Chapman's category isomorphism for arbitrary ARs

by

P. Mrozik (Siegen)

Abstract. In [7] T. A. Chapman proved that there exists a category isomorphism from the full subcategory of the weak proper homotopy category whose objects are complements of Z-sets in the Hilbert cube Q onto the full subcategory of Borsuk's shape category whose objects are Z-sets in Q. We extend Chapman's result from Q to arbitrary ARs, and moreover establish analogous category isomorphism theorems in homotopy theory. Beyond these existence theorems, we specify unique "natural" category isomorphisms and describe their characteristic properties and interrelations.

Introduction. The main concern of this paper is to present versions of Chapman's category isomorphism in which the role of the Hilbert cube, i.e. the role of the *ambient space*, is played by arbitrary ARs, and to characterize these category isomorphisms by certain natural properties. We now give an outline of our results.

Let W be an AR, i.e. an absolute retract for metrizable spaces, which is additionally equipped with a *complete uniform structure* (such always exist, but need not be induced by a metric); we write W^* for the resulting uniform space. Let $\mathrm{C}_{wh}(W^*)$ denote the category whose objects are uniform complements of compact Z-sets in W and whose morphisms are weak complete homotopy classes of complete maps (see § 1 for all definitions), and let $\mathrm{Sh}(W)$ denote the full subcategory of the shape category whose objects are compact Z-sets in W. We prove

THEOREM I (cf. § 2, Theorem 2.3). *There exists a category isomorphism* $T_{wh}\colon \mathrm{C}_{wh}(W^*) \to \mathrm{Sh}(W)$ *such that* $T_{wh}(M) = W-M$ *for each object* M.

We also prove a strong shape analogue of Theorem I (cf. § 2, Theorem 2.3). For compact W, the category $\mathrm{C}_{wh}(W^*)$ can be identified with the full subcategory of the weak proper homotopy category whose objects are topological complements of Z-sets in W; however, it should be emphasized that for *non-compact* W no reasonable results are available in the topological setting. Consider for example $s = \prod_{n=1}^{\infty} \boldsymbol{R}_n$, where each $\boldsymbol{R}_n$ denotes a copy of the real line. It is well known [1] that, for *any* compact $X \subset s$, $s-X$ is homeomorphic to s; thus no category whose objects are topological spaces having the form $s-X$ can be used in a non-trivial category isomorphism.

Theorem I and its strong shape analogue have further extensions — as an ambient space one can take an arbitrary absolute retract for paracompact p-spaces (see [2], [16] concerning the class of absolute retracts for paracompact p-spaces; note that it contains all absolute retracts for (a) metrizable spaces, (b) compact spaces).

In general, there exist *different* category isomorphisms $\theta\colon C_{wh}(W^*) \to \mathrm{Sh}(W)$ such that $\theta(M) = W-M$ for each object M (in fact, this always happens if W contains a Z-set having more than one point). We wish to exhibit a unique category isomorphism $T_{wh}\colon C_{wh}(W^*) \to \mathrm{Sh}(W)$ as "the natural" one. The following analogue of Theorem I in homotopy theory is crucial for this purpose.

Let $\mathrm{UC}_h(W^*)$ denote the category whose objects are uniform complements of compact Z-sets in W and whose morphisms are t-uniformly continuous complete homotopy classes of t-uniformly continuous complete maps (see § 1 for definitions), and let $\mathrm{Top}_h(W)$ denote the full subcategory of the homotopy category of topological spaces Top_h whose objects are compact Z-sets in W.

THEOREM II (cf. § 2, Theorem 2.2). *There exists a unique category isomorphism $R_h\colon \mathrm{UC}_h(W^*) \to \mathrm{Top}_h(W)$ satisfying*

(R1) *$R_h(M) = W-M$ for each object M.*

(R2) *For each morphism $\psi\colon M \to N$ in $\mathrm{UC}_h(W^*)$, there exist maps $f\colon M \to N$ resp. $f'\colon W-M \to W-N$ representing the homotopy classes ψ resp. $R_h(\psi)$ such that f and f' can be pieced together to a continuous $f^*\colon W \to W$.*

Now let $G\colon \mathrm{UC}_h(W^*) \to C_{wh}(W^*)$ be the obvious functor, and let $S\colon \mathrm{Top}_h(W) \to \mathrm{Sh}(W)$ be the shape functor. The particular category isomorphism T_{wh} constructed in the proof of Theorem I has the property $T_{wh} \circ G = S \circ R_h$ (cf. § 2, Theorem 2.3); moreover, in many interesting cases (e.g. $W = I^n$, Q, s) T_{wh} is the *only* functor having this property (cf. § 2, Theorem 2.5). In the general case we obtain a unique characterization of T_{wh} by another property involving the functors R_h (cf. § 2, Theorem 2.8). Roughly speaking, it is the natural behavior of T_{wh}^{-1} on the *map-induced shape morphisms* which enforces a natural behavior on arbitrary shape morphisms.

Details and further results can be found in § 2.

The material in this paper is part of the author's doctoral dissertation written under the supervision of Professor F. W. Bauer at the University of Frankfurt am Main.

1. Preliminaries. Let X be a topological space. A subset $A \subset X$ is said to be *unstable in* X (cf. [19]), if there is a map $H\colon X\times I \to X$ such that $H_0 = 1_X$ and $H_t(X) \subset X-A$ for all $t \in (0,1]$ (for each $t \in I = [0,1]$, $H_t\colon X \to X$ is given by $H_t(x) = H(x,t)$). An *unstable zero-set* $A \subset X$ will be called a *Z-set in* X (recall that A is a *zero-set in* X, if there is a map $u\colon X \to I$ with $A = u^{-1}(0)$). Note that

in standard examples, e.g. in finite or infinite dimensional manifolds, the above definition of a Z-set is equivalent to the definitions given in [1] or [8].

1.1. DEFINITION. A map $f\colon X \to Y$ between uniform spaces X, Y is called *complete*, if $f^{-1}(M) \subset X$ is a complete uniform subspace for every complete uniform subspace $M \subset Y$.

Note that a map $f\colon X \to Y$ between totally bounded uniform spaces is complete iff it is proper (i.e. preimages of compact subsets are compact).

For any uniform space X, let $t(X)$ denote the *family of closed totally bounded uniform subspaces of* X.

1.2. DEFINITION. A map $f\colon X \to Y$ between uniform spaces X, Y is called *t-uniformly continuous*, if the following conditions are satisfied:

(a) For every $A \in t(X)$ there exists $A' \in t(Y)$ such that $f(A) \subset A'$.

(b) For every $A \in t(X)$ and every $B \in t(Y)$ such that $f(A) \subset B$ the map $f_{A,B}\colon A \to B$, $f_{A,B}(x) = f(x)$, is uniformly continuous.

It is easy to see that $f\colon X \to Y$ is t-uniformly continuous iff $f|_A\colon A \to Y$ is uniformly continuous for every $A \in t(X)$; thus if X is totally bounded, then $f\colon X \to Y$ is t-uniformly continuous iff it is uniformly continuous.

We now consider the following categories:

Top, the category whose objects are topological spaces and whose morphisms are continuous functions (= maps);

P, the category whose objects are topological spaces and whose morphisms are proper maps;

UP, the category whose objects are uniform spaces and whose morphisms are uniformly continuous proper maps;

C, the category whose objects are uniform spaces and whose morphism are complete maps;

UC, the category whose objects are uniform spaces and whose morphisms are t-uniformly continuous complete maps.

In each of these categories $\mathfrak{C}$ we have the notions of *homotopy* and *weak homotopy*.

A morphism $H\colon X\times I \to Y$ in $\mathfrak{C}$ is called a *$\mathfrak{C}$-homotopy*. Morphisms $f_0, f_1\colon X \to Y$ in $\mathfrak{C}$ are called *$\mathfrak{C}$-homotopic*, $f_0 \simeq_{\mathfrak{C}} f_1$, if there is a $\mathfrak{C}$-homotopy $H\colon X\times I \to Y$ such that $H_i = f_i$, $i = 0, 1$ (here $H_i\colon X \to Y$ is given by $H_i(x) = H(x, i)$).

Morphisms $f_0, f_1\colon X \to Y$ in $\mathfrak{C}$ are called *weakly $\mathfrak{C}$-homotopic*, $f_0 \simeq_{w\mathfrak{C}} f_1$, if the following condition is satisfied:

($\mathfrak{C}$ = Top) For every ANR (= absolute neighbourhood retract for metrizable spaces) P and every map $r\colon Y \to P$, the maps rf_0 and rf_1 are homotopic;

($\mathfrak{C} = \mathrm{P}$ resp. UP) For every compact $M \subset Y$ there is a continuous resp. uniformly continuous map $H\colon X \times I \to Y$ such that $H^{-1}(M)$ is compact and $H_i = f_i$, $i = 0, 1$;

($\mathfrak{C} = \mathrm{C}$ resp. UC) For every complete $M \subset Y$ there is a continuous resp. t-uniformly continuous map $H\colon X \times I \to Y$ such that $H^{-1}(M)$ is complete and $H_i = f_i$, $i = 0, 1$.

One easily verifies

1.3. (a) The notions of homotopy and weak homotopy in $\mathfrak{C}$ induce equivalence relations on the set $\mathfrak{C}(X, Y)$ of morphisms $f\colon X \to Y$: The $\mathfrak{C}$-homotopy class of $f \in \mathfrak{C}(X, Y)$ is denoted by $[f]_{\mathfrak{C}}$, the weak $\mathfrak{C}$-homotopy class by $[f]_{w\mathfrak{C}}$.

(b) Let $f_0, f_1 \in \mathfrak{C}(X, Y)$ and $g_0, g_1 \in \mathfrak{C}(Y, Z)$: If $f_0 \simeq_{\mathfrak{C}} f_1$ and $g_0 \simeq_{\mathfrak{C}} g_1$, then $g_0 f_0 \simeq_{\mathfrak{C}} g_1 f_1$; if $f_0 \simeq_{w\mathfrak{C}} f_1$ and $g_0 \simeq_{w\mathfrak{C}} g_1$, then $g_0 f_0 \simeq_{w\mathfrak{C}} g_1 f_1$.

Thus in the usual fashion we obtain from $\mathfrak{C}$ a *homotopy category* $\mathfrak{C}_h$ and a *weak homotopy category* $\mathfrak{C}_{wh}$ together with a functor $\Pi\colon \mathfrak{C}_h \to \mathfrak{C}_{wh}$, assigning to the $\mathfrak{C}$-homotopy class $[f]_{\mathfrak{C}}$ the weak $\mathfrak{C}$-homotopy class $[f]_{w\mathfrak{C}}$.

Note that if $\mathfrak{C} = \mathrm{Top}$, then the homotopy category $\mathfrak{C}_h = \mathrm{Top}_h$ is precisely the ordinary homotopy category of topological spaces.

There are also obvious forgetful functors $F_h\colon \mathrm{UP}_h \to \mathrm{P}_h$ and $F_{wh}\colon \mathrm{UP}_{wh} \to \mathrm{P}_{wh}$ (resp. $F_h\colon \mathrm{UC}_h \to \mathrm{C}_h$ and $F_{wh}\colon \mathrm{UC}_{wh} \to \mathfrak{C}_{wh}$) which simply forget uniform continuity (resp. t-uniform continuity).

2. The category isomorphism theorems. Let W be an absolute retract for paracompact p-spaces and W^* be a *complete uniformization of* W (i.e. a complete uniform space W^* such that (a) the sets W^* and W coincide, (b) the topology induced by the uniform structure on W^* coincides with the topology on W). Note that every paracompact topological space *has* a complete uniformization (cf. [11], 8.5.13).

Now let be

(a) $\mathrm{Top}_h(W)$ resp. $\mathrm{Top}_{wh}(W)$ the full subcategories of Top_h resp. Top_{wh} whose objects are compact Z-sets in W;

(b) $\mathrm{C}_h(W^*)$ resp. $\mathrm{C}_{wh}(W^*)$ the full subcategories of C_h resp. C_{wh} whose objects are uniform subspaces $(W-X)^* \subset W^*$, where X is a compact Z-set in W;

(c) $\mathrm{UC}_h(W^*)$ resp. $\mathrm{UC}_{wh}(W^*)$ the full subcategories of UC_h resp. UC_{wh} whose objects are uniform subspaces $(W-X)^* \subset W^*$, where X is a compact Z-set in W;

(d) $\mathrm{sSh}(W)$ resp. $\mathrm{Sh}(W)$ the full subcategories of the *strong shape category* sSh ([3], [5], [6], [9], [14]) resp. the *shape category* Sh ([4], [15]) whose objects are compact Z-sets in W.

Remark. In some of the references given in (d) shape is defined only for compacta. However, once the notion of shape has been introduced for compacta, it extends uniquely to all spaces having the homotopy type of a compactum. The following result shows that this is sufficient for our purpose.

2.1. Proposition. *If W is an absolute retract for paracompact p-spaces, then every compact zero-set in W has the homotopy type of a compactum.*

For the proof see § 3.

Together with the shape categories we have the shape functors $sS\colon \mathrm{Top}_h(W) \to \mathrm{sSh}(W)$ and $S\colon \mathrm{Top}_h(W) \to \mathrm{Sh}(W)$, the functor $\pi\colon \mathrm{sSh}(W) \to \mathrm{Sh}(W)$ satisfying $\pi \circ sS = S$, and the functor $S'\colon \mathrm{Top}_{wh}(W) \to \mathrm{Sh}(W)$ determined by $S' \circ \Pi = S$, where $\Pi\colon \mathrm{Top}_h(W) \to \mathrm{Top}_{wh}(W)$.

The following theorems and propositions are proved in § 3.

2.2. Theorem. *There exist unique functors $R_h\colon \mathrm{UC}_h(W^*) \to \mathrm{Top}_h(W)$ resp. $R_{wh}\colon \mathrm{UC}_{wh}(W^*) \to \mathrm{Top}_{wh}(W)$ satisfying the following conditions:*

(R1) *$R_h(M) = W-M$ (resp. $R_{wh}(M) = W-M$) for each object M.*

(R2) *For each $f \in \mathrm{UC}((W-X)^*, (W-Y)^*)$, the homotopy class $R_h([f]_{\mathrm{UC}}) \in \mathrm{Top}_h(X, Y)$ (resp. the weak homotopy class $R_{wh}([f]_{w\mathrm{UC}}) \in \mathrm{Top}_{wh}(X, Y)$) contains a representative $f'\colon X \to Y$ such that f and f' can be pieced together to a continuous $f^* = f \cup f'\colon W = (W-X) \cup X \to (W-Y) \cup Y = W$. The functors R_h resp. R_{wh} are category isomorphisms which render commutative the diagram*

$$\begin{array}{ccc} \mathrm{UC}_h(W^*) & \xrightarrow{R_h} & \mathrm{Top}_h(W) \\ \downarrow{\scriptstyle \Pi} & & \downarrow{\scriptstyle \Pi} \\ \mathrm{UC}_{wh}(W^*) & \xrightarrow{R_{wh}} & \mathrm{Top}_{wh}(W) \end{array}$$

2.3. Theorem. *There exist category isomorphisms $T_h\colon \mathrm{C}_h(W^*) \to \mathrm{sSh}(W)$ resp. $T_{wh}\colon \mathrm{C}_{wh}(W^*) \to \mathrm{Sh}(W)$ satisfying the following conditions:*

(T1) *$T_h(M) = W-M$ (resp. $T_{wh}(M) = W-M$) for each object M.*

(T2) *The diagram*

$$\begin{array}{ccc} \mathrm{UC}_h(W^*) & \xrightarrow{R_h} & \mathrm{Top}_h(W) \\ \downarrow{\scriptstyle F_h} & & \downarrow{\scriptstyle sS} \\ \mathrm{C}_h(W^*) & \xrightarrow{T_h} & \mathrm{sSh}(W) \end{array} \quad \textit{resp.} \quad \begin{array}{ccc} \mathrm{UC}_{wh}(W^*) & \xrightarrow{R_{wh}} & \mathrm{Top}_{wh}(W) \\ \downarrow{\scriptstyle F_{wh}} & & \downarrow{\scriptstyle S'} \\ \mathrm{C}_{wh}(W^*) & \xrightarrow{T_{wh}} & \mathrm{Sh}(W) \end{array}$$

commutes.

Remark. The condition $T_{wh} \circ F_{wh} = S' \circ R_{wh}$ is equivalent to the condition $T_{wh} \circ G = S \circ R_h$, where $G = F_{wh} \circ \Pi\colon \mathrm{UC}_h(W^*) \to \mathrm{C}_{wh}(W^*)$; this follows from the fact that the functor $\Pi\colon \mathrm{UC}_h(W^*) \to \mathrm{UC}_{wh}(W^*)$ is epimorphic.

In general conditions (T1) and (T2) do not determine unique category isomorphisms T_h resp. T_{wh}. This follows from

2.4. Proposition. *There exist a separable* AR, W, *and nontrivial category isomorphisms $A\colon \mathrm{sSh}(W) \to \mathrm{sSh}(W)$ resp. $B\colon \mathrm{Sh}(W) \to \mathrm{Sh}(W)$ such that $A \circ sS = sS$ resp. $B \circ S = S$.*

However, there are interesting cases in which (T1) and (T2) guarantee uniqueness.

Let Pol_h denote the full subcategory of Top_h whose objects are all spaces having the homotopy type of a polyhedron. Following [15], a Pol_h-*expansion* of a topological space X is a morphism $p\colon X\to \underline{X}$ in the *pro-category* pro-Top_h (cf. [10], [15]), where $\underline{X}$ is an inverse system in Pol_h, with the following universal property: For any inverse system $\underline{Y}$ in Pol_h and any morphism $\underline{h}\colon X\to \underline{Y}$ in pro-Top_h, there exists a unique morphism $\underline{f}\colon \underline{X}\to\underline{Y}$ in pro-Top_h such that $\underline{h}=\underline{f}\circ \underline{p}$. We say W is *large*, if every compact Z-set X in W has a Pol_h-expansion $\underline{p}\colon X\to\underline{X} = \{X_\alpha, p_{\alpha\beta}, L\}$ such that each X_α is homotopy dominated by a compact Z-set in W. Examples for large W are $W=I^n$, Q, s (this can be derived from [15] § 5.3 Corollary 6).

2.5. THEOREM. *Let W be large. Then there exists a unique functor* $T_{wh}\colon \mathrm{C}_{wh}(W^*)\to \mathrm{Sh}(W)$ *satisfying* (T1) *and* (T2); T_{wh} *is a category isomorphism.*

Let us call W *universal*, if each compactum X is homotopy dominated by a compact Z-set X' in W. It is readily verified that each universal W is large, but not conversely (consider $W=I^n$). Examples for universal W are $W=Q$, s.

2.6. THEOREM. *Let W be universal. Then there exists a unique functor* $T_h\colon \mathrm{C}_h(W^*)\to \mathrm{sSh}(W)$ *satisfying* (T1) *and* (T2); T_h *is a category isomorphism.*

It is not known to the author whether the conclusion of 2.6 is valid for more general W, e.g. $W=I^n$.

The reason that T_h resp. T_{wh} are not always uniquely characterized by (T1) and (T2) is that in general there are not enough compact Z-sets in W. To obtain a general uniqueness result we proceed as follows.

Let $\mathfrak{C}$ be one of the categories UC_h, UC_{wh}, C_h, C_{wh}, Top_h, Top_{wh}, sSh, Sh. For each space A, let $p_A\colon A\times Q\to A$ denote the projection map; p_A induces a morphism $p_A(\mathfrak{C})\in\mathfrak{C}(A\times Q, A)$ (e.g. $p_A(\mathrm{C}_h)=[p_A]_{\mathrm{C}}$). It is easy to show that $p_A(\mathfrak{C})$ is an isomorphism in $\mathfrak{C}$.

2.7. PROPOSITION. *There exist unique functors* $i(\mathfrak{C})\colon \mathfrak{C}(W^*)\to\mathfrak{C}(W^*\times Q)$ *resp.* $i(\mathfrak{C})\colon \mathfrak{C}(W)\to\mathfrak{C}(W\times Q)$ *satisfying the following conditions*:

(E1) $i(\mathfrak{C})(A)=A\times Q$ *for each object* A

(E2) *For each morphism* $\psi\colon A\to B$, *the diagram*

$$\begin{array}{ccc} A\times Q & \xrightarrow{\;i(\mathfrak{C})(\psi)\;} & B\times Q\\ {\scriptstyle p_A(\mathfrak{C})}\downarrow & & \downarrow{\scriptstyle p_B(\mathfrak{C})}\\ A & \xrightarrow{\;\psi\;} & B\end{array}$$

commutes.

The functors $i(\mathfrak{C})$ are full embeddings.

2.8. THEOREM. *There exist unique functors* $T_h\colon \mathrm{C}_h(W^*)\to \mathrm{sSh}(W)$ *resp.* $T_{wh}\colon \mathrm{C}_{wh}(W^*)\to \mathrm{Sh}(W)$ *satisfying* (T1) *and*

(T3) *There is an extension* $\theta_h\colon \mathrm{C}_h(W^*\times Q)\to \mathrm{sSh}(W\times Q)$ *of* T_h (*resp.* $\theta_{wh}\colon \mathrm{C}_{wh}(W^*\times Q)\to \mathrm{Sh}(W\times Q)$ *of* T_{wh}) *such that the diagram*

$$\begin{array}{ccc} \mathrm{C}_h(W^*) & \xrightarrow{\;T_h\;} & \mathrm{sSh}(W)\\ {\scriptstyle i(\mathrm{C}_h)}\downarrow & & \downarrow{\scriptstyle i(\mathrm{sSh})}\\ \mathrm{C}_h(W^*\times Q) & \xrightarrow{\;\theta_h\;} & \mathrm{sSh}(W\times Q)\\ {\scriptstyle F_h}\uparrow & & \uparrow{\scriptstyle S}\\ \mathrm{UC}_h(W^*\times Q) & \xrightarrow{\;R_h\;} & \mathrm{Top}_h(W\times Q)\end{array} \quad \textit{resp.} \quad \begin{array}{ccc} \mathrm{C}_{wh}(W^*) & \xrightarrow{\;T_{wh}\;} & \mathrm{Sh}(W)\\ {\scriptstyle i(\mathrm{C}_{wh})}\downarrow & & \downarrow{\scriptstyle i(\mathrm{Sh})}\\ \mathrm{C}_{wh}(W^*\times Q) & \xrightarrow{\;\theta_{wh}\;} & \mathrm{Sh}(W\times Q)\\ {\scriptstyle F_{wh}}\uparrow & & \uparrow{\scriptstyle S'}\\ \mathrm{UC}_{wh}(W^*\times Q) & \xrightarrow{\;R_{wh}\;} & \mathrm{Top}_{wh}(W\times Q)\end{array}$$

commutes.

The functors T_h resp. T_{wh} are category isomorphisms which satisfy (T2) and render commutative the diagram

$$\begin{array}{ccc} \mathrm{C}_h(W^*) & \xrightarrow{\;T_h\;} & \mathrm{sSh}(W)\\ {\scriptstyle \Pi}\downarrow & & \downarrow{\scriptstyle \pi}\\ \mathrm{C}_{wh}(W^*) & \xrightarrow{\;T_{wh}\;} & \mathrm{Sh}(W).\end{array}$$

If the ambient space W is a *compact* absolute retract for paracompact p-spaces, i.e. an absolute retract for compact spaces, we can state our results in a slight different form. Note first that for compact W there exists *exactly one* uniformization W^* of W (which is complete); thus we do not distinguish between W and W^*, but regard W at the same time as a topological and a uniform space. Let be

e) $\mathrm{P}_h(W)$ resp. $\mathrm{P}_{wh}(W)$ the full subcategories of P_h resp. P_{wh} whose objects are topological subspaces $W-X\subset W$, where X is a Z-set in W;

f) $\mathrm{UP}_h(W)$ resp. $\mathrm{UP}_{wh}(W)$ the full subcategories of UP_h resp. UP_{wh} whose objects are uniform subspaces $W-X\subset W$, where X is a Z-set in W.

2.9. PROPOSITION. *If W is compact, then the following pairs of categories can be identified*: $\mathrm{UC}_h(W^*)$ *and* $\mathrm{UP}_h(W)$; $\mathrm{UC}_{wh}(W^*)$ *and* $\mathrm{UP}_{wh}(W)$; $\mathrm{C}_h(W^*)$ *and* $\mathrm{P}_h(W)$; $\mathrm{C}_{wh}(W^*)$ *and* $\mathrm{P}_{wh}(W)$.

This follows immediately from the definitions in § 1, because all uniform subspaces of W are totally bounded.

Let us now consider some ambient spaces which are of particular interest.

For $W=Q$ we obtain the four category isomorphisms $T_h\colon \mathrm{P}_h(Q)\to\mathrm{sSh}(Q)$, $T_{wh}\colon \mathrm{P}_{wh}(Q)\to \mathrm{Sh}(Q)$, $R_h\colon \mathrm{UP}_h(Q)\to \mathrm{Top}_h(W)$ and $R_{wh}\colon \mathrm{UP}_{wh}(Q)\to\mathrm{Top}_{wh}(Q)$. Note that the *existence* of category isomorphisms $\tau\colon \mathrm{P}_h(Q)\to \mathrm{sSh}(Q)$ resp. $\theta\colon \mathrm{P}_{wh}(Q)\to\mathrm{Sh}(Q)$ is well-known (see [10], [14] resp. [7]). However, if we want to characterize *among all category isomorphisms τ resp. θ* a "reasonable" T_h resp. T_{wh}, the functors R_h resp. R_{wh} get involved. Moreover, the characterizing property (T2) allows to describe the relation between the strong shape category and the homotopy category from a new point of view — the category $\mathrm{P}_h(Q)$ arises from $\mathrm{UP}_h(Q)$ simply *by forgetting the uniform structure.*

For the cubes I^n we obtain *explicit finite-dimensional versions* of the category isomorphisms already known for the Hilbert cube (including uniqueness results; it should be mentioned, however, that the mere existence of category isomorphisms τ: $P_h(I^n) \to sSh(I^n)$ resp. θ: $P_{wh}(I^n) \to Sh(I^n)$ follows also from [14] resp. [13]). Additionally we obtain *finite-dimensional versions* for $W = \boldsymbol{R}^n_+ = \{(x_1, \ldots, x_n) \in \boldsymbol{R}^n | x_n \geqslant 0\}$; but observe that for *Euclidean space* $\boldsymbol{R}^n$ our category isomorphisms do *not* produce interesting results, because the empty set is the only Z-set in $\boldsymbol{R}^n$.

For any *infinite-dimensional metrizable linear space* W we obtain *new infinite-dimensional versions* of the category isomorphisms. Particular nice examples are $W =$ *an infinite-dimensional Banach-space*, or $W = s$, the infinite product of lines $\boldsymbol{R}$ (in these cases W has a *natural* complete uniformization W^*). These examples show also that in general we cannot avoid to consider *uniform structures* on the ambient spaces; see the introduction for $W = s$.

3. Proofs. (I) Proof of 2.1

3.1. Lemma. *Let W be an absolute retract for paracompact p-spaces and X be a compact zero-set in W. Then X is homotopy dominated by a compactum Y.*

Proof. It follows from [17] that W can be embedded as a closed subset of a suitable product $C \times I^{\mathfrak{m}}$, where C is a convex subset of a normed linear space and $I^{\mathfrak{m}}$ is a Tychonoff cube of some power $\mathfrak{m}$ (for any cardinal number $\mathfrak{m}$, $I^{\mathfrak{m}}$ is represented by the product $\prod_{\alpha \in A} I_\alpha$, where A is a set of cardinality $\mathfrak{m}$ and each I_α is a copy of I). There is a retraction r: $C \times I^{\mathfrak{m}} \to W$. If p: $C \times I^{\mathfrak{m}} \to C$ denotes the projection map, then $K = p(X) \subset C$ is a compactum. If the restriction of r to $K \times I^{\mathfrak{m}}$ is denoted by R: $K \times I^{\mathfrak{m}} \to W$, then $R^{-1}(X)$ is compact. Obviously X is homotopy dominated by $R^{-1}(X)$. Choose u: $W \to I$ such that $u^{-1}(0) = X$. Then $R^{-1}(X) = (uR)^{-1}(0)$. The map uR: $K \times I^{\mathfrak{m}} \to I$ depends only on countably many coordinates (cf. [11] 2.7.12(c)); thus $(uR)^{-1}(0) = Y \times I^{\mathfrak{n}}$, where Y is a compactum and $\mathfrak{n}$ is a cardinal number not exceeding $\mathfrak{m}$.

Proposition 2.1 follows from 3.1 and a result from [12] stating that every space homotopy dominated by a compactum is homotopy equivalent to a compactum.

(II) Proof of 2.2 and 2.3

Throughout this section let "cl" denote closure.

3.2. Lemma. *Let X be a complete uniform k-space, $A \subset X$ be a Z-set, Y be a complete uniform space, $B \subset Y$ be closed and $V \subset Y$ be an open neighbourhood of B.*

(a) *Every t-uniformly continuous map f: $X - A \to Y - B$ has a unique continuous extension f^+: $X \to Y$. If $f^{-1}(Y - V) \subset X - A$ is complete, then $f^+(A) \subset \mathrm{cl}(V)$; conversely, if $f^+(A) \subset V$, then $f^{-1}(Y - V) \subset X - A$ is complete. Thus f is a complete map iff $f^+(A) \subset B$.*

(b) *If F: $X \to Y$ is continuous, then $F|_{X-A}$: $X - A \to Y$ is t-uniformly continuous.*

Proof. (a) The proof that f has a unique continuous extension f^+ is straightforward (using the facts that X is a k-space and, because A is unstable in X, that each compact $K \subset X$ lies in the closure (rel. X) of some $M \in t(X - A)$). If $f^{-1}(Y - V)$ is complete, then $f^*(V) = X - f^{-1}(Y - V)$ is an open neighbourhood of A in X with $f(f^*(V) - A) \subset V$. Because A is a Z-set in X, $A \subset \mathrm{cl}(f^*(V) - A)$. We obtain

$$f^+(A) \subset f^+\big(\mathrm{cl}(f^*(V) - A)\big) \subset \mathrm{cl}\big(f^+(f^*(V) - A)\big) = \mathrm{cl}\big(f(f^*(V) - A)\big) \subset \mathrm{cl}(V).$$

Conversely, if $f^+(A) \subset V$, then $f^{-1}(Y - V) = (f^+)^{-1}(Y - V)$ is closed in X and a fortiori complete. If f is a complete map, then $f^+(A) \subset \mathrm{cl}(U)$ for each open neighbourhood U of B; it follows $f^+(A) \subset B$; the converse is obvious.

(b) This is straightforward.

3.3. Lemma. *Let W be an absolute retract for paracompact p-spaces and $Y \subset W$ be closed. Two maps f_0, f_1: $X \to Y$ are weakly* Top-*homotopic* (cf. § 1) *iff for every open neighbourhood V of Y in W there exists a homotopy H: $X \times I \to V$ such that $H_i = f_i$, $i = 0, 1$.*

Proof. We only have to observe the following two facts (see [16]):

(a) Every ANR is an absolute neighbourhood retract for paracompact p-spaces;

(b) For every open neighbourhood U of any closed $A \subset W$ there exists an open $U' \subset W$, $A \subset U' \subset U$, which is homotopy equivalent to an ANR.

We also need the following result from [19] (stated in a slightly generalized form).

3.4. Lemma (cf. [19], Lemma 2.1). *Let W be an absolute retract for paracompact p-spaces and X be an unstable subset of W. Then every map f: $B_0 \to W$ defined on a zero-set B_0 of a paracompact p-space B has a continuous extension F: $B \to W$ such that $F(B - B_0) \subset W - X$.*

Let $\mathfrak{F}$ be one of the categories UC, UC_h, UC_{wh}, C_h, C_{wh}. We define a category $\mathfrak{F}^*$ as follows. The objects of $\mathfrak{F}^*$ are all triples (W^*, W, X), where W is an absolute retract for paracompact p-spaces, W^* is a complete uniformization of W and X is a compact Z-set in W. The set of morphisms is defined by

$$\mathfrak{F}^*\big((W_1^*, W_1, X_1), (W_2^*, W_2, X_2)\big) = \mathfrak{F}\big((W_1 - X_1)^*, (W_2 - X_2)^*\big).$$

Obviously there exist canonical full embeddings $\mathfrak{F}(W^*) \subset \mathfrak{F}^*$.

We shall first construct a functor R: $\mathrm{UC}^* \to \mathrm{Top}$.

For each object (W^*, W, X) we define $R(W^*, W, X) = X$. Each morphism $f \in \mathrm{UC}^*\big((W_1^*, W_1, X), (W_2^*, W_2, X_2)\big)$ has a unique continuous extension f^+: $W_1 \to W_2$ such that $f^+(X_1) \subset X_2$ (see 3.2). We define a map $R(f)$: $X_1 \to X_2$ by $R(f)(x) = f^+(x)$. It follows from the uniqueness of the extension that we have defined a *functor* R: $\mathrm{UC}^* \to \mathrm{Top}$.

In the following claims let $f_0, f_1 \in \mathrm{UC}^*\big((W_1^*, W_1, X_1), (W_2^*, W_2, X_2)\big)$. Note that claims (3), (4), (5) are immediate consequences of Lemma 3.4 (tacitly making use also of Lemmas 3.2 and 3.3).

(1) *If $f_0 \simeq_{\mathrm{UC}} f_1$, then $R(f_0) \simeq R(f_1)$*: There is a t-uniformly continuous complete homotopy F: $(W_1 - X_1) \times I \to W_2 - X_2$ such that $F_i = f_i$, $i = 0, 1$. By 3.2, F has a unique continuous extension F^+: $W_1 \times I \to W_2$ such that $F^+(X_1 \times I) \subset X_2$

(observe that each paracompact p-space is a k-space, cf. [2]). This induces a homotopy between $R(f_0)$ and $R(f_1)$.

(2) *If* $f_0 \simeq_{\mathrm{wUC}} f_1$, *then* $R(f_0) \simeq_{\mathrm{wTop}} R(f_1)$: Consider an open neighbourhood V of X_2 in W_2 and choose an open neighbourhood U of X_2 in W_2 such that $\mathrm{cl}(U) \subset V$. Then $W_2 - U$ is complete and there exists a t-uniformly continuous homotopy $F\colon (W_1 - X_1) \times I \to W_2 - X_2$ such that $F^{-1}(W_2 - U)$ is complete and $F_i = f_i$, $i = 0, 1$. By 3.2, the unique continuous extension $F^+\colon W_1 \times I \to W_2$ of F has the property $F^+(X_1 \times I) \subset \mathrm{cl}(U) \subset V$. From 3.3 we deduce $R(f_0) \simeq_{\mathrm{wTop}} R(f_1)$.

(3) *If* $R(f_0) \simeq R(f_1)$, *then* $f_0 \simeq_{\mathrm{UC}} f_1$;

(4) *If* $R(f_0) \simeq_{\mathrm{wTop}} R(f_1)$, *then* $f_0 \simeq_{\mathrm{wUC}} f_1$;

(5) *R is a full functor.*

For the proofs of (3) and (4) observe that $B_0 = W_1 \times \{0, 1\} \cup X_1 \times I$ is a zero-set in $B = W_1 \times I$.

It follows from the above claim that R induces *fully faithful functors* $R_h\colon \mathrm{UC}_h^* \to \mathrm{Top}_h$ *and* $R_{wh}\colon \mathrm{UC}_{wh}^* \to \mathrm{Top}_{wh}$.

As an obvious corollary we obtain Theorem 2.2. Note that uniqueness is a consequence of (R2).

We now prove Theorem 2.3. Let us first consider $W = Q$. It is well-known that the strong shape morphisms $\phi\colon X \to Y$ in $\mathrm{sSh}(Q)$ can be described as approaching homotopy classes (cf. [5], [6], [14], [18]): If X, Y are compacta in Q, then an approaching map f from X to Y is a map $f\colon Q \times (0, 1] \to Q$ with the following property: For every open neighbourhood V of Y in Q there exist an open neighbourhood U of X in Q and a number $t \in (0, 1]$ such that $f(U \times (0, t]) \subset V$. An approaching homotopy H from $X \times I$ to Y is a map $H\colon Q \times I \times (0, 1] \to Q$ with an analogous property (cf. [14]). The approaching homotopy class of an approaching map f is denoted by $[f]_A$. If $g\colon X \to Y$ is a continuous map, we choose an extension $g'\colon Q \times I \to Q$ of g (where $X = X \times \{0\} \subset Q \times I$) and put $g'' = g'|_{Q \times (0,1]}$. Then g'' is an approaching map from X to Y and $[g'']_A$ is a well defined approaching homotopy class depending only on $[g]$; the shape functor $sS\colon \mathrm{Top}_h(Q) \to \mathrm{sSh}(Q)$ is now given by $sS([g]) = [g'']_A$.

Using our notation, we restate Theorem 1 from [14] as follows: *There exists a category isomorphism* $\theta\colon \mathrm{P}_h(Q) \to \mathrm{sSh}(Q)$ *such that* $\theta(M) = Q - M$ *for each object* M. Moreover, if the construction of θ is studied *carefully*, it turns out that for any morphism $[f]_p \in \mathrm{P}_h(Q)(Q - X, Q - Y)$, $\theta([f]_p)$ can be constructed in the following way: Choose a map $H\colon Q \times I \to Q$ such that $H_0 = 1$ and $H_t(Q) \subset Q - X$ for $t \neq 0$, and define $\theta(f, H)\colon Q \times (0, 1] \to Q$ by $\theta(f, H)(x, t) = fH(x, t)$. Then $\theta(f, H)$ is an approaching map from X to Y, and we have $\theta([f]_p) = [\theta(f, H)]_A$. Since Q is compact all uniform subspaces of Q are totally bounded; thus we can identify $\mathrm{P}_h(Q)$ and $\mathrm{C}_h(Q)$ (see § 1). We now write $T_h\colon \mathrm{C}_h(Q) \to \mathrm{sSh}(Q)$ for the category isomorphism constructed in [14]. Using the explicit construction of T_h described above, it is an easy exercise to verify $sS \circ R_h = T_h \circ F_h$.

The shape morphisms $\psi\colon X \to Y$ in $\mathrm{Sh}(Q)$ can be described as fundamental

equivalence classes in the sense of Borsuk [4]. The canonical functor $\pi\colon \mathrm{sSh}(Q) \to \mathrm{Sh}(Q)$ is now given as follows: For any approaching map $f\colon Q \times (0, 1] \to Q$ from X to Y we obtain a fundamental sequence $\{\psi_n\}$ if we define $\psi_n\colon Q \to Q$ by $\psi_n(x) = f(x, 1/n)$, and we have $\pi([f]_A) = \langle\{\psi_n\}\rangle$, where $\langle\ \rangle$ denotes fundamental equivalence class. It is now an easy exercise to show that the category isomorphism $\Omega\colon \mathrm{P}_{wh}(Q) \to \mathrm{Sh}(Q)$ constructed by Chapman [7] has the property $\pi \circ \theta = \Omega \circ \Pi$ where $\Pi\colon \mathrm{P}_h(Q) \to \mathrm{P}_{wh}(Q)$. We identity $\mathrm{P}_{wh}(Q)$ and $\mathrm{C}_{wh}(Q)$ and write $T_{wh}\colon \mathrm{C}_{wh}(Q) \to \mathrm{Sh}(Q)$. Thus $\pi \circ T_h = T_{wh} \circ \Pi$, where $\Pi\colon \mathrm{C}_h(Q) \to \mathrm{C}_{wh}(Q)$. This implies $S \circ R_h = T_{wh} \circ F_{wh} \circ \Pi$, where $\Pi\colon \mathrm{UC}_h(Q) \to \mathrm{UC}_{wh}(Q)$.

Thus 2.3 is proved for $W = Q$. We shall now regard the functors T_h resp. T_{wh} as full embeddings $T_h\colon \mathrm{C}_h(Q) \to \mathrm{sSh}$ resp. $T_{wh}\colon \mathrm{C}_{wh}(Q) \to \mathrm{Sh}$. Recall that for $\mathfrak{F} = \mathrm{UC}_h, \mathrm{UC}_{wh}, \mathrm{C}_h, \mathrm{C}_{wh}$ we can identify $\mathfrak{F}(W^*)$ with a full subcategory of $\mathfrak{F}^*$. If (W^*, W, X) is an arbitrary object of UC_h^*, then by 2.1 X is homotopy equivalent to a compactum X'. We may assume that X' is a Z-set in Q. Since $R_h\colon \mathrm{UC}_h^* \to \mathrm{Top}_h$ is fully faithful, (W^*, W, X) is isomorphic in UC_h^* to (Q, Q, X'). Choose an isomorphism $j(X)\colon (W^*, W, X) \to (Q, Q, X')$ in UC_h^*. We now extend $T_h\colon \mathrm{C}_h(Q) \to \mathrm{sSh}$ to a functor $T_h\colon \mathrm{C}_h^* \to \mathrm{sSh}$. Define $T_h(W^*, W, X) = X$ for each object (W^*, W, X) of C_h^*. For any

$$\psi \in \mathrm{C}_h^*((W_1^*, W_1, X_1), (W_2^*, W_2, X_2))$$

define

$$T_h(\psi) := sSR_h\big(j(X_2)^{-1}\big)T_h(\psi^*)sSR_h\big(j(X_1)\big),$$

where $\psi^* = F_h\big(j(X_2)\big)\psi F_h\big(j(X_1)^{-1}\big)$ is a morphism in $\mathrm{C}_h(Q)$. It is easy to check that $T_h\colon \mathrm{C}_h^* \to \mathrm{sSh}$ is a well-defined fully faithful functor. Obviously, the restriction to $\mathrm{C}_h(W^*)$ determines a category isomorphism $T_h\colon \mathrm{C}_h(W^*) \to \mathrm{sSh}(W)$ satisfying (T1) and (T2). By the same argument we obtain a category isomorphism $T_{wh}\colon \mathrm{C}_{wh}(W^*) \to \mathrm{Sh}(W)$ satisfying (T1) and (T2).

(III) Proof of 2.4. We first define two compacta $K, L \in \boldsymbol{R}^2$. For $a, b \in \boldsymbol{R}^2$ let $[a, b]$ denote the segment connecting a and b. We shall need points $a_0 = (-1; 1)$, $a_1 = (-1; -2)$, $b_0 = (1; 1)$, $b_1 = (1; -2)$ in $\boldsymbol{R}^2$. We define maps

$$u\colon [-1, 0) \cup (0, 1] \to \boldsymbol{R} \quad \text{resp.} \quad V\colon (\to 1, 1) \to \boldsymbol{R}$$

by $u(x) = \sin(1/x)$ resp. $v(x) = \sin(1/1 - |x|)$. Let G resp. H denote the closure (in $\boldsymbol{R}^2$) of the graph of u resp. v. Put $A = [a_0, a_1] \cup [a_1, b_1] \cup [b_0, b_1]$. Then $K = G \cup A$ and $L = H \cup A$ are compacta of dimension 1. The following properties are readily verified.

(1) K and L have the shape of a circle (in fact, there are maps $r\colon K \to S^1$ and $s\colon L \to S^1$ which are strong shape equivalences).

(2) Any map $f\colon K \to L$ resp. $g\colon L \to K$ induces the trivial shape morphism.

(3) Each connected proper subcompactum $X \subset K$ resp. $Y \subset L$ has trivial shape.

(4) For each proper subcompactum $X \subset K$ resp. $Y \subset L$ there exists a subcompactum $X' \subset K$ resp. $Y' \subset L$ having trivial shape such that $X \subset X'$ resp. $Y \subset Y'$.

We embed the topological sum $K+L$ in the two dimensional sphere S^2. If $\mathring{D}^3$ denotes the interior of the three dimensional ball, then $W = \mathring{D}^3 \cup (K+L)$ is a separable AR (W is a convex subset of R^3). $K+L$ is a compact Z-set in W; moreover, a compact subset $X \subset W$ is a Z-set iff $X \subset K+L$. We shall construct a category isomorphism A: $sSh(W) \to sSh(W)$ such that $A \neq 1$ and $A \circ sS = sS$ (the construction of an analogous B: $\mathrm{Sh}(W) \to \mathrm{Sh}(W)$ is similar). For any compact Z-set X in W we choose an equivalence $\Phi_X \in \mathrm{sSh}(X, X)$ as follows:

If K is not contained in X, $\Phi_X = 1_X$; if $K \subset X$, i.e. $X = K+X'$ with $X' \subset L$, $\Phi_X = \Lambda_K + 1_{X'}$, where $\Lambda_K \in \mathrm{sSh}(K, K)$ is the non-trivial shape equivalence determined by the condition $sS([r]) \circ \Lambda_K = sS([-1]) \circ sS([r])$ (here -1: $S^1 \to S^1$ is a map of degree -1). For the above definition one should observe that there exist *finite sums* in the category sSh. Now define $A(X) = X$ for each object X of $\mathrm{sSh}(W)$ and $A(\Psi) = \Phi_Y \Psi \Phi_X^{-1}$ for each morphism $\Psi \in \mathrm{sSh}(X, Y)$. A: $\mathrm{sSh}(W) \to \mathrm{sSh}(W)$ is a well-defined category isomorphism such that $A \neq 1$ (for $\Psi = sS([r])^{-1} \circ sS([s]) \in \mathrm{sSh}(L, K)$, $A(\Psi) \neq \Psi$). Consider $\varphi \in \mathrm{Top}_h(W)(X, Y)$ and represent φ by a map f: $X \to Y$. We have $X = X_1 + X_2$, $Y = Y_1 + Y_2$, where $X_1, Y_1 \subset K$ and $X_2, Y_2 \subset L$. Put $X_{ij} = X_i \cap f^{-1}(Y_j)$. Then $X = X_{11} + X_{12} + X_{21} + X_{22}$, and f induces maps f_{ij}: $X_{ij} \to Y_j$. Using (1)–(4) it is a routine business to verify $A \circ sS([f_{ij}]) = sS([f_{ij}])$. Since the functors A and $sS([\cdot])$ commute with sums, we obtain $A \circ sS([f]) = sS([f])$.

(IV) Proof of 2.5. Let T_{wh}: $\mathrm{C}_{wh}(W^*) \to \mathrm{Sh}(W)$ be a category isomorphism satisfying (T1) and (T2). If F: $\mathrm{C}_{wh}(W^*) \to \mathrm{Sh}(W)$ is another functor satisfying (T1) and (T2), then $G = F \circ T_{wh}^{-1}$: $\mathrm{Sh}(W) \to \mathrm{Sh}(W)$ is a functor satisfying $G \circ S = S$. It is sufficient to show that $G = 1$. Consider compact Z-sets X, Y in W and an arbitrary $\Phi \in \mathrm{Sh}(X, Y)$. Since W is large, Y has a Pol_h-expansion p: $Y \to \underline{Y} = \{Y_\alpha, p_{\alpha\beta}, L\}$ such that each Y_α is homotopy dominated by a compact Z-set K_α in W. Choose $u_\alpha \in \mathrm{Top}_h(Y_\alpha, K_\alpha)$ and $d_\alpha \in \mathrm{Top}_h(K_\alpha, Y_\alpha)$ such that $d_\alpha u_\alpha = 1$. The morphism p consists of a family of homotopy classes $p_\alpha \in \mathrm{Top}_h(Y, Y_\alpha)$, $\alpha \in L$. For any $\alpha \in L$, there exists $f \in \mathrm{Top}_h(X, Y_\alpha)$ such that $S(p_\alpha)\Phi = S(f)$; hence $S(u_\alpha p_\alpha)\Phi = S(u_\alpha f)$. Applying G we obtain

$$S(u_\alpha p_\alpha) G(\Phi) = GS(u_\alpha p_\alpha) G(\Phi) = GS(u_\alpha f) = S(u_\alpha f) = S(u_\alpha p_\alpha)\Phi.$$

Thus $S(p_\alpha) G(\Phi) = S(d_\alpha) S(u_\alpha p_\alpha) G(\Phi) = S(d_\alpha) S(u_\alpha p_\alpha)\Phi = S(p_\alpha)\Phi$. We now have proved $S(p) \circ G(\Phi) = S(p) \circ \Phi$; it follows from the continuity theorem (cf. [15], § 2.3 Theorem 6) that $G(\Phi) = \Phi$. Thus $G = 1$.

(V) Proof of 2.6. It is again sufficient to show that any functor G: $\mathrm{sSh}(W) \to \mathrm{sSh}(W)$ satisfying $G \circ sS = sS$ is the identity. Consider compact Z-sets X, Y in W and an arbitrary $\Phi \in \mathrm{sSh}(X, Y)$. By 2.1, there exist compacta K resp. L homotopy equivalent to X resp. Y. We choose homotopy equivalences u: $X \to K$ and v: $Y \to L$. There is $\Psi \in sSh(K, L)$ such that $sS([v]) \circ \Phi = \Psi \circ sS([u])$. It is well-known [5], [6] that there exist a compactum M and maps f: $K \to M$, i: $L \to M$ such that i is a strong shape equivalence and $sS([i]) \circ \Psi = sS([f])$. Since W is universal, M is homotopy dominated by a compact Z-set Z in W. Choose maps r: $M \to Z$ and s: $Z \to M$ such that $[sr] = 1$. We obtain $sS([riv])\Phi = sS([rfu])$. Application of G gives $sS([riv]) G(\Phi) = GsS([riv]) G(\Phi) = GsS([rfu]) = sS([rfu]) = sS([riv])\Phi$. Since $sS([riv])$ has a left inverse in sSh, $G(\Phi) = \Phi$. Thus $G = 1$.

(VI) Proof of 2.8. We shall show that $W \times Q$ is universal; then 2.8 follows from 2.5, 2.6 and 2.7. Since W is a paracompact p-space, there is a proper map f: $W \to M$ onto a metrizable space M. Thus there exists a compact zero-set $K \subset W$ (e.g. $K = f^{-1}(x)$ for an arbitrary $x \in M$). If X is a compactum, we choose a copy X'' of X which is embedded as a Z-set in Q and define $X' = K \times X''$. X' is a compact Z-set in $W \times Q$ which homotopy dominates X. Thus $W \times Q$ is universal.

References

[1] R. D. Anderson, *Topological properties of the Hilbert cube and the infinite product of open intervals*, Trans. Amer. Math. Soc. 126 (1967), pp. 200–216.

[2] A. V. Arhangel'skiĭ, *On a class of spaces containing all metric and all locally bicompact spaces*, Soviet Math. Dokl. 4 (1963), pp. 1051–1055.

[3] F. W. Bauer, *Some relations between shape constructions*, Cahiers Topol. Geom. Diff. 19 (1978), pp. 337–367.

[4] K. Borsuk, *Theory of shape*, Monografie Matematyczne Tom 59, Warszawa 1975.

[5] A. Calder and H. M. Hasting, *Realizing strong shape equivalences*, J. Pure Appl. Algebra 20 (1981), pp. 129–156.

[6] F. W. Cathey, *Strong shape theory*, Shape Theory and Geom. Top. Proc. (Dubrovnik, 1981), Lecture Notes in Math. 870, Springer, Berlin 1981, pp. 215–238.

[7] T. A. Chapman, *On some applications of infinite-dimensional manifolds to the theory of shape*, Fund. Math. 76 (1972), pp. 181–193.

[8] — *Lectures on Hilbert cube manifolds*, CBMS Regional Conference Series in Mathematics No. 28, AMS Providence 1976.

[9] J. Dydak and J. Segal, *Strong shape theory*, Dissertationes Math. 192 (1982).

[10] D. A. Edwards and H. M. Hastings, *Čech and Steenrod homotopy theories with applications to geometric topology*, Lecture Notes in Math. 542, Springer, New York 1976.

[11] R. Engelking, *General topology*, Monografie Matematyczne Tom 60, Warszawa 1977.

[12] S. Ferry, *Homotopy, simple homotopy and compacta*, Topology 19 (1980), pp. 101–110.

[13] Y. Kodama, J. Ono and T. Watanabe, AR *associated with* ANR-*sequence and shape*, General Topol. Appl. 9 (1978), pp. 71–88.

[14] Y. Kodama and J. Ono, *On fine shape theory*, Fund. Math. 105 (1979), pp. 29–39.

[15] S. Mardešić and J. Segal, *Shape theory*, North-Holland Amsterdam 1982.

[16] S. Mardešić and A. P. Šostak, *On the homotopy type of* ANR*'s for p-paracompacta*, Bull. Acad. Polon. Sci. Sér. Sci. Math. 27 (1979), pp. 803–808.

[17] J. Nagata, *A note on M-spaces and topologically complete spaces*, Proc. Japan Acad. 45 (1969), pp. 541–543.

[18] J. B. Quigley, *An exact sequence from the n-th to the $(n-1)$st fundamental group*, Fund. Math. 77 (1973), pp. 195–210.
[19] R. B. Sher, *Property* SUV$^\infty$ *and proper shape theory*, Trans. Amer. Math. Soc. 190 (1974), pp. 345–356.

UNIVERSITÄT-GESAMTHOCHSCHULE SIEGEN
FACHBEREICH MATHEMATIK
Hölderlinstrasse 3
5900 Siegen
RFN

Received 8 November 1983;
in revised form 4 June 1984

On the Baire order problem for a linear lattice of functions

by

Takeshi Ohno (Shizuoka)

Abstract. Let a be a linear lattice of real valued functions containing the constant functions and $B_1(a)$ be the first Baire class of functions generated by a. Denote by A the smallest complete ordinary function system containing a. Then it follows immediately that $a \subset A \subset B_1(a)$ [3]. Here we show that (1) the condition (*) given by Mauldin in ([3], Th. 4.1) is a necessary and sufficient condition for $B_1(a) = B_1(B_1(a))$, and (2) $A = B_1(a)$ iff A satisfies D-condition.

1. Introduction. Let X be a nonempty set and R^X be the set of all functions from X into the set R of real numbers, forming the lattice ordered R-algebra structure under operations defined pointwise. Let $H \subset R^X$. Then $B_1(H)$ (the first Baire class of H) is the family of all functions in R^X which are pointwise limits on X of sequences from H, $B_2(H) = B_1(B_1(H))$ and in general if $\alpha > 0$ is an ordinal then $B_\alpha(H)$ is the family of pointwise limits of sequences from $\bigcup_{\beta<\alpha} B_\beta(H)$. If ω_1 is the first uncountable ordinal then $B(H) = B_{\omega_1}(H) = B_{\omega_1+1}(H)$, and $B(H)$ is called the Baire class generated by H. H_u denotes the family of all functions in R^X which are uniform limits on X of sequences from H, $LS(H)$ (resp. $US(H)$) the family of all $f \in R^X$ which are pointwise limits of increasing (resp. decreasing) sequences from H and H_b the subset of H consisting of bounded functions.

A subspace H of R^X is called an *ordinary function system* if it is both a linear lattice and algebra which contains the constant functions, and which is closed under inversion (if $f \in H$ and $f > 0$, then $1/f \in H$). An ordinary function system H is called *complete* if it is also closed under uniform limits. If H is a linear lattice containing the constants, then $B_1(H)$ is a complete ordinary function system (See [3]). In [3] Mauldin proved the following.

THEOREM 1.1. *Let $a \subset R^X$ be a linear lattice containing the constants and A be the smallest complete ordinary function system containing a. Then the following hold:*

(1) $a \subset a_u \subset (LS(a) \cap US(a)) \subset A \subset B_1(a) = B_1(A)$.

(2) $(a_u)_b \subset A_b = (LS(a) \cap US(a))_b$.

For a discussion of Baire functions see Mauldin [2] and [3].

Let $H \subset R^X$ be a linear lattice containing the constants, and Let H_b^* denote the dual space of linear space H_b with the norm $\|f\| = \sup_{x \in X} |f(x)|$. Let $\varphi \in H_b^*$. Then