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On the Baire order problem for a linear lattice of functions

by

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Abstract. Let a be a linear lattice of real valued functions containing the constant functions and $B_1(a)$ be the first Baire class of functions generated by a. Denote by A the smallest complete ordinary function system containing a. Then it follows immediately that $a \subset A \subset B_1(a)$ [3]. Here we show that (1) the condition $(*)$ given by Mauldin in ([3], Th. 4.1) is a necessary and sufficient condition for $B_1(a) = B_1(B_1(a))$, and (2) $A = B_1(a)$ iff A satisfies D-condition.

1. Introduction. Let X be a nonempty set and R^X be the set of all functions from X into the set R of real numbers, forming the lattice ordered R-algebra structure under operations defined pointwise. Let $H \subset R^X$. Then $B_1(H)$ (the first Baire class of H) is the family of all functions in R^X which are pointwise limits on X of sequences from H, $B_2(H) = B_1(B_1(H))$ and in general if $\alpha > 0$ is an ordinal then $B_\alpha(H)$ is the family of pointwise limits of sequences from $\bigcup_{\beta<\alpha} B_\beta(H)$. If ω_1 is the first uncountable ordinal then $B(H) = B_{\omega_1}(H) = B_{\omega_1+1}(H)$, and $B(H)$ is called the Baire class generated by H. H_u denotes the family of all functions in R^X which are uniform limits on X of sequences from H, $LS(H)$ (resp. $US(H)$) the family of all $f \in R^X$ which are pointwise limits of increasing (resp. decreasing) sequences from H and H_b the subset of H consisting of bounded functions.

A subspace H of R^X is called an *ordinary function system* if it is both a linear lattice and algebra which contains the constant functions, and which is closed under inversion (if $f \in H$ and $f > 0$, then $1/f \in H$). An ordinary function system H is called *complete* if it is also closed under uniform limits. If H is a linear lattice containing the constants, then $B_1(H)$ is a complete ordinary function system (See [3]). In [3] Mauldin proved the following.

THEOREM 1.1. *Let $a \subset R^X$ be a linear lattice containing the constants and A be the smallest complete ordinary function system containing a. Then the following hold:*

(1) $a \subset a_u \subset (LS(a) \cap US(a)) \subset A \subset B_1(a) = B_1(A)$.

(2) $(a_u)_b \subset A_b = (LS(a) \cap US(a))_b$.

For a discussion of Baire functions see Mauldin [2] and [3].

Let $H \subset R^X$ be a linear lattice containing the constants, and Let H_b^* denote the dual space of linear space H_b with the norm $\|f\| = \sup_{x \in X} |f(x)|$. Let $\varphi \in H_b^*$. Then

φ is *net additive* if for every decreasing net $\{f_\alpha\}\subset H_b$ with $f_\alpha \downarrow 0$ pointwise, $|\varphi|(f_\alpha)\to 0$, and it is *countably additive* if for every decreasing sequence $\{f_n\}\subset H_b$ with $f_n\downarrow 0$ pointwise, $|\varphi|(f_n)\to 0$. A positive linear functional φ is a *lattice homomorphism* if $\varphi(f\vee g)=\varphi(f)\vee\varphi(g)$ for $f,g\in H_b$.

For $f\in R^X$, the zero-set is $Z(f)=\{x\in X: f(x)=0\}$ and cozero-set is $CZ(f)=X\setminus Z(f)$. If H is a subset of R^X, then $Z(H)=\{Z(f): f\in H\}$; $CZ(H)=\{CZ(f): f\in H\}$.

2. Baire order and topologies. Let $a\subset R^X$ be a linear lattice containing the constants. In this section we give some of the necessary and sufficient conditions for the Baire order of a to be no more than 1.

We begin with the simple preliminaries. Let $\hat{X}$ be the decomposition of X obtained by identifying the stationary sets of a_b and $\pi: X\to\hat{X}$ be the natural mapping. For each $x\in X$, $\hat{x}$ denotes the image $\pi(x)$ in $\hat{X}$ and $[x]$ the equivalence class containing x, i.e., $\hat{x}=\pi(x)\in\hat{X}$ and

$$[x]=\pi^{-1}(\hat{x})=\bigcap\{y\in X: f(y)=f(x), f\in a_b\}\subset X.$$

For f in a_b define a function $\hat{f}$ on $\hat{X}$ by $\hat{f}(\hat{x})=f(x)$, $\hat{x}\in\hat{X}$, and set $\hat{a}_b=\{\hat{f}: f\in a_b\}$. Then $\hat{a}_b$ is a function system on $\hat{X}$ which separates the points of $\hat{X}$ and forms a normed linear lattice with respect to the norm $\|\cdot\|_{\hat{X}}$. Let X_a be the set of all lattice homomorphisms on $\hat{a}_b$ which maps 1 into 1. Then X_a is a closed subset of the unit ball in $\hat{a}_b^*$, equipped with ω^*-topology, and hence compact by the Alaoglu theorem. For each $\hat{x}\in\hat{X}$, define $\delta_{\hat{x}}$ by $\delta_{\hat{x}}(\hat{f})=\hat{f}(\hat{x})$, $\hat{f}\in\hat{a}_b$. It is clear that $\delta_{\hat{x}}\in X_a$. Since $\hat{a}_b$ separates the points of $\hat{X}$, the map $\hat{x}\to\delta_{\hat{x}}$ is an injection of $\hat{X}$ into X_a. In the sequel, we will identify $\hat{X}$ with its image under this injection and that treat $\hat{X}$ as a subset of X_a.

Let $C(X_a)$ be the space of all continuous real valued functions on X_a. For each $\hat{f}$ in $\hat{a}_b$ define a function f^β on X_a by $f^\beta(p)=p(\hat{f})$, $p\in X_a$, and set $a_b^\beta=\{f^\beta: \hat{f}\in\hat{a}_b\}$. Then a_b^β is a linear sublattice of $C(X_a)$ which contains the constants and separates the points of X_a. Hence, as is well-known, the uniform closure of a_b^β in $C(X_a)$ is identical with $C(X_a)$. It is clear that $(\hat{a}_b)_u=\widehat{(a_u)}_b$ on $\hat{X}$, and by [2], $(\hat{a}_b)_u$ is an SW-algebra ([16]). Since each $\hat{f}$ in $(\hat{a}_b)_u$ is the uniform limit of a sequence $\{\hat{f}_n\}$ from $\hat{a}_b$, each p in X_a can be uniquely extended to $(\hat{a}_b)_u$ preserving lattice-operations by the obvious definition: $p(\hat{f})=\lim_n p(\hat{f}_n)$. Hence we may assume that each p in X_a is a lattice homomorphism defined on $(\hat{a}_b)_u$. It then follows from ([5], Th. 1), that $p\in X_a$ if and only if p is a positive linear algebra homomorphism on $(\hat{a}_b)_u$ which maps 1 into 1. Furthermore, using the argument given by Kirk ([6], Th. 1.1), it follows immediately that $\hat{X}$ is dense in X_a with respect to $\sigma(X_a,\hat{a}_b)$-topology. Thus every $\hat{f}$ in $(\hat{a}_b)_u$ has the unique continuous extension f^β from $\hat{X}$ to X_a by $f^\beta(p)=p(\hat{f})$, $p\in X_a$, and $(\hat{a}_b)_u$ is algebraically, topologically and lattice isomorphic to $C(X_a)$ under $\hat{f}\to f^\beta$. When we treat X as a topological space, its topology will mean the weak topology induced by the functions of a_b.

We need the following lemma which is similar to a lemma of Nagami [8].

LEMMA 1. *If $f\in B(a)$, then there is a sequence $\{f_n\}$ of functions from a such that $f(x)=f(y)$ whenever $f_n(x)=f_n(y)$ for all n.*

Proof. Use the argument given in ([8], Lemma).

Now we obtain the following result which extends the theorem given by Mauldin in ([3], Th. 4.1).

THEOREM 2. *The following are equivalent:*

(A) *For each $\varphi\in a_b^*$, there is a countable subset $\{x_n\}$ of X and a sequence $\{\alpha_n\}$ from l^1 such that $\varphi(f)=\sum_n\alpha_n f(x_n)$ for all $f\in a_b$*–(*).

(B) (1) *X is a compact space (not necessarily Hausdorff).*

(2) *The Baire order of a is no more than 1; i.e., $B_1(a)=B(a)$.*

(3) *For each $f\in B(a)$, the image set $f(X)$ is countable.*

(C) (1) *X is a compact space (not necessarily Hausdorff).*

(2) *For each $f\in a_u$, the image set $f(X)$ is countable.*

(D) *X_a is a totally disconnected compact space without perfect subsets.*

Proof. (A) → (B): (1) By assumption (A) it is easy to see that every $\varphi\in a_b^*$ is net additive. Hence by ([4], (14) Th.), it is immediate that X is compact (not necessarilly Hausdorff). (2) Let f be a function in $B(a)$. Then by Lemma 1 there is a sequence $\{f_i\}$ from a such that $f(x)=f(y)$ whenever $f_i(x)=f_i(y)$ for all i. Let $\{f_{ij}\}_{j=1}^\infty$ be a sequence from a_b converging pointwise to each f_i. Let a_0 be a norm closed and separable subalgebra of $(a_u)_b$ spanned by $\{f_{ij}\}_{i,j=1}^\infty$, and let a_0 contain constants. Then it is clear that a_0 is a closed subspace of $C(X)_b$. Let $\tilde{X}$ be the quotient of X obtained by identifying the stationary sets of a_0, and π be the quotient map of X onto $\tilde{X}$. Denote by $\tilde{x}$ the image $\pi(x)$ in $\tilde{X}$. For each $g\in a_0$ define a function $\tilde{g}$ on $\tilde{X}$ by $\tilde{g}(\tilde{x})=g(x)$, $x\in\tilde{x}$, and set $\tilde{a}_0=\{\tilde{g}: g\in a_0\}$. Then from the result of (1), X is compact, and by the above preliminaries it follows that $X_{a_0}=\tilde{X}$ and $C(X_{a_0})=\tilde{a}_0$. Hence $C(X_{a_0})$ is separable with respect to the sup norm. From this, it follows that X_{a_0} is a compact metrizable. Let $\tilde{\varphi}$ be a functional in $C(X_{a_0})^*$, and define a functional φ in a_0^* by $\varphi(g)=\tilde{\varphi}(\tilde{g})$ for all $g\in a_0$. Since $a_0\subset(a_u)_b\subset C(X)_b$, by Hahn Banach Theorem there is a bounded linear functional Φ on $(a_u)_b$ which coincides with φ on a_0. Then by assumption (A) there is a countable subset $\{x_n\}$ of X and a sequence $\{\alpha_n\}$ from l^1 such that

(i) $\Phi(g)=\sum_n\alpha_n g(x_n)$ for all $g\in a_b$

Since each g in $(a_u)_b$ is the uniform limit of a sequence $\{g_n\}$ of functions from a_b, Φ has the same form (i) on $(a_u)_b$, and particulary, it follows that

(ii) $\Phi(g)=\varphi(g)=\sum_n\alpha_n g(x_n)$ for all $g\in a_0$.

Hence $\tilde{\Phi}$ has also the same form (ii). This means that X_{a_0} is almost discrete space (See [9]) and by Babiker ([9], Cor. 4.3), X_{a_0} is at most countable. Hence it follows by Mauldin ([2], Th. 12) that $B_1(C(X_{a_0}))=R^{X_{a_0}}$. Define a function $\tilde{f}$ on X_{a_0} by $\tilde{f}(\tilde{x})=f(x)$ for all $\tilde{x}\in X_{a_0}$. Since $f(x)=f(y)$ whenever $f_{ij}(x)=f_{ij}(y)$ for all i,j,

$\tilde{f}$ is well-defined on X_{a_0} and $\tilde{f} \in B_1(C(X_{a_0}))$. Let $\{\tilde{g}_n\}$ be a sequence from $C(X_{a_0})$ converging pointwise to $\tilde{f}$. Then it is clear that $g_n(x) = \tilde{g}_n(\pi(x)) \in a_0$ and $g_n(x) \to f(x)$ $(n \to \infty)$. Hence $f \in B_1(a)$. Thus we have $B(a) = B_1(a_u) = B_1(a)$. (3) Finally, since X_{a_0} is at most countable, it follows immediately that $f(X)$ is countable.

(B) → (C): Since $a_u \subset B(a)$, (2) is obvious from (3) of assumption (B).

(C) → (D): From (1) of assumption (C), it is clear that $\hat{X} = X_a$ and X_a is a compact Hausdorff. Let K be a closed connected subset of X_a and p be a point of K. If $q \in K \setminus \{p\}$, then there is a continuous function $\hat{f}$ on X_a such that $\hat{f}(p) = 0$, $\hat{f}(q) = 1$ and $0 \leqslant f \leqslant 1$ on X_a. Since $(\widehat{a_u})_b = C(X_a)$ it follows from (2) of assumption (C) that $\hat{f}(X_a)$ is countable. Let $\alpha \notin \hat{f}(X_a)$ for some $0 < \alpha < 1$, and define a function g by $g = 0$ on $[0, \alpha)$ and $g = 1$ on $(\alpha, 1]$. It is clear that $h = g \cdot \hat{f}$ is continuous on X_a. Let $F_1 = K \cap h^{-1}(0)$ and $F_2 = K \cap h^{-1}(1)$. Then F_1, F_2 are closed subsets of K, and it follows that $p \in F_1$, $q \in F_2$, $K = F_1 \cup F_2$ and $F_1 \cap F_2 = \emptyset$. But this is a contradiction, and hence $K = \{p\}$. Thus X_a is totally disconnected, and by Gillman and Jerison ([10], 16.17), X_a has a base of clopen sets. Next, suppose that X_a contains a nonempty perfect set P and let $\{P_0, P_1\}$ be a partition of P where P_0, P_1 are relatively clopen infinite subsets of P. Since P_0, P_1 are open in P, it follows immediately that P_0, P_1 are perfect. Since X_a has a base of clopen sets, there are clopen subsets X_0 and X_1 of X_a such that $P_0 \subset X_0$, $P_1 \subset X_1$, $X_0 \cap X_1 = \emptyset$ and $X_0 \cup X_1 = X_a$. Repeating the above reasoning, and using the argument given by Pelczynski and Semadeni in ([7], Lemma 2), we have a continuous function $\hat{f}$ from X_a onto the Cantor discontinuum C. Then it is clear that $f = \hat{f} \cdot \pi \in a_u$ and $f(X) = C$. But this contradicts the fact that $f(X)$ is countable. Hence X_a contains no perfect nonempty subset.

(D) → (A): Let φ be a bounded linear functional on a_b. Without loss of generality, we suppose that $\varphi \geqslant 0$. Define $\hat{\pi}(\varphi)$ by $\hat{\pi}(\varphi)(\hat{g}) = \varphi(\hat{g} \cdot \pi)$ for all $\hat{g} \in \hat{a}_b$, and let $0 \leqslant \Phi$ be a extension of $\hat{\pi}(\varphi)$ on $C(X_a) = (a_u)_b$. Since X_a is compact space without perfect subsets, by Rudin ([11], Th. 6), there is a countable subset $\{x_n\}$ of X_a and a sequence $\{\alpha_n\}$ from l^1 such that $\Phi(\hat{f}) = \sum_n \alpha_n \hat{f}(\hat{x}_n)$ for all $\hat{f} \in C(X_a)$. Hence it is immediate that $\varphi(f) = \sum_n \alpha_n f(x_n)$ for $f \in a_b$, and the proof is complete.

As an obvious corollary of Theorem 2, we have;

Corollary 3. *Let X be a completely regular space. Then the following are equivalent*:

(A) *For each $\varphi \in C^*(X)_b$ there is a countable subset $\{x_n\}$ of X and a sequence $\{\alpha_n\}$ from l^1 such that $\varphi(f) = \sum_n \alpha_n f(x_n)$ for all $f \in C(X)_b$.*

(B) (1) *X is a compact Hausdorff space.*

(2) *The Baire order of $C(X)$ is no more than* 1.

(3) *If $f \in B(C(X))$, then $f(X)$ is countable.*

(C) (1) *X is a compact Hausdorff space.*

(2) *If $f \in C(X)$, then $f(X)$ is countable.*

(D) *X is a totally disconnected compact space without perfect subsets.*

3. Dominant condition. We first need some preliminary. Two functions f and g of R^X are *disjoint* if $|f| \wedge |g| = 0$, and a *disjoint system* $\{f_i\}$ of R^X is a pairwise disjoint collection of nonnegative functions f_i of R^X.

Let H be a linear sublattice of R^X. H is *σ-laterally complete* if the ordered supremum of every disjoint sequence of H exists, and *σ-complete* if every countable subset of H with an upper bound in H has a ordered supremum in H.

A sequence $\{f_n\}$ of R^X is *uniformly bounded* if there is a constant function α such that $|f_n| \leqslant \alpha$ for all n. A linear sublattice H is said to be *bounded σ-closed* in R^X if every uniform bounded sequence of H has its pointwise supremum in H. If H is boundedly σ-closed in R^X, then it follows easily that H_b is σ-complete. But its converse is in general false. A sequence $\{f_n\}$ of H *order-converges* to f in H if there is a decreasing sequence $\{u_n\}$ of H with $\bigwedge_n u_n = 0$ in H and $|f_n - f| \leqslant u_n$ for all n. A sequence $\{f_n\}$ of H is *order-Cauchy* if there is a decreasing sequence $\{u_n\}$ of H with $\bigwedge_n u_n = 0$ in H and $|f_{n+m} - f_n| \leqslant u_n$ for all m, n. A linear sublattice H is said to be *order-Cauchy complete* if every order-Cauchy sequence in H order-converges in H to some element of H.

Let a be a linear sublattice of R^X containing the constants and $Z_\delta(a)$ be the family of all countable intersections of zero sets in $Z(a)$. Let A denote the smallest complete ordinary function system containing a. Then it follows by Mauldin ([3], Th. 3.3), that A is the set of all functions f of R^X such that $f^{-1}(F) \in Z_\delta(a)$ for every closed subset F of R. In the sequel, we shall suppose that linear sublattice a contains the constants and separates points of X. This is no restriction for our considerations, since if a does not separate points of X, as observed in Section 2, we may consider the quotient of X obtained by identifying the points of X not separated by a, and obtain a function system which has the same structure as a. Let X_A denotes the set of all nonzero lattice (algebra) homomorphisms of A_b onto R. Let τ_A be the relative $\sigma(A_b^*, A_b)$-topology on X embedded into X_A. Since A_b is an SW-algebra, it follows that X_A is the compactification of (X, τ_A) for which $f \in C_b(X, \tau_A)$ has a continuous extension f^β on X_A if and only if $f \in A_b$. However, in general, $A_b \neq C_b(X, \tau_A)$ and $A \neq C(X, \tau_A)$ (See [6]).

Definition. We shall say that A satisfies *D-condition* if it satisfies the following: (D) for each $g \in LS(a)$ there is an $f_g \in A$ such that $f_g \geqslant g$ on X.

Since $LS(a) \subset B_1(a) \subset US(LS(a))$ (See [3]), A satisfies D-condition if and only if for each $g \in B_1(a)$ there is an $f \in A$ such that $f \geqslant g$. We can now prove the following result.

Theorem 4. *Let A be the smallest complete ordinary function system containing a. Then the following are equivalent*:

(A) *A satisfies D-condition.*

(B) (1) *A is σ-laterally complete.*

(2) *A is σ-complete.*

(C) (1) *X is P-space, i.e., a space in which every G_δ set is open.*

(2) *A is σ-complete.*

(D) A *is bounded* σ*-closed.*

(E) A *is closed under pointwise convergence of sequences*; $A = B_1(A) = B_1(a)$.

Proof. (A) → (B): (1) Let $\{f_n\}$ be a disjoint sequence of A, and let $g = \sum_n nf_n$. It is clear that $g \in B_1(a)$. Since A satisfies D-condition there is a function $f \in A$ such that $f \geq g$ and $f \geq 1$. It then follows that for every $n, p > 1$,

$$\sum_{k=n+1}^{n+p} f_k \cdot f^{-1} \leq \sum_{k=n+1}^{n+p} (k/n) f_k \cdot f^{-1} \leq 1/n \text{ and } \sum_{k=n+1}^{n+p} f_k \cdot f^{-1} \in A.$$

Let $h = \sum_n f_n \cdot f^{-1}$. Since A is uniformly closed, it is immediate that $h \in A$, and $\sum_n f_n = h \cdot f \in A$. Hence A is σ-laterally complete. (2) Since $C(X_A) \cong A_b \subset A$, it is clear that every disjoint sequence of $C(X_A)$ with an upper bound has its ordered supremum in $C(X_A)$. Hence it follows from ([12], Prop. 2) and ([10], 3N) that $C(X_A)$ is σ-complete. This means that A_b is σ-complete. Let $h(x) = x/(1+|x|)$ on R. If $f, f_n \in A$ and $f_n \leq f$ for all n, then it is clear that $h \cdot f_n$, $h \cdot f \in A_b$ and $h \cdot f_n \leq h \cdot f$. Since A_b is σ-complete, it follows immediately that $g = \bigvee_n h \cdot f_n \in A_b \subset A$. Hence $\bigvee_n f_n = h^{-1} \cdot g = g/(1-|g|) \in A$, and it follows that A is σ-complete.

(B) → (C): (1) It is sufficient to show that every point in X is a P-point (See [10], 4L). If $p \in X$ is not a P-point, then there is a G_δ-set G of X containing p such that for a sequence $\{U_n\}$ of open neighborhood of p in X_A, $G = X \cap (\bigcap_n U_n)$ is not neighborhood of p. Let $Z(f^\beta)$ be a zero set of X_A such that $Z(f^\beta) \subset \bigcap_n U_n$ and $p \in Z(f^\beta) \cap X \subset G$, where $f \in A_b$ and $0 \leq f \leq 1$. Let $G_n = \{x \in X\colon f(x) > 1/n\}$ and $Z_n = \{x \in X\colon f(x) \geq 1/n\}$. Then for each n there is an $f_n \in A_b$ such that $f_n = 1$ on $Z_{n+1} \setminus G_n$, $f_n = 0$ on $Z_{n-1} \cup (X \setminus G_{n+2})$ and $0 \leq f_n \leq 1$. Let $g_1 = \sum_n nf_{3n-2}$, $g_2 = \sum_n nf_{3n-1}$ and $g_3 = \sum_n nf_{3n}$. Since $\{nf_{3n-2}\}$ $\{nf_{3n-1}\}$ and $\{nf_{3n}\}$ are disjoint sequences of A, it follows from (1) of assumption (B) that $g_i \in A$ for each $i = 1, 2, 3$. It then is not difficult to verify that each g_i is continuous with respect to the topology (X, τ_A). Let $g = g_1 \vee g_2 \vee g_3$. Since $g \in A$ and $g(p) = 0$, there is a neighborhood W of p such that $g(x) < 1$ for all $x \in W$. Since $Z(f^\beta) \cap X$ is not neighborhood of p, it follows that $(X \setminus Z_n) \cap W \setminus Z(f^\beta) \neq \emptyset$ for all n. Hence for every $x \in (X \setminus Z_{3n-2}) \cap W \setminus Z(f^\beta)$, $g(x) = g_1(x) \vee g_2(x) \vee g_3(x) \geq n$. But this contradicts the fact that $g(x) < 1$ on W. Hence X is P-space. (2) of (C) is immediate from (2) of (B).

(C) → (D): We first show that pointwise monotone convergence in A is equivalent to monotone order convergence in A. It is clear that pointwise monotone convergence implies order convergence. Let $\{u_n\}$ be a decreasing sequence $u_n \downarrow$ in A with $\bigwedge_n u_n = 0$ in A, and suppose there is an $x_0 \in X$ and $\varepsilon > 0$ such that $u_n(x_0) > \varepsilon$ for all n. By Theorem 1.1, (2) there is an $f_n \in a$ such that $0 \leq f_n \leq u_n$ and $f_n(x_0) > \varepsilon$ for all n. Let $Z_n = \{x \in X\colon f_n(x) \geq \varepsilon\}$ and $Z = \bigcap_n Z_n$. By (1) of assumption (C),

Z is an open zero set containing x_0, and $Z \in Z(A)$. Let h be a function in A_b such that $h(x_0) = \varepsilon/2$, $h = 0$ on $X \setminus Z$ and $0 \leq h \leq \varepsilon/2$ on X. Then it follows that $u_n \geq f_n \geq h \neq 0$ for all n. But this is a contradiction, and hence $\{u_n\}$ converges pointwise to 0. From this, if $f_n \in A$ and $f_n \uparrow f \in A$ in order convergence, then it follows that f is the pointwise limit of $\{f_n\}$. Let $\{f_n\}$ be an uniformly bounded sequence in A. Then by (2) of assumption (C) and the argument above it is simple to verify that $\bigvee_n f_n = \sup_n f_n(x) = f \in A$. Thus A is bounded σ-complete.

(D) → (E): Since A is uniformly closed, it is clear that $Z_\delta(a) = Z(A)$. Let $f \in A$ and χ_f be the characteristic function of $CZ(f)$. Since $(n|f|) \wedge 1 \leq 1$ for all n, it follows from assumption (D) that $\chi_f = \sup (n|f|) \wedge 1 = \bigvee_n (n|f|) \wedge 1 \in A$ and also $1 - \chi_f \in A$. Hence $CZ(A) = Z(A)$ and $Z(A)$ is σ-algebra. Let $g \in LS(A)$ and $\{f_n\}$ be a increasing sequence from A converging to g. Then, for every real number γ, $\{x \in X\colon g(x) > \gamma\} = \bigcup_n \{x \in X\colon f_n(x) > \gamma\} \in Z(A)$. Also, if $h \in US(A)$ and $\{f_n\}$ is a decreasing sequence from A converging to h, then

$$\{x \in X\colon h(x) \geq \gamma\} = \bigcap_n \{x \in X\colon f_n(x) \geq \gamma\} \in Z(A).$$

This means that $LS(A) \cup US(A) \subset A$, and hence $B_1(a) = B_1(A) = LS(US(A)) \cap \cap US(LS(A)) \subset A$. Thus it follows that $A = B_1(A) = B_1(a)$.

(E) → (A): This is obvious, and the proof is complete.

As a corollary of Theorem 4, we have;

Corollary 5. *If* A *satisfies* D*-condition, then* A *is order Cauchy complete.*

Proof. It is immediate from the argument given in (D) and (E) of Th. 4.

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Polyhedral-shape concordance implying homeomorphic complements

by

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Abstract. If two compacta X and Y satisfy the inessential loops condition in the interior of a piecewise linear m-manifold M, $m \geqslant 6$, and are shape concordant in M, and if X has the shape of a compact k-polyhedron, $k \leqslant m-3$, then $M-X$ is homeomorphic to $M-Y$.

A compact subset X of a manifold N satisfies the *inessential loops condition* (abbreviated ILC) if for each neighborhood U of X, there is a neighborhood V of X such that each loop in $V-X$ which is null homotopic in V is null homotopic in $U-X$. Throughout the paper, $I = [0, 1]$.

Two compacta X_0 and X_1 (satisfying ILC) in the interior of a manifold M is said to be (ILC) *shape concordant* if there is a compactum Z (satisfying ILC) in $M \times I$ such that $X_i \times \{i\} = Z \cap (M \times \{i\}) \hookrightarrow Z$ is a shape equivalence for each $i = 0, 1$. Similarly, if X_0, X_1 and Z are polyhedra in the corresponding PL manifolds, we can define the notion of *polyhedral concordance*.

Sher has proved in [S] that "*If X and Y are* ILC *compacta in a* PL *manifold* M^m ($\partial M = \emptyset$ and $m \geqslant 6$) *and* ILC *shape concordant in M by Z, and if X has the shape of a k-polyhedron* ($k \leqslant m-3$), *then* $M-X \cong M-Y$." In this note, we will show that if X and Y satisfy ILC in M and are shape concordant, then $M-X$ is still homeomorphic to $M-Y$ without assuming that Z satisfies ILC in $M \times I$ (Theorem 3.4).

For standard notions and notations in piecewise linear (abbreviated PL) topology as: simplicial collapse, PL homeomorphism, ambient isotopy, singular set $S(f)$ of a PL map f, derived neighborhood, boundary ∂Q of a PL manifold Q, $\mathrm{Fr}_Q N$, $\mathrm{Int}_Q N, \ldots$, we refer to [Hd]. Given an open subset W of ∂Q, by an *open collar* of W in Q, we mean the image of a PL open embedding $h\colon W \times [0, 1) \to Q$ such that $h(x, 0) = x$ for all $x \in W$. If Q is a PL manifold, $\dot{Q}$ denotes $Q - \partial Q$.

We will suppress the base points from our notations of homotopy groups. If $(X, A) \subset (Y, B)$, the homomorphism $\pi_q(X, A) \to \pi_q(Y. B)$ will be the inclusion-induced one if it is not specified otherwise.

We assume that the reader is familiar with the fundamentals of shape [B] and ANR-systems [M–S].