap H$.

The proof of the claim can be repeated after that of Proposition 2, except that the set $\mathcal{T}$ of Lemma 3 has to be constructed so that $\mathcal{T} \cap \im(x_0) = \emptyset$, for each $i$. Thus to assure this we require that the sets $\mathcal{T}$ constructed in the proof of Proposition 3, in addition to conditions (a)-(f),

$$(g)\quad E(i) \cap \im(x_0) = \emptyset.$$ 

The previous construction works in this setting for the sets $K(\sigma) = \im(x_0) \cap \cap f^{-1}(y)$ being 0-dimensional it follows that the sets $S(f)$ separating $A$ from $B$ in $f^{-1}(y)$ may be taken to be disjoint from $K(\sigma)$.

Theorem 3. Assume that any map $I' \to R^n$ is approximable by maps with images missing $S(f)$, the singular set of $f$. If $\sigma + \dim(f) + \dim(f) < n$ then the set $\{\sigma \in C(I', R^n) : \sigma \text{ is } 1\text{-to-}1\}$ is dense in $C(I', R^n)$.

Proof. We consider two cases

1. $\dim(f) = 0$. Fix $\beta : I' \to R^n$ and $\sigma > 0$; we shall construct an $\alpha : I' \to R^n$ such that $\dim(\sigma, \beta) < \sigma$ and $\alpha \in G(\sigma)$, where

$$G(\sigma) = \{\sigma \in C(I', R^n) : \sigma \text{ and } f(\im(\sigma)) \text{ are } (1, \sigma)-\text{maps}\}.$$ 

The assertion will then follow by Baire category theorem.

The construction of $\alpha$. Let $\mathcal{I}$ be any triangulation of $I'$ which is so fine that any set $\beta(\sigma), \sigma \in \mathcal{I}$, is contained in an open ball $U(\sigma)$ in $R^n$ of radius $\sigma/2$. Let

$$\{\sigma : I' \to R^n, S(f)\}$$

be a dense subset of $C(I', R^n)$ consisting of embeddings. By induction and Baire category theorem we may assume that $\beta$ is such that

(a) $\beta(\sigma) = \mathcal{I}$ is 1-to-1, and
(b) $\beta(\sigma) \cap \cup \{\im(\sigma) : \sigma \in N\} = \emptyset$.

By Corollary 1 there is a $\sigma$-compact set $A = S(f)$ such that $\dim(A) = E(p/2)$ and $f(A) \cap (R^n \setminus A)$ is 1-to-1. Write

$$B = A \cup \beta(\sigma) \cup \beta(\sigma) \cup \beta(\sigma)$$

We have $\dim(f) \leqslant \dim(f) + s - 1$ and so $\dim(B) \leqslant \max(E(p/2), \sigma - 1) < n - \sigma$ (we are using assumptions on $\sigma$ and results in [EE, pp. 136 and 143].

By Lemma 4 there are compacta $C(\sigma) = U(\sigma) \setminus B, \sigma \in \mathcal{I}$, such that

$$\beta(\sigma)$$

is null-homotopic in every neighborhood of $C(\sigma)$. With $C = \cup \{C(\sigma) : \sigma \in \mathcal{I}\}$ it is transparent that $f(C)$ is 1-to-1. By compactness of $C$ there is a neighborhood $N$ of $C$ in $R^n$ such that $f[N]$ is a (1, $p$)-map; the properties of $C(\sigma)$ then allow us to construct a map $\alpha : I' \to R^n$ such that $\alpha(\sigma) \in U(\sigma) \setminus N$ and $\alpha(\sigma) = \beta(\sigma)$, for each $\sigma \in \mathcal{I}$. By general position we may additionally require that $\alpha$ be an embedding and hence a desired member of $G(\sigma)$ with dist $(\alpha, B) < \sigma$. (Here, property (b) and Lemma 2 are used to construct $\alpha$ so that $\alpha(\sigma) \cap \im(x_0) \cap \beta(\sigma) = \emptyset$, for every $\sigma \in \mathcal{I}$.)

(2) The general case. We embed $\mathcal{I}$ standardly in $I'$, where $t = n - \dim(f)$, and extend a given $\beta : I' \to R^n$ to a map $\alpha : I' \to R^n$. Using Proposition 3 and assumptions on $S(f)$, along with Baire theorem, we get a map $\alpha : I' \to R^n$ which is as close to $\alpha$ as we wish and satisfies the following conditions:

1. $E(\alpha) = 0$;
2. $e(\alpha) \cap S(f) = \emptyset$, where $e = \{x \in I' : all 2 \text{ co-ordinates of } x \text{ are rational}\}$;
3. $S(f) \cap W = \emptyset$.

Then $S(f) \cap W = \emptyset$ and we may apply the special case (1) above, obtaining a map $\alpha : I' \to I'$ such that

1. $f(\alpha) \cap \mathcal{I} = \emptyset$;
2. $\alpha(\sigma)$ is 1-to-1.

It is clear that $\alpha = \mathcal{I}$ closely approximates $\beta$ and $f\alpha$ is 1-to-1.

Remark 3. With assumptions on $\mathcal{I}$ in Theorem 3 replaced by $\sigma + \dim(f) < n$, it follows by similar arguments that each map $I' \to R^n$ is approximable by maps $\alpha : I' \to R^n$ with $f\alpha$ being finite-to-1, e.g. $q$-to-1 where $q = p(\sigma - n-1)^{-1}$.

Remark 4. The results of this section remain valid with $\mathcal{I}$ being replaced by any (separable metric) space $P$ of dimension $\leqslant \sigma$. (In this setting, $C(P, R^n)$ is given the limitation topology generated by all balls in supremum-metric on $C(P, R^n)$.) A proof follows by considering
first the case where \( P \) is a locally finite polyhedron of dimension \( \leq s \) and then approximating maps \( P \to R^n \) by compositions \( P \to K \to R^n \) with \( K \) a polyhedron as above.

§ 5. A relation to the problem of characterization of \( Q \)-manifolds. In this section all spaces are locally compact and \( Q = [-1, 1]^n \), the Hilbert cube. \( Q \)-manifolds may be characterized using the following properties

1. any map \( I^n \times \{1, 2\} \to X \) is approximable by maps sending \( I^n \times \{1\} \) and \( I^n \times \{2\} \) to disjoint sets;

namely, \( X \) is a \( Q \)-manifold iff \( X \in \text{ANR} \) and \( X \in (o)_2 \), for each \( n \). (See [T] and [E]).

On the other hand it has been conjectured by J. W. Cannon that the property \((o)_2\) distinguishes \( n \)-manifolds among \( \text{ANR} \)'s having a finite dimension \( \geq 5 \) and local homology groups that of \( R^n \). This conjecture has been turned into a theorem by results of R. D. Edwards (see [E2]) and of F. Quinn [Q]. Returning to \( Q \)-manifolds, no characterization of \( Q \)-manifolds in terms of \((o)_2\) and local homology groups is known, although R. D. Davis and J. Walsh [DW] have characterized them using \((o)_2\) and certain homology analogues of \((o)_2\), \( n \geq 3 \). (C.f. also [LW]).

Applying duality-type results (see [DW], p. 414, for more general statements) it is easy to see that the \( Q \)-manifold analogue of Cannon's conjecture can equivalently be formulated in the following way which exhibits directly its dimension-theoretic aspect:

(C) Let \( X \in \text{ANR} \cap (o)_2 \) be such that \( H_4(I^n, X, x) = 0 \) for each \( x \in X \) and let \( n \geq 3 \). Then, any map \( I^n \to X \) is approximable by maps sending \( I^n \) to \( n \)-dimensional sets.

In this section we provide a further illustration of this aspect of the characterization problem by using Theorem 1 in the proof of the following

**Proposition 4**. Suppose \( X \in \text{ANR} \cap (o)_2 \). Then, \( X \) is a \( Q \)-manifold iff, for every \( n \), there is a map \( g: I^n \to Y \) such that \( \int Y < \infty \) and every fiber \( g^{-1}(y) \), \( y \in Y \), is a \( (n-1) \)-set in \( Y \).

Here, we say that a closed set \( K \subseteq X \) is a \( (n-1) \)-set (resp. a \( (n-2) \)-set) in \( X \) iff \( H_i(U, U \cap X) = 0 \) (resp. \( H_{n-1}(U, X) = 0 \)) for every \( i \leq n \) and every open set \( U \subseteq X \).

**Proof**. We omit some details as our aim is mainly to present some of the motivation for the results of this paper and for the questions stated in § 6. Fix \( f: I^n \to X \) embeddably \( I^n \) into \( R^n \), and let \( f: R^n \to X \) be a map extending \( f_0 \). Let \( g: I^n \to Y \) be a map such that \( \int Y < \infty \) and every fiber of \( g \) is a \( (n-1) \)-set in \( Y \). Replacing \( Y \) by \( Y \times I^n \) and using the property \((o)_2\) of \( I^n \) it can be achieved that each fiber of \( g \) is a \( (n-1) \)-set in \( X \). (A closed set \( K \) in an \( ANR \)-space \( X \) has property \( Z_2 \) in it if it has properties \( Z_2 \) and \( Z_2 \).) We need the following

**Lemma 5**. If \( g \) is as above then given \( u: I^n \to X \) and \( v: I^n \to X \), there is a map \( w: I^n \to X \) which is as close to \( e \) as we wish and satisfies \( g((x)) \cap g((x) \times I^n) = \emptyset \), for each \( x \in I^n \).

The proof of the Lemma reduces to constructing \( I^n \)-preserving maps \( I^n \times I^n \to I^n \times X \) with images disjoint from \( K = \{ \left( x \right) \times g^{-1}(y) \mid x \in I^n \} \). Since \( K \) intersects each fiber \((x) \times X \) along a \( Z_2 \)-set this can be done analogously as in [W].

Returning to the proof of the Proposition, let \( A \) denote the family of all \( n \)-element subsets of \( \{1, \ldots, 2n\} \). For \( A = \emptyset \) let \( p_1: R^{2n} \to R^n \) be the projection, let \( V(A) \) be a fixed dense countable subset of \( R^n \) and let \( q_{\alpha, \beta}: R^n \to R^{2n} \) be linear sections of \( p_{\alpha, \beta} \) with \( \{ \left( \frac{1}{\sqrt{2}}(x + y) \right) \mid \alpha < \beta \} \). Using Lemma 5 and the Baire category theorem, or a convergence procedure, we get a map \( h: R^n \to X \) which is in the compact open topology as close to \( f \) as we wish and satisfies the following condition

(i) Given \( A, B \in \emptyset \) we have \( gh((x) \times v_\alpha(x)) \cap gh((x) \times I^n) = \emptyset \), for every \( x \in R^n \) and \( \alpha \in (B) \).

Employing \((o)_2\), we may also require that \( h|W \) be 1-to-1, where \( W = \{ x \in R^n : \text{at most } 2 \text{ co-ordinates of } x \text{ are irrational} \} \).

If \( F = (gh)^{-1}(y) \) is any fiber of \( gh \) then either \( F \) is in the Menger-Nöbeling set

\[ R^{2n} \setminus \bigcup \{ \left( \frac{1}{\sqrt{2}}(x + y) \right) \mid \alpha < \beta \} \]

and hence disjoint from \( F \), or else \( F \) is of the form \( \left( \frac{1}{\sqrt{2}}(x + y) \right) \) for some \( \alpha \), \( \beta \in \emptyset \), \( A \in \emptyset \) and \( x \in R^n \). In the latter case it follows from (i) that \( F \) is contained in another set

\[ \{ \left( \frac{1}{\sqrt{2}}(x + y) \right) \mid \alpha < \beta \} \setminus \bigcup \{ \left( \frac{1}{\sqrt{2}}(x + y) \right) \mid \alpha < \beta \} \]

of the Menger-Nöbeling type. Hence in either case \( \dim F \leq n-1 \) and by Theorem 1 there is a set \( K \subseteq F \subseteq R^{2n} \) such that \( \dim(R^{2n} \setminus K) \leq n-1 \) and \( gh \mid K \) is \( p \)-to-1, where \( p = \dim Y \). By Hurewicz theorem [E, p. 136] we infer that \( \dim(R^{2n} \setminus K) < \infty \) and so there is a \( G_2 \)-set \( L \subseteq R^{2n} \) such that \( L \cap K \subseteq W \) and \( \dim \left( L \right) < \infty \). By Lemma 4, the inclusion \( I^n \subseteq R^{2n} \) is approximable by maps with \( \dim \left( L \right) = \infty \). It is clear that \( h|L \) may serve as the desired approximation to \( f_0 \) having a finite-dimensional image.

§ 6. Some questions. The requirement \( \int Y < \infty \) is caused directly by a similar assumption in Theorem 1. As without it Proposition 4 would provide a proof of (C), it is of interest to investigate the following "limit" statements of results of §§ 2, 3 (all spaces are compact and \( Q = I^n \)):

1. Let \( f: I^n \to Y \) be a map with \( \dim(f) = 0 \), let \( n \in N \) and in case \( n > 1 \) assume that \( S(f) \) is a countable union of \( Z_2 \)-sets. Is then any map \( I^n \to Q \) approximable by maps \( s: I^n \to Q \) for which \( fs(I^n) \) is a countable union of finite-dimensional compacta?

2. Is there an \( n \in N \) such that whenever \( f: X \to Y \) and \( \dim(f) = 0 \), \( \dim X = n \), then there exists a set \( A \subseteq X \) such that \( \dim(X \setminus A) < n-1 \) and \( f(A) \) is a countable union of finite-dimensional compacta?
Incidentally, if $X = I^n \times (\text{Cantor set})$ then a set $A$ as above exists, see [Bu]. In 1 and in 2 the words "countable u. of f.d. compacta" could be replaced by "weakly infinite-dimensional" to obtain alternative version of potential interest; the latter property is assured if $f$ is $\omega_1$-to-$1$ on $A$ or on $\alpha(I^n)$.

The problem of exhibiting infinite-dimensional versions of Theorems 1–3 seems to be interesting also because if there are no such then these theorems could serve as guidelines how to assure infinite-dimensionality of certain decomposition spaces, e.g. the question whether dimension is preserved by $1$-dimensional CE-maps (cf. [KW]) is equivalent to:

3. Is there a CE-map $f : I^n \to Y$ such that $\dim(f) = 1$ and any $\sigma$-compact set with $\dim(f|_{\sigma\times A}) = 1$ intersects a fiber of $f$ in more than $n$ points?

Conceivably, $f$ as above may be constructed so that each arc connecting a given pair of points of $I^n$ intersects a fiber of $f$ in at least 2 points ($n \geq 5$; compare Theorem 2).

Finally, in connection with Theorem 2 and the problem of detecting property (a) in product spaces, let us ask the following:

4. Let $f : I^n \to Y \in \text{ANR}$ be a map with $\dim(f) = 0$ and $n = \dim Y \geq 4$. Is the set $\{ x \in C(I^n) : fx \text{ is } 1\text{-to}-1 \}$ connected and locally connected?

References