L-space without any uncountable 0-dimensional subspace

by

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Abstract. J. Rolem in [Ro] posed the following question: "Is there a left separated L-space of type 0, 0-dimensional?". The paper contains a negative answer to this question. The construction gives also (under the assumption that there exists a cardinal x such that 2^x = x^+\) an answer to two questions of Arkhange\'lskii [Ar; Problems 11 and 12, p. 81]: "Is every regular left separated space 0-dimensional?" and "Does every completely regular space have a dense 0-dimensional subspace?"

Terminology and notation. Our terminology related to topology and set theory follows [En1] and [Ku] respectively.

A topological space X is called left (right) separated if there is a well-ordering < of X such that every initial segment of X under < is closed (open).

hL(X) = sup {\{C \subseteq X : C is right separated\} + \omega}.

By an L-space we will understand a regular space which is hereditarily Lindelöf and not hereditarily separable. Let us recall that a space X is hereditarily Lindelöf if and only if hL(X) = \omega.

Let H(A, B) be the set of all finite functions from a set A into a set B and let \mathcal{B} be a standard countable basis for the unit interval I = (0, 1) not containing the empty set. For x ∈ H(A, B) the standard basic set in I^A given by x is denoted by [x], i.e., [x] = \bigcap \{ F \times (0, \delta) : x(a) \in F, a \in dom(x)\}.

For a family \mathcal{F} of subsets of a set X let \tau(\mathcal{F}) denote the topology on X determined by \mathcal{F} as a subbasis.

Auxiliary lemmas. The first lemma is a generalization of the Hurewicz Theorem [En2; Thm. 1.5.20, p. 81], that the Hilbert cube I^\omega is not the countable sum of 0-dimensional subspaces.

Let \mathcal{B} be the family of all sets of the form F × I^{\omega \setminus A} where A ∈ [\omega]^\omega and F is a 0-dimensional subset of I^\omega.

Lemma 1. If D_\alpha ∈ \mathcal{B} for \alpha < \omega then I^\omega \neq \bigcup \{ D_\alpha : \alpha < \omega \}.
Proof. Let \( D_n = F_n \times I^{\omega \setminus \omega} \) for \( n < \omega \) where \( F_n \) is a 0-dimensional subset of \( I^n \). We can choose a one-to-one sequence \( \{k_n : n < \omega\} \) s.t. \( k_n \in A_n \).

Let \( S_n \subseteq I^{\omega \setminus \omega} \) be a partition between the faces \( I^{\omega \setminus (\omega \setminus \{k_n\})} \) (\( i = 0, 1 \)) s.t. \( S_n \cap F_n = \emptyset \) for \( n < \omega \) and let \( L_n = S_n \times I^{\omega \setminus \omega} \). Hence \( L_n \) is a partition between \( W^{(i)} = I^{\omega \setminus (\omega \setminus \{k_n\})} \) (\( i = 0, 1 \)) s.t. \( L_n \cap D_n = \emptyset \).

For \( x \in [\omega]^{\omega} \) let \( I(x) = \{ f \in I^{\omega} : \forall y < x (f(y) < k_y) \} \). The intersection \( L_n \cap I(x) \) is a partition in \( I(x) \) between \( W^{(i)} \cap I(x) \) (\( i = 0, 1 \)) for \( n < \omega \).

Hence (see [En2; Thm. 1.3.1, p. 72]) \( \bigcap \{ L_n : n < \omega \} = \bigcap \{ I(x) : x \in \omega \} = \emptyset \).

So, the family \( \{ L_n : n < \omega \} \) of closed subsets of \( I^n \) has the finite intersection property.

The space \( I^n \) being compact, \( \bigcap \{ L_n : n < \omega \} \neq \emptyset \). But

\[ \bigcap \{ L_n : n < \omega \} \subseteq I^n \cup \{ D_n : n < \omega \} \text{ and so } I^n \neq \bigcup \{ D_n : n < \omega \}. \]

As an easy corollary to the above lemma we obtain the following:

**Lemma 2.** Let \( K = \prod_{n < \omega} I^n \) s.t. \( J_n = \langle a_n, b_n \rangle \) where \( 0 < a_n < b_n < 1 \) for \( n < \omega \) and let \( D_n \in \mathcal{A} \) for \( n < \omega \). Then

\[ K \cup \{ D_n : n < \omega \} \neq \emptyset. \]

The next lemma is basic in our construction. We use the following notation:

\[ \mathcal{E} = \{ \langle a_n : n < \omega \rangle : \langle e_n : n < \omega \rangle \in H(\omega, \mathcal{A}) \text{ for } n < \omega \} \text{ and } \text{dom}_{a_n} \cap \text{dom}_{e_n} = \emptyset \text{ for } n < k < \omega \}. \]

**Lemma 3.** Let \( D_n \in \mathcal{A} \) and \( E_n \in \mathcal{E} \) for \( n < \omega \). Then

\[ \bigcap \{ E_n : n < \omega \} \cup \{ D_n : n < \omega \} \neq \emptyset. \]

Proof. Let \( E_n = \bigcup_{k < \omega} \text{dom}_{a_k} \cap \text{dom}_{e_k} = \emptyset \) for \( k < l < \omega \) and \( n < \omega \). Let \( \langle k_n : n < \omega \rangle \) be a sequence s.t.

\[ \text{dom}_{a_{k_n}} \cap \text{dom}_{e_{k_n}} = \emptyset \text{ for } n < m < \omega. \]

Then

\[ L = \bigcap_{n < \omega} \{ k_n \} = \bigcap \{ \bigcup_{n < \omega} \{ k_n \} \} = \bigcap E_n \]

and by (**) there is a set \( K \subseteq L \) as in the assumption of Lemma 2. Hence

\[ \bigcap \{ E_n : n < \omega \} \cup \{ D_n : n < \omega \} \neq \emptyset. \]

**Lemma 4.** Let \( Y \) be a topological space with basis \( \mathcal{B} \), \( Z \) a 0-dimensional subspace of \( Y \), s.t. \( hL(Z) = \omega \) and \( \mathcal{A} \) a countable family of open sets. Then there exists a \( \mathcal{B} \subseteq \mathcal{A} \) s.t. \( |\mathcal{B}| \leq \omega \) and

\[ \forall U \in \mathcal{B}, \forall z \in U \cap Z \exists \{ z \in U \cup V \cap Z \text{ is clopen in } (Z, \tau(\mathcal{A})) \}. \]

Proof. Let \( U \in \mathcal{B} \) and \( z \in U \cap Z \). Then by 0-dimensionality of \( Z \) there exist open sets \( V(z, U) \) and \( W(z, U) \) s.t.

\[ V(z, U) \cap W(z, U) = \emptyset, \quad Z \subseteq V(z, U) \cup W(z, U) \quad z \in V(z, U) \cup W(z, U) \]

and

\[ Z \cup W(z, U). \]

Moreover, by \( hL(Z) = \omega \), we can choose \( V(z, U) \) and \( W(z, U) \) s.t. \( V(z, U) \cap W(z, U) = \emptyset \), for some countable \( \mathcal{A} \subseteq \mathcal{A} \). Thus \( V(z, U) \cap Z \) is clopen in \( (Z, \tau(\mathcal{A})) \).

Let \( Z(U) \in [\omega]^{\omega} \) be s.t. \( U \cap Z \subseteq \bigcup \{ V(z, U) : z \in Z(U) \} \) and let \( \mathcal{A}(U) = \bigcup \{ \mathcal{A}(z) : z \in Z(U) \} \).

Then \( Z(U) \subseteq \mathcal{A}(U) \) satisfies (**) (**) \( \mathcal{B} \subseteq \mathcal{A}(U) \).

The example. The basic idea of our construction is taken from Hurewicz’s example (under CH) of an uncountable space \( X \in I^n \) without a 0-dimensional subspace (see [En2; Example 1.4.21, p. 82]) and from the construction of an HFC-set from CH (see [Ro1]).

**Theorem.** Let us assume CH. Then there exists a left separated space \( X \in I^{\omega \setminus \omega} \) of power \( \omega_1 \) s.t. \( hL(X) = \omega \) and without any uncountable 0-dimensional subspace.

Proof. Let \( \mathcal{G} \) be a family of all sets of the form \( G \times I^{\omega \setminus \omega} \) where \( \omega \leq \omega \) and \( G \in \omega \)-dimensional \( G \) s.t. \( I^n \) then \( \omega \leq \omega \). Moreover, let \( \mathcal{G} \) be the family of all sets of the form \( \langle \{ a_n : n < \omega \rangle : e_n \in H(\omega, \mathcal{A}) \text{ for } n < \omega \} \) s.t. \( D_n \in \mathcal{A}(z) \) for \( n < \omega \) and \( \text{dom}_{a_n} \cap \text{dom}_{e_n} = \emptyset \) for \( n < k < \omega \) and \( k < l < \omega \).

We define \( X = \{ f : \omega \leq \zeta < \omega, \zeta \leq \zeta \} \).

and we choose \( f : \zeta \) s.t.

\[ f = \bigcup_{\zeta} \{ \langle a_n, e_n : n < \omega \rangle : \langle B_n \rangle \in \mathcal{B}(z) \\} \]

where \( \langle B_n \rangle : \zeta \leq \zeta \) is a projection. The point as in (**) can be chosen by Lemma 3 and the fact that \( I^2 \) and \( I^2 \) are homeomorphic.

The properties of the space \( X \).

1. \( X \) is left separated.

Proof. Let \( x, y \in X \), \( x \neq y \) and \( \langle y, W \rangle \in \mathcal{B}(x) \). Then \( U_y = \bigcup \{ \{ x : \langle y, W \rangle \in \mathcal{B}(z) \} \} \) is open and \( U_x \cap X = \emptyset \) (**) \( hL(X) = \omega \).
Proof. Let us assume that $hL(X) > \omega$. Then there exists a sequence
\[ \langle x_\xi : \xi < \omega_1 \rangle \] of elements of $H(\omega_2, \mathfrak{s})$ s.t.
\[ X \cap [x_\xi] \neq \varnothing \quad \text{for} \quad \xi < \omega_1. \]

By the $A$-lemma [Ku; Thm. 1.6, Ch. II], we can assume that $\text{dom} x_\eta \cap x_\zeta$ = $\varnothing$ for $\eta < \zeta < \omega_1$. Hence $E = \cup \{[x_\eta] : n < \omega\} \in \mathfrak{s}$. Let $E = E_1$. Then, by (**)\[ \{x_\zeta : \zeta < \omega_1\} \subseteq E_1 = \cup \{[x_\eta] : n < \omega\}. \]

So, there exists a $\xi < \omega_1$ s.t.
\[ X \cap [x_\xi] \neq \varnothing, \]
contradicting (o).

(3) If $Z \subseteq X$ is 0-dimensional then $|Z| \leq \omega$.

Proof. Let $\mathfrak{s}_\lambda = \{[x] \in \mathfrak{m} : \lambda \in H(\lambda, \mathfrak{s})\}$ for $\lambda < \omega_1$.

By Lemma 4 we can define an increasing sequence $\langle \lambda_\alpha : \alpha < \omega \rangle$ s.t.
\[ \lambda_\omega = \omega \quad \text{and, for} \quad n < \omega, \]
\[ \forall U \in \mathfrak{s}_\lambda \exists Z \subseteq U \cup Z \text{ is clopen in} \ (Z, \tau(\mathfrak{s}_\lambda)). \]

Hence, if $\lambda = \cup \{\lambda_\alpha : n < \omega\} < \omega_1$ then $Z$ is 0-dimensional in topology $\tau(\mathfrak{s}_\lambda)$, i.e.,
\[ p_\lambda(Z) \text{ is 0-dimensional in} \ I^\lambda. \]

So (compare [En2; Thm. 1.2, p. 15]) there exists a 0-dimensional $G_\delta$-set $D$ in topology $\tau(\mathfrak{s}_\lambda)$ s.t.
\[ Z \subseteq D. \] But $D \in \mathfrak{s}_\lambda$, i.e., $D = D_1$ for some $\xi < \omega_1$. Hence, by (**)\[ Z \subseteq D_1 \cap X \subseteq \{x_\zeta : \omega < \zeta < \xi\}. \]

By the fact that every completely regular regular space of power less then continuum is 0-dimensional (see [Ro]) we obtain

**Corollary 1.** The continuum hypothesis is equivalent to the statement: "there exists an $L$-space without any uncountable 0-dimensional subspace"

**Corollary 2.** Let us assume CH. Then there exists a completely regular left separated space of type $\omega_1$ without any 0-dimensional subspace of power $\omega_1$. In particular, such space does not contain any dense 0-dimensional subspace.

Remark. If we assume that there exists a cardinal $\mathfrak{s} = \omega_1$, then using our construction, we can define a left separated space $Y \subseteq \mathfrak{m}^\omega$ of type $\mathfrak{s}$ s.t.
\[ hL(Y) \leq \omega \]
and without 0-dimensional subspaces of power $\omega_1$. In particular, $Y$ does not contain dense 0-dimensional subspaces.

Our construction for $\mathfrak{s} > \omega$ differs from that for $\mathfrak{s} = \omega$ only in the proof of Lemma 1: for $\mathfrak{s} > \omega$ the proof is based on slightly different methods (see [Mi; Corollary A, p. 282]).