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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ALABAMA
University, AL 35486

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L-space without any uncountable 0-dimensional subspace

by

K. Ciesielski (Warszawa)

Abstract. J. Roitman in [Ro] posed the following question: "Under CH is every left separated L-space of type ω_1 0-dimensional?". The paper contains a negative answer to this question. The construction gives also (under the assumption that there exists a cardinal $\varkappa$ s.t. $2^\varkappa = \varkappa^+$) an answer to two questions of Arkhangel'skiǐ [Ar; Problems 11 and 12, p. 81]: "Is every regular left separated space 0-dimensional?" and "Does every completely regular space have a dense 0-dimensional subspace?".

Terminology and notation. Our terminology related to topology and set theory follows [En1] and [Ku] respectively.

A topological space Z is called left (right) separated if there is a well-ordering $<$ of Z s.t. every initial segment of Z under $<$ is closed (open).

$$hL(X) = \sup\{|Z|:\ Z\subset X \text{ is right separated}\}+\omega.$$

By an L-space we will understand a regular space which is hereditarily Lindelöf and not hereditarily separable. Let us recall that a space X is heraditarily Lindelöf if and only if $hL(X) = \omega$.

Let $H(A, B)$ be the set of all finite functions from a set A into a set B and let $\mathscr{B}$ be a standard countable basis for the unit interval $I = \langle 0, 1\rangle$ not containing the empty set. For $\varepsilon \in H(A, \mathscr{B})$ the standard basic set in I^A given by ε is denoted by $[\varepsilon]$, i.e., $[\varepsilon] = \bigcap\{I^{A\setminus(a)}\times(\varepsilon(a))^{(a)}:\ a\in \operatorname{dom}\varepsilon\}$.

For a family $\mathscr{F}$ of subsets of a set X let $\tau(\mathscr{F})$ denote the topology on X determined by $\mathscr{F}$ as a subbasis.

Auxiliary lemmas. The first lemma is a generalization of the Hurewicz Theorem [En2; Thm. 1.8.20, p. 81], that the Hilbert cube I^ω is not the countable sum of 0-dimensional subspaces.

Let $\mathscr{D}$ be the family of all sets of the form $F\times I^{\omega\setminus A}$ where $A\in[\omega]^\omega$ and F is a 0-dimensional subset of I^A.

Lemma 1. *If $D_n\in\mathscr{D}$ for $n<\omega$ then $I^\omega \neq \bigcup\{D_n:\ n<\omega\}$.*

Proof. Let $D_n = F_n \times I^{\omega \setminus A_n}$ for $n < \omega$ where F_n is a 0-dimensional subset of I^{A_n}. We can choose a one-to-one sequence $\langle k_n \colon n < \omega \rangle$ s.t. $k_n \in A_n$.

Let $S_n \subset I^{A_n}$ be a partition between the faces $I^{A_n \setminus \{k_n\}} \times \{i\}^{\{k_n\}}$ $(i = 0, 1)$ s.t. $S_n \cap F_n = \emptyset$ for $n < \omega$ and let $L_n = S_n \times I^{\omega \setminus A_n}$. Hence L_n is a partition between $W_n^i = I^{\omega \setminus \{k_n\}} \times \{i\}^{\{k_n\}}$ $(i = 0, 1)$ s.t. $L_n \cap D_n = \emptyset$.

For $x \in [\omega]^{<\omega}$ let $I(x) = \{f \in I^\omega \colon (\forall k < \omega)[(\forall n \in x)(k \neq k_n) \to f(k) = 0]\}$. The intersection $L_n \cap I(x)$ is a partition in $I(x)$ between $W_n^i \cap I(x)$ $(i = 0, 1)$ for $n \in x$. Hence (see [En2; Thm. 1.8.1, p. 72]) $\bigcap \{L_n \colon n \in x\} \supset \bigcap \{L_n \cap I(x) \colon n \in x\} \neq \emptyset$. So, the family $\{L_n \colon n < \omega\}$ of closed subsets of I^ω has the finite intersection property. The space I^ω being compact, $\bigcap \{L_n \colon n < \omega\} \neq \emptyset$. But

$$\bigcap \{L_n \colon n < \omega\} \subset I^\omega \setminus \bigcup \{D_n \colon n < \omega\}, \text{ and so } I^\omega \neq \bigcup \{D_n \colon n < \omega\}.$$

As an easy corollary to the above lemma we obtain the following

LEMMA 2. *Let* $K = \prod_{n<\omega} J_n \subset I^\omega$ *s.t.* $J_n = \langle a_n, b_n \rangle$ *where* $0 \leqslant a_n < b_n \leqslant 1$ *for* $n < \omega$ *and let* $D_n \in \mathscr{D}$ *for* $n < \omega$. *Then*

$$K \setminus \bigcup \{D_n \colon n < \omega\} \neq \emptyset.$$

The next lemma is basic in our construction. We use the following notation:

$$\mathscr{E} = \{\bigcup_{n<\omega} [\varepsilon_n] \colon \varepsilon_n \in H(\omega, \mathscr{B}) \text{ for } n < \omega \text{ and } \operatorname{dom}\varepsilon_n \cap \operatorname{dom}\varepsilon_k = \emptyset \text{ for } n < k < \omega\}.$$

LEMMA 3. *Let* $D_n \in \mathscr{D}$ *and* $E_n \in \mathscr{E}$ *for* $n < \omega$. *Then*

$$\bigcap \{E_n \colon n < \omega\} \setminus \bigcup \{D_n \colon n < \omega\} \neq \emptyset.$$

Proof. Let $E_n = \bigcup_{k<\omega} [\varepsilon_k^n]$ where $\operatorname{dom}\varepsilon_l^n \cap \operatorname{dom}\varepsilon_k^n = \emptyset$ for $k < l < \omega$ and $n < \omega$. Let $\langle k_n \colon n < \omega \rangle$ be a sequence s.t.

(*) $$\operatorname{dom}\varepsilon_{k_n}^n \cap \operatorname{dom}\varepsilon_{k_m}^m = \emptyset \quad \text{for} \quad n < m < \omega.$$

Then

$$L = \bigcap_{n<\omega} [\varepsilon_{k_n}^n] \subset \bigcap_{n<\omega} \Big(\bigcup_{k<\omega} [\varepsilon_k^n]\Big) = \bigcap_{n<\omega} E_n$$

and by (*) there is a set $K \subset L$ as in the assumption of lemma 2. Hence

$$\bigcap \{E_n \colon n < \omega\} \setminus \bigcup \{D_n \colon n < \omega\} \neq \emptyset.$$

LEMMA 4. *Let* Y *be a topological space with basis* $\tilde{\mathscr{B}}$, Z *a* 0-*dimensional subspace of* Y, *s.t.* $hL(Z) = \omega$ *and* $\mathscr{B}_0$ *a countable family of open sets. Then there exists a* $\mathscr{B}_1 \subset \tilde{\mathscr{B}}$ *s.t.* $|\mathscr{B}_1| \leqslant \omega$ *and*

(*) $$\forall U \in \mathscr{B}_0 \forall z \in U \cap Z \exists V [z \in V \subset U \,\&\, V \cap Z \text{ is clopen in } (Z, \tau(\mathscr{B}_1))].$$

Proof. Let $U \in \mathscr{B}_0$ and $z \in U \cap Z$. Then by 0-dimensionality of Z there exist open sets $V_1(z, U)$ and $V_2(z, U)$ s.t.

$$V_1(z, U) \cap V_2(z, U) = \emptyset, \quad Z \subset V_1(z, U) \cup V_2(z, U) \quad z \in V_1(z, U) \subset U$$

and

$$Z \setminus U \subset V_2(z, U).$$

Moreover, by $hL(Z) = \omega$, we can choose $V_1(z, U)$ and $V_2(z, U)$ s.t. $V_1(z, U)$, $V_2(z, U) \in \tau(\mathscr{B}(z, U))$, for some countable $\mathscr{B}(z, U) \subset \tilde{\mathscr{B}}$. Thus $V_1(z, U) \cap Z$ is clopen in $(Z, \tau(\mathscr{B}(z, U)))$.

Let $Z(U) \in [Z]^{\leqslant\omega}$ be s.t. $U \cap Z \subset \bigcup \{V_1(z, U) \colon z \in Z(U)\}$ and let $\mathscr{B}(U) = \bigcup \{\mathscr{B}(z, U) \colon z \in Z(U)\}$. Then

$$\forall z \in U \cap Z \; \exists V \; [z \in V \subset U \,\&\, V \cap Z \text{ is clopen in } (Z, \tau(\mathscr{B}(U)))].$$

Hence $\mathscr{B}_1 = \bigcup \{\mathscr{B}(U) \colon U \in \mathscr{B}_0\}$ satisfies (*).

The example. The basic idea of our construction is taken from Hurewicz's example (under CH) of an uncountable space $X \subset I^\omega$ without an uncountable 0-dimensional subspace (see [En2; Example 1.8.21, p. 82]) and from the construction of an HFC-set from CH (see [Ro]).

THEOREM. *Let us assume* CH. *Then there exists a left separated space* $X \subset I^{\omega_1}$ *of power* ω_1 *s.t.* $hL(X) = \omega$ *and without any uncountable* 0-*dimensional subspace.*

Proof. Let $\tilde{\mathscr{D}}$ be a family of all sets of the form $G \times I^{\omega_1 \setminus \alpha} \subset I^{\omega_1}$ where $\omega \leqslant \alpha < \omega_1$ and G is a 0-dimensional G_δ-set in I^α. Then $|\tilde{\mathscr{D}}| = 2^\omega = \omega_1$. So, let $\langle D_\zeta \colon \omega \leqslant \zeta < \omega_1 \rangle$ be an enumeration of $\tilde{\mathscr{D}}$ s.t. if $D_\zeta = G \times I^{\omega_1 \setminus \alpha}$ where G is 0-dimensional in I^α then $\alpha \leqslant \zeta$. Moreover, let $\tilde{\mathscr{E}}$ be the family of all sets of the form $\bigcup \{[\varepsilon_n] \colon n < \omega\}$ where $\varepsilon_n \in H(\omega_1, \mathscr{B})$ for $n < \omega$ and $\operatorname{dom}\varepsilon_n \cap \operatorname{dom}\varepsilon_k = \emptyset$ for $n < k < \omega$. Then $|\tilde{\mathscr{E}}| = 2^\omega = \omega_1$. So, let $\langle E_\zeta \colon \omega \leqslant \zeta < \omega_1 \rangle$ be an enumeration of $\tilde{\mathscr{E}}$ s.t. if $E_\zeta = \bigcup \{[\varepsilon_n] \colon n < \omega\}$ where $\operatorname{dom}\varepsilon_n \cap \operatorname{dom}\varepsilon_k = \emptyset$ for $n < k < \omega$ then $\operatorname{dom}\varepsilon_n \subset \zeta$ for $n < \omega$.

We define $X = \{f_\zeta \colon \omega \leqslant \zeta < \omega_1\} \subset I^{\omega_1}$ s.t.

(*) $$f_\zeta(\alpha) = \begin{cases} 1 & \text{for} \quad \zeta = \alpha, \\ 0 & \text{for} \quad \zeta < \alpha \end{cases}$$

and we choose $f_\zeta \restriction \zeta$ s.t.

(**) $$f_\zeta \restriction \zeta \in p_\zeta\big(\bigcap \{E_\xi \colon \omega \leqslant \xi < \zeta\} \setminus \bigcup \{D_\xi \colon \omega \leqslant \xi < \zeta\}\big)$$

where $p_\zeta \colon I^{\omega_1} \to I^\zeta$ is a projection. The point as in (**) can be chosen by Lemma 3 and the fact that I^ζ and I^ω are homeomorphic.

The properties of the space X.

(1) *X is left separated.*

Proof. Let $1 \in W \in \mathscr{B}$, $0 \notin W$ and $\varepsilon_\eta = \{\langle \eta, W \rangle\} \in H(\omega_1, \mathscr{B})$. Then $U_\xi = \bigcup \{[\varepsilon_\eta] \colon \xi \leqslant \eta < \omega_1\}$ is open and $U_\xi \cap X = \{f_\eta \colon \xi \leqslant \eta < \omega_1\}$.

(2) $hL(X) = \omega$.

Proof. Let us assume that $hL(X) > \omega$. Then there exists a sequence $\langle \varepsilon_\xi : \xi < \omega_1 \rangle$ of elements of $H(\omega_1, \mathscr{B})$ s.t.

(o) $$X \cap [\varepsilon_\xi] \setminus \bigcup_{\eta < \xi} [\varepsilon_\eta] \neq \emptyset \quad \text{for} \quad \xi < \omega_1 .$$

By the Δ-lemma [Ku; Thm. 1.6, Ch. II], we can assume that $\operatorname{dom}\varepsilon_\eta \cap \operatorname{dom}\varepsilon_\xi = \emptyset$ for $\eta < \xi < \omega_1$. Hence $E = \bigcup \{[\varepsilon_n]: n < \omega\} \in \tilde{\mathscr{E}}$. Let $E = E_\zeta$. Then, by (**),

$$\{f_\xi : \zeta \leqslant \xi < \omega_1\} \subset E_\zeta = \bigcup \{[\varepsilon_n]: n < \omega\}.$$

So, there exists a $\xi < \omega_1$ s.t.

$$X \cap [\varepsilon_\xi] \setminus \bigcup_{\eta < \xi} [\varepsilon_\eta] = \emptyset ,$$

contradicting (o).

(3) *If $Z \subset X$ is 0-dimensional then $|Z| \leqslant \omega$.*

Proof. Let $\mathscr{B}_\lambda = \{[\varepsilon] \subset I^{\omega_1}: \varepsilon \in H(\lambda, \mathscr{B})\}$ for $\lambda < \omega_1$.

By Lemma 4 we can define an increasing sequence $\langle \lambda_n \in \omega_1 : n < \omega \rangle$ s.t. $\lambda_0 = \omega$ and, for $n < \omega$,

$$\forall U \in \mathscr{B}_{\lambda_n} \forall z \in U \cap Z \ \exists V [z \in V \subset U \ \& \ V \cap Z \text{ is clopen in } (Z, \tau(\mathscr{B}_{\lambda_{n+1}}))] .$$

Hence, if $\lambda = \bigcup \{\lambda_n : n < \omega\} < \omega_1$ then Z is 0-dimensional in topology $\tau(\mathscr{B}_\lambda)$, i.e., $p_\lambda(Z)$ is 0-dimensional in I^λ. So (compare [En2; Thm. 1.2.14, p. 15]) there exists a 0-dimensional G_δ-set D in topology $\tau(\mathscr{B}_\lambda)$ s.t. $Z \subset D$. But $D \in \tilde{\mathscr{D}}$, i.e., $D = D_\xi$ for some $\xi < \omega_1$. Hence, by (**),

$$Z \subset D_\xi \cap X \subset \{f_\zeta : \omega \leqslant \zeta < \xi\} .$$

By the fact that every completely regular space of power less then continuum is 0-dimensional (see [Ro]) we obtain

Corollary 1. *The continuum hypothesis is equivalent to the statement: "there exists an L-space without any uncountable 0-dimensional subspace"*

Corollary 2. *Let us assume* CH. *Then there exists a completely regular left separated space of type ω_1 without any 0-dimensional subspace of power ω_1. In particular, such space does not contain any dense 0-dimensional subspace.*

Remark. If we assume that there exists a cardinal $\varkappa > \omega$ s.t. $2^\varkappa = \varkappa^+$ then, using our construction, we can define a left separated space $X \subset I^{\varkappa^+}$ of type $\varkappa^+$ s.t. $hL(X) \leqslant \varkappa$ and without 0-dimensional subspaces of power $\varkappa^+$. In particular, X does not contain dense 0-dimensional subspaces.

Our construction for $\varkappa > \omega$ differs from that for $\varkappa = \omega$ only in the proof of Lemma 1: for $\varkappa > \omega$ the proof is based on slightly different methods (see [Mi; Corollary A, p. 282]).

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INSTITUTE OF MATHEMATICS
WARSAW UNIVERSITY
PKiN
00-901 Warszawa

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