On Carathéodory type selections

by

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Abstract. Roughly speaking, certain "random" analogues of the theorem on paracompactness of metric spaces and Michael's theorem on continuous selections are given for Polish spaces. The proofs resort to the results on measurability of multivalued mappings due to Himmelberg [8].

Let T be a measurable space, X be a Polish space, Y be a Banach space and F be a multivalued mapping from $T\times X$ into Y with closed convex values. We are interested in the existence of Carathéodory's selections for F, i.e., single-valued mappings f from $T\times X$ into Y such that $f(t,x)\in F(t,x)$ for $(t,x)\in T\times X$, $f(\cdot,x)$ is measurable for $x\in X$ and $f(t,\cdot)$ is continuous for $t\in T$. There are several papers containing results of that sort ([3], [4], [5], [7], [9], [12]). In those papers in general, either the measurable structure of T is generated by a topology ([3], [4], [5]) or X is assumed to be locally compact ([7], [9]). In [12] assumptions concerning T and X are most general; however, certain special conditions are imposed upon F.

In this paper we show that the original proof of Michael's theorem on continuous selection can be modified so as work with Carathéodory type selection. This is possible owing to a generalization of Dieudonné's result on paracompactness of second-countable metrizable spaces (Theorem 1). I am indebted to the referee for the suggestion how to simplify the proof of this theorem.

1. Preliminaries. Throughout the paper T denotes a measurable space with a σ-algebra $\mathscr{A}$. T is called complete if there is a complete σ-finite measure defined on $\mathscr{A}$. X denotes a Polish space and ϱ is a complete metric for X. By $\mathscr{B}(X)$ we denote the σ-algebra of Borel subsets of X, by $\mathscr{A}\times\mathscr{B}(X)$ the product σ-algebra on $T\times X$. Let 2^X be the family of all subsets of X. A relation $F\subset T\times X$ is denoted by $F\colon T\to 2^X$ and is called a multivalued mapping. We write

$$\mathrm{Gr}(F)=\{(t,x)\in T\times X\colon\ x\in F(t)\},$$
$$F^{-1}(B)=\{t\in T\colon\ F(t)\cap B\neq\emptyset\},\quad \text{for}\quad B\subset X.$$

F has measurable graph if $\mathrm{Gr}(F)\in\mathscr{A}\times\mathscr{B}(X)$. F is $\mathscr{B}$-measurable (resp. measurable, weakly measurable) if $F^{-1}(B)\in\mathscr{A}$ for each Borel (resp. closed, open) subset of X, see [8]. For a multivalued mapping $F\colon T\times X\to 2^Y$, where Y is a topological space, are defined with respect to $\mathscr{A}\times\mathscr{B}(X)$ various kinds of measurability. $G\colon X \to 2^Y$ is lower semicontinuous if the set $G^{-1}(C)$ is open for every open subset C of Y.

Let $\bar{R}$ denote the extended positive half-line $[0, +\infty]$ metrized by

$$\varrho_0(a, b) = \left|\frac{a}{1+a} - \frac{b}{1+b}\right|, \quad \left(\text{putting } \frac{a}{1+a} = 1 \text{ for } a = +\infty\right).$$

LEMMA 1. *Let $F\colon T \to 2^X$ be a multivalued mapping. Consider the statements:*

(a) *F has measurable graph,*

(b) *F is $\mathscr{B}$-measurable,*

(c) *F is measurable,*

(d) *F is weakly measurable,*

(e) *$\bar{F}$ defined by $\bar{F}(t) = \overline{F(t)}$ is weakly measurable,*

(f) *the function $f\colon T\times X \to \bar{R}$ defined by*

$$f(t, x) = d(x, F(t)) = \begin{cases} \inf\{\varrho(x, y)\colon y \in F(t)\} & \text{if} \quad F(t) \neq \emptyset, \\ +\infty & \text{if} \quad F(t) = \emptyset, \end{cases}$$

is measurable in t for each fixed $x \in X$,

($\bar{a}$) *$\bar{F}$ has measurable graph.*

Then:

(i) (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Leftrightarrow$ (e) $\Leftrightarrow$ (f) $\Rightarrow$ ($\bar{a}$),

(ii) *if T is complete then also* (a) $\Rightarrow$ (b).

For the proof see [8]: (i) (b) $\Rightarrow$ (c) $\Rightarrow$ (d) is given by Proposition 2.1, (d) $\Leftrightarrow$ (e) by Proposition 2.6, (d) $\Leftrightarrow$ (*f*) $\Rightarrow$ ($\bar{a}$) by Theorem 3.3, (ii) is given by Theorem 3.4.

LEMMA 2. *Let $F, G\colon T \to 2^X$ be multivalued mappings with measurable graphs. Then:*

(i) *$F'\colon T \to 2^X$ defined by $F'(t) = X\setminus F(t)$ has measurable graph;*

(ii) *$H\colon T \to 2^X$ defined by $H(t) = F(t) \cap G(t)$ has measurable graph.*

This is a consequence of the equalities: $\mathrm{Gr}(F') = (T\times X)\setminus \mathrm{Gr}(F)$, $\mathrm{Gr}(H) = \mathrm{Gr}(F) \cap \mathrm{Gr}(G)$.

LEMMA 3 ([8], Proposition 2.3). *Let J be a set, finite or countable, and let $F_n\colon T \to 2^X$ be a multivalued mapping for each $n \in J$. Then, if each F_n is measurable, so is the multivalued mapping $\bigcup_n F_n\colon T \mapsto 2^X$ defined by $(\bigcup_n F_n)(t) = \bigcup_n F_n(t)$.*

LEMMA 4 ([10], Proposition 3). *Let Y be a metric space, $f\colon T\times X \to 2^X$ be a mapping measurable in t and continuous in x, V be an open subset of Y. Let $G\colon T \to 2^X$ be a weakly measurable mapping with nonempty closed values. Then the multivalued mapping $F\colon T \to 2^X$ defined by $F(t) = \{x \in G(t)\colon f(t, x) \in V\}$ is weakly measurable.*

2. Random partition of unity.

THEOREM 1. *Let T be a complete measurable space, X be a Polish space. Given a family of multivalued mappings $U_i\colon T \to 2^X$, $i \in N$, such that, for each i, U_i has measurable graph and $\{U_i(t)\}_{i\in N}$ is an open cover of X for every $t \in T$. Then there exists a family of multivalued mappings with measurable graphs $V_i^m\colon T \to 2^X$, $i, m \in N$, $i \leqslant m$, such that $\{V_i^m(t)\}_{i\leqslant m}$ is a locally finite open cover of X for every $t \in T$ and $V_i^m(t) \subset U_i(t)$ for $i \leqslant m$, $t \in T$.*

Proof. We shall adopt the proof of Lemma 5.2.4 from [6] (p. 394). Define $f_i\colon T\times X \to [0, 1]$ by

$$f_i(t, x) = \frac{d(x, X\setminus U_i(t))}{1+d(x, X\setminus U_i(t))} \quad \text{iff} \quad U_i(t) \neq X,$$

$$f_i(t, x) = 1 \quad \text{iff} \quad U_i(t) = X.$$

Since for each i the multivalued mapping U_i' has measurable graph, $f_i(\cdot, x)$ is measurable for each $i \in N$, $x \in X$. Clearly $f_i(t, \cdot)$ is continuous for $t \in T$, $i \in N$. Letting $f(t, x) = \sum_{i=1}^{\infty} (\tfrac{1}{2})^i f_i(t, x)$ we define a function $f\colon T\times X \to [0, 1]$ measurable in t and continuous in x. Define $V^m, W^m, V_i^m\colon T \to 2^X$ by

$$V^m(t) = \{x \in X\colon f(t, x) > 1/m\}, \qquad W^m(t) = \{x \in X\colon f(t, x) \geqslant 1/m\},$$

$$V_i^m(t) = U_i(t) \cap (V^{m+1}(t)\setminus W^{m-1}(t)), \qquad \text{where } m = 1, 2, \ldots,$$

$$1 \leqslant i \leqslant m, \ W^0(t) = \emptyset \text{ for } t \in T.$$

For every $t \in T$ the family $\{V_i^m(t)\}_{i\leqslant m}$ is a star-finite open cover of X, and so it is a locally finite open cover of X, (see [6], p. 394). Clearly $V_i^m(t) \subset U_i(t)$.

Since f is measurable in both variables jointly (see e.g. [8], Theorem 6.1), we see that $\mathrm{Gr}(V^m) = f^{-1}(1/m, 1]$ and $\mathrm{Gr}(W^m) = f^{-1}[1/m, 1]$ belong to $\mathscr{A}\times\mathscr{B}(X)$. Since $\mathrm{Gr}(V_i^m) = \mathrm{Gr}(U_i) \cap \mathrm{Gr}(V^{m+1}) \cap \mathrm{Gr}(W^{m-1})'$, each V_i^m has measurable graph.

Remark 1. Under the assumptions of Theorem 1 one can also prove the existence of a family of multivalued mappings satisfying the same requirements but with closed values.

COROLLARY 1. *Let T, X and a family $\{U_i\}_{i\in N}$ be as in Theorem 1. Then there exists a family of functions $p_i^m\colon T\times X \to [0, 1]$, $i, m \in N$, $i \leqslant m$, such that:*

(i) *$p_i^m(\cdot, x)$ is measurable for $x \in X$, $i, m \in N$;*

(ii) *the family $\{p_i^m(t, \cdot)\}_{i\leqslant m}$ forms a locally finite partition of unity on X for every $t \in T$ and $\{x \in X\colon p_i^m(t, x) > 0\} \subset U_i(t)$ for $i \leqslant m$.*

Proof. Let a family $\{V_i^m\}_{i,m\in N, i\leqslant m}$ be chosen for $\{U_i\}_{i\in N}$ according to the assertion of Theorem 1. Letting

$$f_i^m(t, x) = \frac{d(x, X\setminus V_i^m(t))}{1+d(x, X\setminus V_i^m(t))} \quad \text{iff} \quad V_i^m(t) \neq X,$$

$$f_i^m(t, x) = 1 \quad \text{iff} \quad V_i^m(t) = X$$

and

$$p_i^m(t,x) = \frac{f_i^m(t,x)}{\sum_{l=1}^{\infty} \sum_{k \leqslant l} f_k^l(t,x)},$$

we define a family of functions satisfying (i) and (ii).

3. A selection theorem. Now we state a selection theorem based on Michael's result ([11], Theorem 3.2″, see also [2], Theorem 7.1).

LEMMA 5. *Let T be complete, let $(Y, |\cdot|)$ be a separable Banach space and $\mathrm{CCl}(Y)$ be the family of all nonempty closed convex subsets of Y. Suppose that a multivalued mapping $F\colon T \times X \to \mathrm{CCl}(Y)$ is weakly measurable (with respect to $\mathscr{A} \times \mathscr{B}(X)$) and lower semicontinuous in x for each $t \in T$. Then for any $\varepsilon > 0$ there exists a mapping $f_\varepsilon\colon T \times X \to Y$ such that:*

(i) *$f_\varepsilon(t, \cdot)$ is continuous for each $t \in T$, the set*

$$G_\varepsilon(t,x) = \{y \in F(t,x)\colon |y - f_\varepsilon(t,x)| < \varepsilon\}$$

is nonempty for $(t,x) \in T \times X$ and $\bar{G}_\varepsilon\colon T \times X \to \mathrm{CCl}(Y)$ defined by $\bar{G}_\varepsilon(t,x) = \overline{G_\varepsilon(t,x)}$ is lower semicontinuous in x;

(ii) *f_ε is measurable and $\bar{G}_\varepsilon$ is weakly measurable.*

Proof. Let $\{y_i\}_{i \in N}$ be a dense subset of Y. Let us put $B_i = \{y \in Y\colon |y - y_i| < \varepsilon\}$ and define $U_i\colon T \to 2^X$ by $U_i(t) = \{x \in X\colon F(t,x) \cap B_i \neq \emptyset\}$.

Since $\{B_i\}_{i \in N}$ covers Y and $F(t, \cdot)$ is lower semicontinuous for each t, it follows that $\{U_i(t)\}_{i \in N}$ is an open cover of X for each t. Since

$$\{(t,x)\colon x \in U_i(t)\} = \{(t,x)\colon F(t,x) \cap B_i \neq \emptyset\}$$

and F is weakly measurable, each U_i has measurable graph. By Corollary 1 there exists a family of functions $p_i^m\colon T \times X \to [0,1]$, $i, m \in N$, $i \leqslant m$, which are measurable in t and continuous in x and such that $\{p_i^m(t, \cdot)\}_{i \leqslant m}$ is a locally finite partition of unity satisfying

$$\{x \in X\colon p_i^m(t,x) > 0\} \subset U_i(t) \quad \text{for} \quad i \leqslant m,\ t \in T.$$

Letting $f_\varepsilon(t,x) = \sum_{m=1}^{\infty} \sum_{i \leqslant m} p_i^m(t,x) y_i$ we define a mapping $f_\varepsilon\colon T \times X \to Y$. By the proof of Michael's Theorem (see [11], Lemma 4.1, Proposition 2.5, Proposition 2.3) f_ε and $\bar{G}_\varepsilon$ satisfies (i).

Since $f_\varepsilon(t,x) = \lim_{n \to \infty} S_n(t,x)$, where $S_n(t,x) = \sum_{m=1}^{n} \sum_{i \leqslant m} p_i^m(t,x) y_i$, and $S_n\colon T \times X \to Y$ is measurable for every n, f_ε is measurable. It remains to show that $\bar{G}_\varepsilon$ is weakly measurable. Let $g\colon (T \times X) \times Y \to R$ be defined by $g(t,x,y) = |y - f_\varepsilon(t,x)|$. It is easy to check that $g(t,x,\cdot)$ is continuous for $(t,x) \in T \times X$ and $g(\cdot,\cdot,y)$ is measurable for $y \in Y$. We have $G_\varepsilon(t,x) = \{y \in F(t,x)\colon g(t,x,y) < \varepsilon\}$; thus by Lemma 4 G_ε is weakly measurable. Hence $\bar{G}_\varepsilon$ is weakly measurable (Lemma 1).

THEOREM 2. *Let $(T, \mathscr{A})$ be a complete measurable space, X be a Polish space, $(Y, |\cdot|)$ be a separable Banach space, $\mathrm{CCl}(Y)$ be the family of all nonempty closed convex subsets of Y. Suppose that a multivalued mapping $F\colon T \times X \to \mathrm{CCl}(Y)$ is weakly measurable and $F(t, \cdot)\colon X \to \mathrm{CCl}(Y)$ is lower semicontinuous for each fixed $t \in T$. Then there exists a mapping $f\colon T \times X \to Y$ such that:*

(i) *$f(t, \cdot)\colon X \to Y$ is continuous for each fixed $t \in T$;*

(ii) *f is measurable;*

(iii) *$f(t,x) \in F(t,x)$ for $(t,x) \in T \times X$.*

Proof. We will follow the proof of Theorem 3.2″ (A) from [11]. Using Lemma 5 we define inductively a sequence (f_n) of mappings from $T \times X$ into Y, which are continuous in x and measurable and such that

(1) $$|f_n(t,x) - f_{n-1}(t,x)| \leqslant 2/2^{n-1}, \quad n = 2, 3, \ldots, (t,x) \in T \times X,$$

(2) $$d(f_n(t,x), F(t,x)) < 1/2^n, \quad n = 1, 2, \ldots, (t,x) \in T \times X.$$

Applying Lemma 5 for $\varepsilon = \frac{1}{2}$ we get a mapping f_1 satisfying (2). Suppose that we have $f_1, f_2, \ldots, f_k$ satisfying (1) and (2). Define

$$G_{k+1}(t,x) = \{y \in F(t,x)\colon |y - f_k(t,x)| < 1/2^k\}$$

and

$$\bar{G}_{k+1}(t,x) = \overline{G_{k+1}(t,x)}.$$

Clearly $\bar{G}_{k+1}(t,x) \subset F(t,x)$. By the inductive assumption (2), $G_{k+1}(t,x) \neq \emptyset$ for $(t,x) \in T \times X$, so $\bar{G}_{k+1}$ is a multivalued mapping from $T \times X$ into $\mathrm{CCl}(Y)$. By Lemma 5, $\bar{G}_{k+1}$ is lower semicontinuous in x and weakly measurable. Again by Lemma 5 for $\varepsilon = 1/2^{k+1}$ there exists a mapping $f_{k+1}\colon T \times X \to Y$ continuous in x and measurable and such that

$$\{y \in \bar{G}_{k+1}(t,x)\colon |y - f_{k+1}(t,x)| < 1/2^{k+1}\} \neq \emptyset.$$

Thus

$$d(f_{k+1}(t,x), F(t,x)) < 1/2^{k+1} \quad \text{and} \quad |f_{k+1}(t,x) - f_k(t,x)| \leqslant 2/2^k.$$

Now observe that, by (1), (f_n) is a uniformly Cauchy sequence of mappings which are continuous in x and measurable. Let f be the limit of (f_n). Certainly f satisfies (i) and (ii). From (2) it follows that f satisfies (iii).

4. A representation. Now we state a necessary and sufficient condition for a multivalued mapping $F\colon T \times X \to \mathrm{CCl}(Y)$ to be lower semicontinuous in x and weakly measurable. We adopt the proof of Michael's result on representation for lower semicontinuous multivalued mappings ([11], Lemma 5.2, see also [1], Lemma 2). The analogous result for measurable multivalued mappings is due to Castaing (see [8], Theorem 5.6).

THEOREM 3. *Let T, X, Y be as in Theorem 2. A multivalued mapping $F\colon T \times X \to \mathrm{CCl}(Y)$ is lower semicontinuous in x and weakly measurable if and only if* *there exists a countable family $\mathscr{F}$ of mappings $f\colon T\times X\to Y$ satisfying conditions (i)–(iii) of Theorem 2 and such that $F(t,x)=\overline{\{f(t,x)\colon f\in\mathscr{F}\}}$ for $(t,x)\in T\times X$.*

Proof. If $F(t,x)=\overline{\{f(t,x)\colon f\in\mathscr{F}\}}$ for $(t,x)\in T\times X$ and each f satisfies (i)–(iii) of Theorem 2, then the weak measurability of F follows easily from the measurability of all $f\in\mathscr{F}$, via Lemma 3 and Lemma 1. The lower semicontinuity of $F(t,\cdot)$ is a consequence of the equality

$$\{x\in X\colon F(t,x)\cap V\neq\emptyset\}=\bigcup_{f\in\mathscr{F}}\{x\in X\colon f(t,x)\in V\},$$

valid for every open subset V of Y.

Suppose now that F is lower semicontinuous in x and weakly measurable. Let $\{y_i\}_{i\in N}$ be a dense subset of Y and let

$$B_i^m=\{y\in Y\colon |y-y_i|<1/2^m\}\quad\text{for}\quad i,m\in N.$$

Letting

$$U_i^m(t)=\{x\in X\colon F(t,x)\cap B_i^m\neq\emptyset\}$$

we define the multivalued mapping $U_i^m\colon T\to 2^X$ with open values and measurable graph for every $i,m\in N$. Hence, similarly as in the proof of Theorem 1, we can see that each of multivalued mappings $W_{ij}^m\colon T\to 2^X$, $i,j,m\in N$, defined by

$$W_{ij}^m(t)=\{x\in X\colon d(x,X\setminus U_i^m(t))\geqslant 1/j\},$$

has closed values and measurable graph. It is evident that $\bigcup_j \mathrm{Gr}(W_{ij}^m)=\mathrm{Gr}(U_i^m)$. Define the multivalued mappings $G_{ij}^m\colon T\times X\to 2^Y$ letting

$$G_{ij}^m(t,x)=F(t,x)\cap B_i^m\quad\text{iff}\quad (t,x)\in\mathrm{Gr}(W_{ij}^m),$$
$$G_{ij}^m(t,x)=F(t,x)\quad\text{iff}\quad (t,x)\in(T\times X)\setminus\mathrm{Gr}(W_{ij}^m).$$

Clearly, each G_{ij}^m has nonempty convex values. We show that each G_{ij}^m is weakly measurable and $G_{ij}^m(t,\cdot)$ is lower semicontinuous for every $t\in T$. Let V be an open subset of Y. We have

$$(G_{ij}^m)^{-1}(V)=\big(\mathrm{Gr}(W_{ij}^m)\cap F^{-1}(B_i^m\cap V)\big)\cup\big(((T\times X)\setminus\mathrm{Gr}(W_{ij}^m))\cap F^{-1}(V)\big).$$

Hence, the weak measurability of F and the measurability of $\mathrm{Gr}(W_{ij}^m)$ implies the weak measurability of G_{ij}^m. For any $t\in T$ we have

$$\begin{aligned}&\{x\colon G_{ij}^m(t,x)\cap V\neq\emptyset\}\\&=\big(W_{ij}^m(t)\cap\{x\colon F(t,x)\cap B_i^m\cap V\neq\emptyset\}\big)\cup\big((X\setminus W_{ij}^m(t))\cap\{x\colon F(t,x)\cap V\neq\emptyset\}\big)\\&=\{x\colon F(t,x)\cap B_i^m\cap V\neq\emptyset\}\cup\big((X\setminus W_{ij}^m(t))\cap\{x\colon F(t,x)\cap V\neq\emptyset\}\big),\end{aligned}$$

since $\{x\colon F(t,x)\cap B_i^m\cap V\neq\emptyset\}\subset\{x\colon F(t,x)\cap V\neq\emptyset\}$. Hence, by the lower semicontinuity of $F(t,\cdot)$ and since the set $X\setminus W_{ij}^m(t)$ is open, it follows that $G_{ij}^m(t,\cdot)$ is lower semicontinuous.

Now, define the multivalued mappings $F_{ij}^m\colon T\times X\to \mathrm{CCl}(Y)$, $i,j,m\in N$, letting

$$F_{ij}^m(t,x)=\overline{G_{ij}^m(t,x)}.$$

Each F_{ij}^m is weakly measurable (see Lemma 1), and for every $t\in T$ the mapping $F_{ij}^m(t,\cdot)$ is lower semicontinuous (see [11], Proposition 2.3). In virtue of Theorem 2, for each F_{ij}^m we can choose a selector satisfying (i)–(iii) of this Theorem. Let f_{ij}^m be such selector for F_{ij}^m and let

$$\mathscr{F}=\{f_{ij}^m\colon i,j,m\in N\}.$$

The standard reasoning gives the equality

$$F(t,x)=\overline{\{f_{ij}^m(t,x)\colon f_{ij}^m\in\mathscr{F}\}}\quad\text{for}\quad (t,x)\in T\times X.$$

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