An ordering of normal ultrafilters

by

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Abstract. Suppose $\varkappa$ is supercompact, and $\lambda \geqslant \varkappa$. Let $\mathrm{Nuf}_\varkappa(\lambda)$ be the collection of all normal ultrafilters on $P_\varkappa(\lambda)$. For any $U \in \mathrm{Nuf}_\varkappa(\lambda)$, let i_U be the elementary embedding associated with U.

Def. For $U, W \in \mathrm{Nuf}_\varkappa(\lambda)$, $U <_0 W$ iff $i_U(\varkappa) < i_W(\varkappa)$.

Def. If $A \subseteq \mathrm{Nuf}_\varkappa(\lambda)$, then $I(A) = \{i_U(\varkappa)\colon U \in A\}$.

We establish various facts regarding the $<_0$ ordering and the mapping I. We show that $I(\mathrm{Nuf}_\varkappa(\lambda))$ is cofinal below $(2^{\lambda^\varkappa})^+$ and in particular $I\{U \in \mathrm{Nuf}_\varkappa(\lambda)\colon U \text{ is extendible}\}$ is cofinal below $(2^{\lambda^\varkappa})^+$. This provides an alternative proof of the theorem of Magidor on the number of extendible elements of $\mathrm{Nuf}_\varkappa(\lambda)$. In addition, we establish connections between the $<_0$ ordering and the degree of extendibility of elements of $\mathrm{Nuf}_\varkappa(\lambda)$, and we show that an open question concerning the number of normal ultrafilters on a measurable cardinal which concentrate on nonmeasurable cardinals is related to questions involving the $<_0$ ordering.

0. Introduction. Let $\varkappa$ be a supercompact cardinal, and fix a cardinal $\lambda \geqslant \varkappa$. Solovay, Reinhardt, and Kanamori ([5]) established that there exist $2^{2^{\lambda^{\underset{\smile}{\varkappa}}}}$ many normal ultrafilters on $P_\varkappa(\lambda)$. In this paper, we examine a natural ordering of this collection of normal ultrafilters. This ordering is defined in terms of the action of the elementary embedding associated with each normal ultrafilter.

In Section 1, we standardize our notation and review some of the known results that we shall need. In Section 2, we define and prove various facts about the ordering. We also relate the position of a normal ultrafilter in the ordering to its degree of extendibility, and improve on a result established in [1]. In Section 3, we consider what happens to the ordering in the special case that $2^{2^{\lambda^{\underset{\smile}{\varkappa}}}} > (2^{\lambda^{\underset{\smile}{\varkappa}}})^+$. This will yield information about the lower levels of the ordering. Finally, in Section 4, we consider the top levels of the ordering. This relates to the notion of an extendible normal ultrafilter.

1. Preliminaries. We work in ZFC throughout. We remind the reader that a cardinal $\varkappa$ is said to be λ-supercompact iff there exists an elementary embedding $i\colon V \to M$ where V is the universe of all sets, M is an inner model closed under $\lambda^{\underset{\smile}{\varkappa}}$ sequences, $\varkappa$ is the first ordinal moved by i, and $i(\varkappa) > \lambda$. $\varkappa$ is supercompact iff $\varkappa$ is λ-supercompact for all $\lambda \geqslant \varkappa$.

Equivalently, $\varkappa$ is λ-supercompact iff there exists a normal ultrafilter on $P_\varkappa(\lambda) = \{X \subseteq \lambda\colon |X| < \varkappa\}$. One goes from the normal ultrafilter characterization of supercompactness to the embedding characterization by taking ultrapowers of V with respect to the normal ultrafilter, taking the transitive collapse to get M, and composing the associated embeddings to get i. This construction will be central to this paper. For details, we refer the reader to [5].

If U is a normal ultrafilter on $P_\varkappa(\lambda)$, M_U denotes the inner model and i_U the elementary embedding associated with U. Thus, $i_U\colon V \to M_U$.

Throughout the paper, α and β shall be used to denote ordinals, while γ, δ, η, $\varkappa$, and λ shall be reserved for cardinals. $\lambda^{\underset{\sim}{\varkappa}}$ denotes the cardinal $\sup_{\gamma<\varkappa}(\lambda^\gamma)$, or, equivalently, the cardinal of $P_\varkappa(\lambda)$. Cardinal exponentiation is always associated from the top. Thus, for example $2^{2^{\lambda^{\underset{\sim}{\varkappa}}}}$ denotes $2^{(2^{(\lambda^{\underset{\sim}{\varkappa}})})}$. If N is a set of ordinals, $\mathrm{OT}(N)$ denotes the order type of N. If I_0 and I_1 are sets of ordinals, we shall write $I_0 < I_1$ to denote the fact that if $\alpha \in I_0$ and $\beta \in I_1$, then $\alpha < \beta$. For any set P, $|P|$ denotes the cardinality of P.

Next, we develope some notation.

Definition 1a. $\mathrm{Nuf}_\varkappa(\lambda) = \{U\colon U$ is a normal ultrafilter on $P_\varkappa(\lambda)\}$.

Solovay, Reinhardt, and Kanamori ([5]), p. 92) established that if $\varkappa$ is $2^{\lambda^{\underset{\sim}{\varkappa}}}$-supercompact for some cardinal $\lambda \geqslant \varkappa$, then $|\mathrm{Nuf}_\varkappa(\lambda)| = 2^{2^{\lambda^{\underset{\sim}{\varkappa}}}}$.

Definition 1b. For $A \subseteq \mathrm{Nuf}_\varkappa(\lambda)$, $I(A) = \{i_U(\varkappa)\colon U \in A\}$.

If $\varkappa \leqslant \lambda \leqslant \gamma$ and $W \in \mathrm{Nuf}_\varkappa(\gamma)$, then there is a natural way to define $W \restriction \lambda \in \mathrm{Nuf}_\varkappa(\lambda)$, the "restriction" of W to $P_\varkappa(\lambda)$. This normal ultrafilter is defined by $Y \in W \restriction \lambda$ if $Y \subseteq P_\varkappa(\lambda)$ and for some $X \in W$, $Y = \{Z \cap \lambda\colon Z \in X\}$.

There is a natural elementary embedding $k\colon M_{W\restriction\lambda} \to M_W$. For the definition and basic properties of k, the reader is again refered to [5]. A fact which will be important for us is that the first ordinal moved by k is $[(2^{\lambda^{\underset{\sim}{\varkappa}}})^+]_{M_{W\restriction\lambda}}$. For the proof of this fact see [4], p. 340. In addition, a straightforward induction establishes that every set of rank less than $[(2^{\lambda^{\underset{\sim}{\varkappa}}})^+]_{M_{W\restriction\lambda}}$ is fixed by k.

Definition 1c. For $U \in \mathrm{Nuf}_\varkappa(\lambda)$ and $\gamma \geqslant \lambda$, U is *γ-extendible* if $U = W \restriction \lambda$ for some $W \in \mathrm{Nuf}_\varkappa(\gamma)$. $U \in \mathrm{Nuf}_\varkappa(\lambda)$ is *extendible* if it is γ-extendible for every $\gamma \geqslant \lambda$.

The notion of restrictions of normal ultrafilters leads in a natural way to a certain tree structure on the collection of normal ultrafilters associated with a supercompact cardinal. This structure is studied in [1].

2. The ordering. We assume from now on, unless otherwise noted, that $\varkappa$ is a fixed supercompact cardinal. In addition, we fix some cardinal $\lambda \geqslant \varkappa$.

In this section we consider the elements of $\mathrm{Nuf}_\varkappa(\lambda)$. We shall study the following ordering on $\mathrm{Nuf}_\varkappa(\lambda)$:

Definition 2a. For $U, W \in \mathrm{Nuf}_\varkappa(\lambda)$, $U <_0 W$ iff $i_U(\varkappa) < i_W(\varkappa)$. $U =_0 W$ iff $i_U(\varkappa) = i_W(\varkappa)$.

We shall connect this ordering with the notion of the degree of extendibility of elements of $\mathrm{Nuf}_\varkappa(\lambda)$. Note that $I(\mathrm{Nuf}_\varkappa(\lambda))$ is simply the set of ordinals associated with each position in the $<_0$ ordering. We shall show that many distinct, non-overlapping subsets of $I(\mathrm{Nuf}_\varkappa(\lambda))$ can be specified such that normal ultrafilters associated with different subsets have different degrees of extendibility.

First, we establish "where" $I(\mathrm{Nuf}_\varkappa(\lambda))$ is.

Lemma 2b. $I(\mathrm{Nuf}_\varkappa(\lambda)) \subseteq (2^{\lambda^{\underset{\sim}{\varkappa}}})^+ \setminus 2^{\lambda^{\underset{\sim}{\varkappa}}}$.

Proof. Suppose $U \in \mathrm{Nuf}_\varkappa(\lambda)$. We must show that $2^{\lambda^{\underset{\sim}{\varkappa}}} < i_U(\varkappa) < (2^{\lambda^{\underset{\sim}{\varkappa}}})^+$. Since M_U is closed under $\lambda^{\underset{\sim}{\varkappa}}$ sequences, it follows that $[P_\varkappa(\lambda)]_{M_U} = P_\varkappa(\lambda)$ and $[P(P_\varkappa(\lambda))]_{M_U} = P(P_\varkappa(\lambda))$. Thus, $2^{\lambda^{\underset{\sim}{\varkappa}}} = |P(P_\varkappa(\lambda))| \leqslant |P(P_\varkappa(\lambda))|_{M_U} = (2^{\lambda^{\underset{\sim}{\varkappa}}})_{M_U}$. But, in M_U, $i_U(\varkappa)$ is supercompact and thus inaccessible. Therefore, $(2^{\lambda^{\underset{\sim}{\varkappa}}})_{M_U} < i_U(\varkappa)$, and it follows that $2^{\lambda^{\underset{\sim}{\varkappa}}} < i_U(\varkappa)$.

Next, we show that $i_U(\varkappa) < (2^{\lambda^{\underset{\sim}{\varkappa}}})^+$. By definition, $i_U(\varkappa) = \mathrm{OT}(\{f\colon P_\varkappa(\lambda) \to \varkappa\}/U)$. Thus, $|i_U(\varkappa)| = |\{f\colon P_\varkappa(\lambda) \to \varkappa\}/U| \leqslant |\{f\colon P_\varkappa(\lambda) \to \varkappa\}| = \varkappa^{\lambda^{\underset{\sim}{\varkappa}}} = 2^{\lambda^{\underset{\sim}{\varkappa}}}$. It follows that $i_U(\varkappa) < (2^{\lambda^{\underset{\sim}{\varkappa}}})^+$. ■

Our main lemma is the following:

Lemma 2c. *If $W \in \mathrm{Nuf}_\varkappa(2^{\gamma^{\underset{\sim}{\varkappa}}})$ for some $\gamma \geqslant \lambda$, k is the usual elementary embedding $k\colon M_{W\restriction\lambda} \to M_W$, and $k(\alpha) = \gamma$ for some ordinal α, then,*

a. *$I\{U \in \mathrm{Nuf}_\varkappa(\lambda)\colon U$ is γ-extendible$\}$ is cofinal below $(2^{\lambda^{\underset{\sim}{\varkappa}}})^+$.*

b. *$I\{U \in \mathrm{Nuf}_\varkappa(\lambda) \cap M_{W\restriction\lambda}\colon U$ is γ-extendible$\}$ is cofinal below $[(2^{\lambda^{\underset{\sim}{\varkappa}}})^+]_{M_{W\restriction\lambda}}$.*

Proof. Suppose W, γ, k, and α are as in the statement of the lemma. Assume by way of contradiction that $I\{U \in \mathrm{Nuf}_\varkappa(\lambda)\colon U$ *is γ-extendible*$\}$ is not cofinal below $(2^{\lambda^{\underset{\sim}{\varkappa}}})^+$. Since M_W is closed under $2^{\gamma^{\underset{\sim}{\varkappa}}}$ sequences, $\mathrm{Nuf}_\varkappa(\gamma) \subseteq M_W$. Suppose $U \in \mathrm{Nuf}_\varkappa(\lambda)$. Then, since $\mathrm{Nuf}_\varkappa(\gamma) \subseteq M_W$, U is γ-extendible if and only if it is γ-extendible in M_W. Also, since all functions from $P_\varkappa(\lambda)$ into $\varkappa$ are present in M_W, $i_U(\varkappa)$ is correctly computed inside M_W. Then, noting that by closure considerations, $[(2^{\lambda^{\underset{\sim}{\varkappa}}})^+]_{M_W} = (2^{\lambda^{\underset{\sim}{\varkappa}}})^+$, we may conclude that $M_W \models I\{U \in \mathrm{Nuf}_\varkappa(\lambda)\colon U$ is γ-extendible$\}$ is not cofinal below $(2^{\lambda^{\underset{\sim}{\varkappa}}})^+$.

Next, we use the elementarity of $k\colon M_{W\restriction\lambda} \to M_W$. Since $\varkappa, \lambda < [(2^{\lambda^{\underset{\sim}{\varkappa}}})^+]_{M_{W\restriction\lambda}}$ it follows that $k(\varkappa) = \varkappa$ and $k(\lambda) = \lambda$. Then, since $k(\alpha) = \gamma$, $M_{W\restriction\lambda} \models I\{U \in \mathrm{Nuf}_\varkappa(\lambda)\colon U$ is α-extendible$\}$ is not cofinal below $(2^{\lambda^{\underset{\sim}{\varkappa}}})^+$. Pick $\beta < [(2^{\lambda^{\underset{\sim}{\varkappa}}})^+]_{M_{W\restriction\lambda}}$ such that $M_{W\restriction\lambda} \models I\{U \in \mathrm{Nuf}_\varkappa(\lambda)\colon U$ is α-extendible$\} \subseteq \beta$. Then, since $k(\beta) = \beta$,

$$M_W \models I\{U \in \mathrm{Nuf}_\varkappa(\lambda)\colon U \text{ is } \gamma\text{-extendible}\} \subseteq \beta.$$

Using closure considerations as previously, we conclude that (in V), $I\{U \in \mathrm{Nuf}_\varkappa(\lambda)\colon U$ is γ-extendible$\} \subseteq \beta$. But $W \restriction \lambda = (W \restriction \gamma) \restriction \lambda$ and thus $W \restriction \lambda$ is a γ-extendible element of $\mathrm{Nuf}_\varkappa(\lambda)$. Therefore, $i_{W\restriction\lambda}(\varkappa) \in I\{U \in \mathrm{Nuf}_\varkappa(\lambda)\colon U$ is γ-extendible$\} \subseteq \beta < [(2^{\lambda^{\underset{\sim}{\varkappa}}})^+]_{M_{W\restriction\lambda}} < i_{W\restriction\lambda}(\varkappa)$, where the last inequality follows from the inaccessibility (in $M_{W\restriction\lambda}$) of $i_{W\restriction\lambda}(\varkappa)$. This is a contradiction and thus part a has been established.

Next, for part b, we note that by part a and by closure considerations, $M_W \models I\{U \in \mathrm{Nuf}_\varkappa(\lambda)\colon U$ is γ-extendible$\}$ is cofinal below $(2^{\lambda^{\underset{\sim}{\varkappa}}})^+$. By the elementarity of k, noting that $k(\alpha) = \gamma$, it follows that $M_{W\restriction\lambda} \models I\{U \in \mathrm{Nuf}_\varkappa(\lambda)\colon U \text{ is } \alpha\text{-extendible}\}$ is cofinal below $(2^{\lambda^{\varkappa}})^+$. Thus, $I\{U \in \mathrm{Nuf}_\varkappa(\lambda) \cap M_{W\restriction\lambda}\colon M_{W\restriction\lambda} \models U \text{ is } \alpha\text{-extendible}\}$ is cofinal below $[(2^{\lambda^{\varkappa}})^+]_{M_{W\restriction\lambda}}$. We therefore have only to show that if $U \in \mathrm{Nuf}_\varkappa(\lambda) \cap M_{W\restriction\lambda}$ and $M_{W\restriction\lambda} \models U$ is α-extendible, then, (in V), U is γ-extendible. This follows easily though, since, by elementarity, if $M_{W\restriction\lambda} \models U$ is α-extendible, then, $M_W \models U$ is γ-extendible ($k(U) = U$ since $\mathrm{rank}(U) < [(2^{\lambda^{\varkappa}})^+]_{M_{W\restriction\lambda}}$). By the closure properties of M_W, it follows that (in V), U is γ-extendible. ■

The following corollary is immediate from part a of the lemma:

Corollary 2d. *$I(\mathrm{Nuf}_\varkappa(\lambda))$ is cofinal below* $(2^{\lambda^{\varkappa}})^+$.

Cardinals γ that satisfy the conditions of Lemma 2c will be extremely important for us. Thus we consider

Definition 2e. A cardinal γ is $(\varkappa-\lambda)$ *specifiable* if for any $W \in \mathrm{Nuf}_\varkappa(2^{\gamma^{\varkappa}})$, $k(\alpha) = \gamma$ for some ordinal α, where k is the usual elementary embedding, $k\colon M_{W\restriction\lambda} \to M_W$.

As an example, let γ be the β^{th} inaccessible cardinal above λ, where β is some ordinal with $\beta \leqslant 2^{\lambda^{\varkappa}}$ (Assume, for this example, that such a cardinal exists). We claim that γ is $(\varkappa-\lambda)$ specifiable. Let W be any element of $\mathrm{Nuf}_\varkappa(2^{\gamma^{\varkappa}})$ and let $k\colon M_{W\restriction\lambda} \to M_W$ be the usual elementary embedding. By closure considerations, $M_W \models \gamma$ is the βth inaccessible cardinal above λ. Thus, $M_W \models$ There exists a βth inaccessible cardinal above λ. By elementarity, since $k(\beta) = \beta$ and $k(\lambda) = \lambda$, $M_{W\restriction\lambda} \models$ There exists a βth inaccessible cardinal above λ. Pick α such that $M_{W\restriction\lambda} \models \alpha$ is the βth inaccessible cardinal above λ. Then, it clearly follows from the elementarity of k that $k(\alpha) = \gamma$.

This example suggests the following:

Definition 2f. A cardinal γ has *property* $A(\varkappa-\lambda)$ if there is a formula $\phi(x, x_1, x_2, \dots x_n)$ and ordinals $\alpha_1, \alpha_2, \dots, \alpha_n \leqslant 2^{\lambda^{\varkappa}}$ such that given any inner model M which is closed under $2^{\gamma^{\varkappa}}$ sequences, $M \models \gamma$ is the unique set y such that $\phi(y, \alpha_1, \alpha_2, \dots \alpha_n)$.

Lemma 2g. *If a cardinal γ has property $A(\varkappa-\lambda)$, it is $(\varkappa-\lambda)$ specifiable.*

Proof. Suppose γ has property $A(\varkappa-\lambda)$, and $\phi(x, x_1, x_2, \dots x_n)$, $\alpha_1, \alpha_2, \dots \alpha_n$ are as in the definition. Pick any $W \in \mathrm{Nuf}_\varkappa(2^{\gamma^{\varkappa}})$. Then, $M_W \models \gamma$ is the unique set y such that $\phi(y, \alpha_1, \alpha_2, \dots \alpha_n)$. Thus, $M_W \models$ there is a unique ordinal y such that $\phi(y, \alpha_1, \alpha_2, \dots \alpha_n)$. Since $\alpha_1, \alpha_2, \dots \alpha_n \leqslant 2^{\lambda^{\varkappa}}$, $k\colon M_{W\restriction\lambda} \to M_W$ fixes each of these ordinals. Then, by the elementarity of k, $M_{W\restriction\lambda} \models$ there is a unique ordinal y such that $\phi(y, \alpha_1, \alpha_2, \dots \alpha_n)$. Pick α such that $M_{W\restriction\lambda} \models \phi(\alpha, \alpha_1, \alpha_2, \dots \alpha_n)$. Then, it clearly follows from the elementarity of k that $k(\alpha) = \gamma$. ■

Property $A(\varkappa-\lambda)$ provides a simple means of showing that many cardinals which can be described in a "reasonable" way (loosely speaking, this means that they can be specified using ordinals less than or equal to $2^{\lambda^{\varkappa}}$) are $(\varkappa-\lambda)$ specifiable. For the most part, we shall be directly using the $(\varkappa-\lambda)$ specifiability of certain cardinals.

It is not hard to see, using property $A(\varkappa-\lambda)$ and Lemma 2g that for each ordinal $\alpha \leqslant 2^{\lambda^{\varkappa}}$, the αth cardinal bigger than λ is $(\varkappa-\lambda)$ specifiable. Also, for example, for each $\alpha \leqslant 2^{\lambda^{\varkappa}}$, the αth cardinal bigger than λ which is the limit of strong limit cardinals is $(\varkappa-\lambda)$ specifiable. Thus,

Lemma 2h. *The order type of the set of all cardinals which are $(\varkappa-\lambda)$ specifiable is strictly greater than* $2^{\lambda^{\varkappa}}$.

As we shall see later, this set will have order type strictly between $2^{\lambda^{\varkappa}}$ and $(2^{\lambda^{\varkappa}})^+$.

We shall soon identify a subset of $I(\mathrm{Nuf}_\varkappa(\lambda))$ with each cardinal which is $(\varkappa-\lambda)$ specifiable. First though, we must develop some additional notation.

Definition 2i. For each cardinal $\gamma \geqslant \lambda$, we define $U(\varkappa-\lambda, \gamma)$ to be some γ-extendible element of $\mathrm{Nuf}_\varkappa(\lambda)$ having the property that $U(\varkappa-\lambda, \gamma) \leqslant_0 W$ for any $W \in \mathrm{Nuf}_\varkappa(\lambda)$ which is γ-extendible.

There is no claim that there is a unique way to choose $U(\varkappa-\lambda, \gamma)$. We think of $U(\varkappa-\lambda, \gamma)$ as a minimal (with respect to $\leqslant_0$) γ-extendible element of $\mathrm{Nuf}_\varkappa(\lambda)$.

The subsets of $I(\mathrm{Nuf}_\varkappa(\lambda))$ that we will be interested in will be subsets of the form $I(A_\gamma)$, for A_γ given as follows:

Definition 2j. For each cardinal γ which is $(\varkappa-\lambda)$ *specifiable*,

$$A_\gamma = \{U \in \mathrm{Nuf}_\varkappa(\lambda) \cap M_{U(\varkappa-\lambda, 2^{\gamma^{\varkappa}})}\colon U \text{ is } \gamma\text{-extendible}\}.$$

Lemma 2k. *For any cardinal γ which is $(\varkappa-\lambda)$ specifiable, $I(A_\gamma)$ is cofinal below* $[(2^{\lambda^{\varkappa}})^+]_{M_{U(\varkappa-\lambda, 2^{\gamma^{\varkappa}})}}$.

Proof. This is immediate from Lemma 2c, part b. ■

Theorem 2l. *Suppose γ is a cardinal which is $(\varkappa-\lambda)$ specifiable. Then*

a. *for each $U \in A_\gamma$, U is γ-extendible, but not $2^{\gamma^{\varkappa}}$-extendible,*

b. $|I(A_\gamma)| = 2^{\lambda^{\varkappa}}$,

c. $|A_\gamma| = 2^{\lambda^{\varkappa}}$.

The proof of Theorem 2l requires the following lemma, the proof of which can be found in [3], p. 190.

Lemma 2m. *If $U, U' \in \mathrm{Nuf}_\varkappa(\lambda)$ and $U \in M_{U'}$, then $U <_0 U'$.*

Proof of Theorem 2l. Fix $U \in A_\gamma$. By the definition of A_γ, U is γ-extendible. Also by the definition of A_γ, $U \in M_{U(\varkappa-\lambda, 2^{\gamma^{\varkappa}})}$. Thus, by Lemma 2m, $U <_0 U(\varkappa-\lambda, 2^{\gamma^{\varkappa}})$. Then, by the definition of $U(\varkappa-\lambda, 2^{\gamma^{\varkappa}})$, U is not $2^{\gamma^{\varkappa}}$-extendible. This establishes part a.

Next, for part b, we first note that by Lemma 2k, $I(A_\gamma)$ is cofinal below $[(2^{\lambda^{\varkappa}})^+]_{M_{U(\varkappa-\lambda, 2^{\gamma^{\varkappa}})}}$. But clearly $A_\gamma \in M_{U(\varkappa-\lambda, 2^{\gamma^{\varkappa}})}$ and thus $M_{U(\varkappa-\lambda, 2^{\gamma^{\varkappa}})} \models I(A_\gamma)$ is cofinal below $(2^{\lambda^{\varkappa}})^+$. But, since $M_{U(\varkappa-\lambda, 2^{\gamma^{\varkappa}})} \models (2^{\lambda^{\varkappa}})^+$ is a regular cardinal, it follows that $M_{U(\varkappa-\lambda, 2^{\gamma^{\varkappa}})} \models |I(A_\gamma)| = (2^{\lambda^{\varkappa}})^+$. Thus, (in V),

$$|I(A_\gamma)| = |[(2^{\lambda^{\varkappa}})^+]_{M_{U(\varkappa-\lambda, 2^{\gamma^{\varkappa}})}}|.$$

Claim. $|[(2^{\lambda^{\varkappa}})^+]_{M_{U(\varkappa-\lambda, 2^{\gamma^{\varkappa}})}}| = 2^{\lambda^{\varkappa}}$.

Proof of Claim. Since $M_{U(\kappa-\lambda,2^{\gamma^{\underset{\smile}{\kappa}}})}$ is closed under $\lambda^{\underset{\smile}{\kappa}}$ sequences, it follows that $P(P_\kappa(\lambda)) = [P(P_\kappa(\lambda))]_{M_{U(\kappa-\lambda,2^{\gamma^{\underset{\smile}{\kappa}}})}}$. Thus $2^{\lambda^{\underset{\smile}{\kappa}}} \leqslant [2^{\lambda^{\underset{\smile}{\kappa}}}]_{M_{U(\kappa-\lambda,2^{\gamma^{\underset{\smile}{\kappa}}})}}$ and we have $2^{\lambda^{\underset{\smile}{\kappa}}} \leqslant [2^{\lambda^{\underset{\smile}{\kappa}}}]_{M_{U(\kappa-\lambda,2^{\gamma^{\underset{\smile}{\kappa}}})}} < [(2^{\lambda^{\underset{\smile}{\kappa}}})^+]_{M_{U(\kappa-\lambda,2^{\gamma^{\underset{\smile}{\kappa}}})}} < i_{U(\kappa-\lambda,2^{\gamma^{\underset{\smile}{\kappa}}})}(\kappa) < (2^{\lambda^{\underset{\smile}{\kappa}}})^+$, where the last inequality follows from Lemma 2b, and the second to last inequality follows from the inaccessibility, in $M_{U(\kappa-\lambda,2^{\gamma^{\underset{\smile}{\kappa}}})}$, of $i_{U(\kappa-\lambda,2^{\gamma^{\underset{\smile}{\kappa}}})}(\kappa)$. This establishes the claim, which in turn establishes part b of the theorem.

Clearly $|A_\gamma| \geqslant |I(A_\gamma)| = 2^{\lambda^{\underset{\smile}{\kappa}}}$. Since

$$A_\gamma \subseteq \mathrm{Nuf}_\kappa(\lambda) \cap M_{U(\kappa-\lambda,2^{\gamma^{\underset{\smile}{\kappa}}})} = [\mathrm{Nuf}_\kappa(\lambda)]_{M_{U(\kappa-\lambda,2^{\gamma^{\underset{\smile}{\kappa}}})}}$$

we may conclude that $|A_\gamma| \leqslant |[2^{2^{\lambda^{\underset{\smile}{\kappa}}}}]_{M_{U(\kappa-\lambda,2^{\gamma^{\underset{\smile}{\kappa}}})}}| = 2^{\lambda^{\underset{\smile}{\kappa}}}$. The last equality can be established in the same manner as the previous claim was established. Thus $|A_\gamma| = 2^{\lambda^{\underset{\smile}{\kappa}}}$. This establishes part c of theorem. ■

This theorem yields subsets of $I(\mathrm{Nuf}_\kappa(\lambda))$, each of size $2^{\lambda^{\underset{\smile}{\kappa}}}$, such that the associated elements of $\mathrm{Nuf}_\kappa(\lambda)$ have certain specified degrees of extendibility.

The following corollary establishes that the $I(A_\gamma)$ "usually" do not overlap.

COROLLARY 2n. *For cardinals γ and η, both $(\kappa-\lambda)$ specifiable with $2^{\gamma^{\underset{\smile}{\kappa}}} \leqslant \eta$, $I(A_\gamma) < I(A_\eta)$.*

Proof. Pick $U_\gamma \in A_\gamma$ and $U_\eta \in A_\eta$. We must show that $U_\gamma <_0 U_\eta$. By Lemma 2k, $i_{U_\gamma}(\kappa) < [(2^{\lambda^{\underset{\smile}{\kappa}}})^+]_{M_{U(\kappa-\lambda,2^{\gamma^{\underset{\smile}{\kappa}}})}}$. But, by the inaccessibility, in $M_{U(\kappa-\lambda,2^{\gamma^{\underset{\smile}{\kappa}}})}$ of $i_{U(\kappa-\lambda,2^{\gamma^{\underset{\smile}{\kappa}}})}(\kappa)$, it follows that

$$[(2^{\lambda^{\underset{\smile}{\kappa}}})^+]_{M_{U(\kappa-\lambda,2^{\gamma^{\underset{\smile}{\kappa}}})}} < i_{U(\kappa-\lambda,2^{\gamma^{\underset{\smile}{\kappa}}})}(\kappa),$$

and thus $U_\gamma <_0 U(\kappa-\lambda, 2^{\gamma^{\underset{\smile}{\kappa}}})$. But, it follows from the definition of $U(\kappa-\lambda, 2^{\gamma^{\underset{\smile}{\kappa}}})$ and the fact that U_η is η-extendible (and therefore $2^{\gamma^{\underset{\smile}{\kappa}}}$-extendible) that $U(\kappa-\lambda, 2^{\gamma^{\underset{\smile}{\kappa}}}) \leqslant_0 U_\eta$. Thus, $U_\gamma <_0 U_\eta$. ■

Of course, if the GCH holds, it follows immediately from Corollary 2n that if cardinals γ and η are $(\kappa-\lambda)$ specifiable and $\gamma < \eta$, then $I(A_\gamma) < I(A_\eta)$.

What emerges is a picture of the set of ordinals $I(\mathrm{Nuf}_\kappa(\lambda))$ with the following structure:

We distinguish a certain collection of cardinals γ, those which are $(\kappa-\lambda)$ specifiable. The order type of this collection is strictly greater than $2^{\lambda^{\underset{\smile}{\kappa}}}$. Each such γ yields a distinct ordinal $[(2^{\lambda^{\underset{\smile}{\kappa}}})^+]_{M_{U(\kappa-\lambda,2^{\gamma^{\underset{\smile}{\kappa}}})}}$. This will give us a sequence of ordinals below $(2^{\lambda^{\underset{\smile}{\kappa}}})^+$ of order type strictly greater than $2^{\lambda^{\underset{\smile}{\kappa}}}$. Each such ordinal $[(2^{\lambda^{\underset{\smile}{\kappa}}})^+]_{M_{U(\kappa-\lambda,2^{\gamma^{\underset{\smile}{\kappa}}})}}$ has the property that there is a set A_γ of normal ultrafilters on $P_\kappa(\lambda)$ such that each element of A_γ is γ-extendible, but not $2^{\gamma^{\underset{\smile}{\kappa}}}$-extendible, and $I(A_\gamma)$ is cofinal below $[(2^{\lambda^{\underset{\smile}{\kappa}}})^+]_{M_{U(\kappa-\lambda,2^{\gamma^{\underset{\smile}{\kappa}}})}}$.

We do not have the answer to the following:

OPEN QUESTION 2o: For a cardinal γ which is $(\kappa-\lambda)$ specifiable, is $I(A_\gamma)$ a subinterval of $I(\mathrm{Nuf}_\kappa(\lambda))$?

In other words, can we have $U, U' \in A_\gamma$ such that for some $W \in \mathrm{Nuf}_\kappa(\lambda)$, $U <_0 W <_0 U'$, but $W \neq_0 U''$ for any $U'' \in A_\gamma$?

In section 4, we shall deal with the question of how high in the $I(\mathrm{Nuf}_\kappa(\lambda))$ order the $I(A_\gamma)$ go, for $(\kappa-\lambda)$ specifiable cardinals γ. This, of course, relates to the question of how many $(\kappa-\lambda)$ specifiable cardinals there are.

Before closing this section, we consider two special cases of Lemma 2c and Theorem 2l. For the following corollary, we do not assume that κ is fully supercompact.

COROLLARY 2p. *If κ is 2^κ supercompact, then $I\{U: U$ is a normal ultrafilter on κ (in the measurable cardinal sense)$\}$ is cofinal below $(2^\kappa)^+$.*

Proof. Clearly κ is $(\kappa-\kappa)$ specifiable. Then, Corollary 2d implies that $I(\mathrm{Nuf}_\kappa(\kappa))$ is cofinal below $(2^{\kappa^{\underset{\smile}{\kappa}}})^+$. But $\kappa^{\underset{\smile}{\kappa}} = \kappa$ since κ is inaccessible, and, as far as the associated elementary embeddings are concerned, normal ultrafilters on $P_\kappa(\kappa)$ are equivalent to normal ultrafilters on κ (see, e.g., [5], p. 89). ■

In [1], we proved that for cardinals $\lambda \geqslant \kappa$ which are "specifiable" from κ (a notion which is close to what we would presently call $(\kappa-\kappa)$ specifiable), there exist $2^{2^{\lambda^{\underset{\smile}{\kappa}}}}$ many elements of $\mathrm{Nuf}_\kappa(\lambda)$ which are not $2^{2^{\lambda^{\underset{\smile}{\kappa}}}}$-extendible. We also showed that for any $\lambda \geqslant \kappa$, there exists (at least) one element of $\mathrm{Nuf}_\kappa(\lambda)$ which is not $2^{\lambda^{\underset{\smile}{\kappa}}}$-extendible. We improve this last result using our present methods.

COROLLARY 2q. *For any cardinal $\lambda \geqslant \kappa$, there exist (at least) $2^{\lambda^{\underset{\smile}{\kappa}}}$ many elements of $\mathrm{Nuf}_\kappa(\lambda)$ which are not $2^{\lambda^{\underset{\smile}{\kappa}}}$-extendible.*

Proof. Since λ is certainly $(\kappa-\lambda)$ specifiable, it follows from Theorem 2l that $|A_\lambda| = 2^{\lambda^{\underset{\smile}{\kappa}}}$ and if $U \in A_\lambda$, then U is an element of $\mathrm{Nuf}_\kappa(\lambda)$ which is not $2^{\lambda^{\underset{\smile}{\kappa}}}$-extendible. ■

Of course, if the GCH holds, then we have shown that there are $2^{\lambda^{\underset{\smile}{\kappa}}} = \lambda^+$ many elements of $\mathrm{Nuf}_\kappa(\lambda)$ which are not extendible at all.

We have not completely answered the question of the number of elements of $\mathrm{Nuf}_\kappa(\lambda)$ which are not $2^{\lambda^{\underset{\smile}{\kappa}}}$-extendible. We still have the following:

OPEN QUESTION 2r. Can it be shown that there exist $2^{2^{\lambda^{\underset{\smile}{\kappa}}}}$ (or $(2^{\lambda^{\underset{\smile}{\kappa}}})^+$) many elements of $\mathrm{Nuf}_\kappa(\lambda)$ which are not $2^{\lambda^{\underset{\smile}{\kappa}}}$-extendible, and, if so, is the set of ordinals obtained by applying I to this collection cofinal below $(2^{\lambda^{\underset{\smile}{\kappa}}})^+$?

3. The bottom of the ordering and connections with a result of Laver. By Corollary 2d, $|I(\mathrm{Nuf}_\kappa(\lambda))| = (2^{\lambda^{\underset{\smile}{\kappa}}})^+$. As noted in Section 1, $|\mathrm{Nuf}_\kappa(\lambda)| = 2^{2^{\lambda^{\underset{\smile}{\kappa}}}}$. In this section, we consider the situation in which $2^{2^{\lambda^{\underset{\smile}{\kappa}}}} > (2^{\lambda^{\underset{\smile}{\kappa}}})^+$ For technical reasons, we shall want $cf(2^{2^{\lambda^{\underset{\smile}{\kappa}}}}) > (2^{\lambda^{\underset{\smile}{\kappa}}})^+$.

First, we consider the question of consistency. Assume that κ is supercompact, $\lambda \geqslant \kappa$, and $\eta > 2^{2^{\lambda^{\underset{\smile}{\kappa}}}}$ is regular. Let $H = \{f: f$ is a partial function from $2^{\lambda^{\underset{\smile}{\kappa}}} \times \eta$ into 2 and $|f| < 2^{\lambda^{\underset{\smile}{\kappa}}}\}$. Then, forcing with H yields a model in which $2^{2^{\lambda^{\underset{\smile}{\kappa}}}} = \eta$. In addition, since H is $cf(2^{\lambda^{\underset{\smile}{\kappa}}})$-closed, no new subsets of any cardinal $\gamma < cf(2^{\lambda^{\underset{\smile}{\kappa}}})$ are added. It follows that in the forcing extension, κ is γ-supercompact for each cardinal γ satisfying $\gamma^{\underset{\smile}{\kappa}} = |P_\kappa(\gamma)| < cf(2^{\lambda^{\underset{\smile}{\kappa}}})$. By Konig's theorem, this inequality is true for $\gamma = \lambda$. Thus, the forcing extension satisfies that κ is λ-supercompact.

It turns out that something much stronger is possible. In [2], Laver established the following:

THEOREM 3a. *If $\varkappa$ is supercompact, then there is a $\varkappa$ c.c. partial ordering Q with $|Q| = \varkappa$ such that in V^Q, $\varkappa$ is supercompact, and remains supercompact upon forcing with any $\varkappa$-directed closed partial ordering.*

The partially ordered set H defined above is a $\varkappa$-directed closed partial ordering. Thus, Laver's theorem yields the following:

COROLLARY 3b. *Con* (ZFC+$\varkappa$ *is supercompact) implies Con* (ZFC+$\varkappa$ *is supercompact + for some cardinal* $\lambda > \varkappa$, $cf(2^{2^{\lambda^{\underset{\smile}{\varkappa}}}}) > (2^{\lambda^{\underset{\smile}{\varkappa}}})^+$).

We assume for the remainder of this section that $\varkappa$ is supercompact and $cf(2^{2^{\lambda^{\underset{\smile}{\varkappa}}}}) > (2^{\lambda^{\underset{\smile}{\varkappa}}})^+$.

By the first paragraph of this section, the following is immediate:

THEOREM 3c. *For some ordinal α with $2^{\lambda^{\underset{\smile}{\varkappa}}} < \alpha < (2^{\lambda^{\underset{\smile}{\varkappa}}})^+$, $A = \{U \in \mathrm{Nuf}_\varkappa(\lambda)\colon i_U(\varkappa) = \alpha\}$ has cardinality $2^{2^{\lambda^{\underset{\smile}{\varkappa}}}}$.*

Thus, for any U, $W \in A$, $U =_0 W$. Also note that by the definition of A, $I(A) = \{\alpha\}$.

One consequence of Theorem 3c is that since, by Lemma 2m, if $U =_0 W$, then $U \notin M_W$, the ordering given by "$U \lhd W$ iff $U \in M_W$ (see [5], p. 92) is not a linear ordering. We do not know whether this ordering can be linear if $2^{2^{\lambda^{\underset{\smile}{\varkappa}}}} = (2^{\lambda^{\underset{\smile}{\varkappa}}})^+$.

Next, we show how the assumption that $cf(2^{2^{\lambda^{\underset{\smile}{\varkappa}}}}) > (2^{\lambda^{\underset{\smile}{\varkappa}}})^+$ implies that a result similar to Theorem 3c holds inside a certain model. This will have certain consequences for the "bottom" of the $I(\mathrm{Nuf}_\varkappa(\lambda))$ ordering.

THEOREM 3d. *For some ordinal β with $2^{\lambda^{\underset{\smile}{\varkappa}}} < \beta < [(2^{\lambda^{\underset{\smile}{\varkappa}}})^+]_{M_{U(\varkappa-\lambda, 2^{\lambda^{\underset{\smile}{\varkappa}}})}}$, $B = \{U \in \mathrm{Nuf}_\varkappa(\lambda)\colon i_U(\varkappa) = \beta$ and U is not $2^{\lambda^{\underset{\smile}{\varkappa}}}$-extendible$\}$ has cardinality at least $2^{\lambda^{\underset{\smile}{\varkappa}}}$.*

Proof. Fix $W \in \mathrm{Nuf}_\varkappa(2^{\lambda^{\underset{\smile}{\varkappa}}})$ such that $U(\varkappa-\lambda, 2^{\lambda^{\underset{\smile}{\varkappa}}}) = W \restriction \lambda$. Since M_W is closed under $2^{\lambda^{\underset{\smile}{\varkappa}}}$-sequences, $\mathrm{Nuf}_\varkappa(\lambda) \subseteq M_W$, and thus $\mathrm{Nuf}_\varkappa(\lambda) = [\mathrm{Nuf}_\varkappa(\lambda)]_{M_W} \in M_W$. Let A be as in the statement of Theorem 3c. Then, since each $i_U(\varkappa)$ for $U \in \mathrm{Nuf}_\varkappa(\lambda)$ is correctly computed in M_W, it follows (by the axiom of separation) that $A \in M_W$. Thus, $M_W \models$ For some ordinal β with $2^{\lambda^{\underset{\smile}{\varkappa}}} < \beta < (2^{\lambda^{\underset{\smile}{\varkappa}}})^+$ and some set $C \subseteq \mathrm{Nuf}_\varkappa(\lambda)$ with $2^{\lambda^{\underset{\smile}{\varkappa}}} < |C| \leqslant 2^{2^{\lambda^{\underset{\smile}{\varkappa}}}}$, $I(C) = \{\beta\}$. By the elementarity of $k\colon M_{U(\varkappa-\lambda, 2^{\lambda^{\underset{\smile}{\varkappa}}})} \to M_W$, it follows that $M_{U(\varkappa-\lambda, 2^{\lambda^{\underset{\smile}{\varkappa}}})} \models$ for some ordinal β with $2^{\lambda^{\underset{\smile}{\varkappa}}} < \beta < (2^{\lambda^{\underset{\smile}{\varkappa}}})^+$, and some set $C \subseteq \mathrm{Nuf}_\varkappa(\lambda)$ with $2^{\lambda^{\underset{\smile}{\varkappa}}} < |C| \leqslant 2^{2^{\lambda^{\underset{\smile}{\varkappa}}}}$, $I(C) = \{\beta\}$. Fix a specific ordinal β and a specific set $C \subseteq \mathrm{Nuf}_\varkappa(\lambda)$ such that $C \in M_{U(\varkappa-\lambda, 2^{\lambda^{\underset{\smile}{\varkappa}}})}$ and $M_{U(\varkappa-\lambda, 2^{\lambda^{\underset{\smile}{\varkappa}}})} \models 2^{\lambda^{\underset{\smile}{\varkappa}}} < \beta < (2^{\lambda^{\underset{\smile}{\varkappa}}})^+$, $2^{\lambda^{\underset{\smile}{\varkappa}}} < |C| \leqslant 2^{2^{\lambda^{\underset{\smile}{\varkappa}}}}$, and $I(C) = \{\beta\}$. It follows that (in V), $2^{\lambda^{\underset{\smile}{\varkappa}}} < \beta < [(2^{\lambda^{\underset{\smile}{\varkappa}}})^+]_{M_{U(\varkappa-\lambda, 2^{\lambda^{\underset{\smile}{\varkappa}}})}}$, and $|C| = 2^{\lambda^{\underset{\smile}{\varkappa}}}$ (this last equality can be established in a manner similar to the claim in the proof of Theorem 2l). Finally, we note that for $U \in \mathrm{Nuf}_\varkappa(\lambda) \cap M_{U(\varkappa-\lambda, 2^{\lambda^{\underset{\smile}{\varkappa}}})}$, it follows from Lemma 2m that $U <_0 U(\varkappa-\lambda, 2^{\lambda^{\underset{\smile}{\varkappa}}})$ and then, by the definition of $U(\varkappa-\lambda, 2^{\lambda^{\underset{\smile}{\varkappa}}})$, it follows that U is not $2^{\lambda^{\underset{\smile}{\varkappa}}}$-extendible. The theorem follows by noting that if B is as in the statement of the theorem, then $C \subseteq B$. ∎

We note that we cannot hope, by our present methods, to improve Theorem 3d to "$|B| = 2^{2^{\lambda^{\underset{\smile}{\varkappa}}}}$" since $|(2^{2^{\lambda^{\underset{\smile}{\varkappa}}}})_{M_{U(\varkappa-\lambda, 2^{\lambda^{\underset{\smile}{\varkappa}}})}}| = 2^{\lambda^{\underset{\smile}{\varkappa}}}$.

By the theorem, we have a $2^{\lambda^{\underset{\smile}{\varkappa}}}$-way tie, somewhere near the bottom (i.e., before $[(2^{\lambda^{\underset{\smile}{\varkappa}}})^+]_{M_{U(\varkappa-\lambda, 2^{\lambda^{\underset{\smile}{\varkappa}}})}}$ of $I(\mathrm{Nuf}_\varkappa(\lambda))$.

OPEN QUESTION 3e. Can this tie occur on the bottom level? In other words, can the β of Theorem 3d be the least element of $I(\mathrm{Nuf}_\varkappa(\lambda))$?

An affirmative answer to this question would relate to a major topic in the study of normal ultrafilters, as shown by the following:

COROLLARY 3f. *If the β of the theorem is the least element of $I(\mathrm{Nuf}_\varkappa(\lambda))$, then*

a. *there exist (at least) $2^{\lambda^{\underset{\smile}{\varkappa}}}$ many elements U of $\mathrm{Nuf}_\varkappa(\lambda)$ such that $M_U \models \varkappa$ is not λ-supercompact, and*

b. *in particular, if $\lambda = \varkappa$, there exist (at least) $2^\varkappa$ many normal ultrafilters on $\varkappa$ which concentrate on non-measurable cardinals.*

Proof. For part a, assume that β is the least element of $\mathrm{Nuf}_\varkappa(\lambda)$, and B is as in the statement of the theorem. It follows from Lemma 2m that if $U \in B$ and $W \in \mathrm{Nuf}_\varkappa(\lambda)$, then $W \notin M_U$ and thus $M_U \models$ There are no normal ultrafilters on $P_\varkappa(\lambda)$. Hence, $M_U \models \varkappa$ is not λ-supercompact.

For part b, we note that if $\lambda = \varkappa$, then, by the above, for any $U \in B$, $M_U \models \varkappa$ is not $\varkappa$-supercompact. Thus, $M_U \models \varkappa$ is not measurable, and it follows that U concentrates on non-measurables. Finally, we note that $|B| = 2^{\lambda^{\underset{\smile}{\varkappa}}} = 2^{\varkappa^{\underset{\smile}{\varkappa}}} = 2^\varkappa$. ∎

4. The top of the ordering and connections with a result of Magidor. We begin this section by considering the question of how high in $I(\mathrm{Nuf}_\varkappa(\lambda))$ the $I(A_\gamma)$ go, for γ a $(\varkappa-\lambda)$ specifiable cardinal. We have already pointed out that the order type of the $(\varkappa-\lambda)$ specifiable cardinals is strictly greater than $2^{\lambda^{\underset{\smile}{\varkappa}}}$.

THEOREM 4a. *$\bigcup\{I(A_\gamma)\colon \gamma$ is a $(\varkappa-\lambda)$ specifiable cardinal$\}$ is bounded strictly below $(2^{\lambda^{\underset{\smile}{\varkappa}}})^+$. Equivalently, the order type of the $(\varkappa-\lambda)$ specifiable cardinals is strictly less than $(2^{\lambda^{\underset{\smile}{\varkappa}}})^+$.*

Note. The equivalence expressed in the theorem follows from Theorem 2l, part b, and Corollary 2n.

Before beginning the proof, we establish the following:

CLAIM. *There exists a $U \in \mathrm{Nuf}_\varkappa(\lambda)$ such that U is extendible.*

Proof of Claim. Suppose by way of contradiction that no $U \in \mathrm{Nuf}_\varkappa(\lambda)$ is extendible. Then, for each $U \in \mathrm{Nuf}_\varkappa(\lambda)$, let δ_U be the least cardinal δ such that U is not δ-extendible. Since $\mathrm{Nuf}_\varkappa(\lambda)$ is a set (of cardinality $2^{2^{\lambda^{\underset{\smile}{\varkappa}}}}$), $\sup\{\delta_U\colon U \in \mathrm{Nuf}_\varkappa(\lambda)\}$ is a cardinal. Pick any η bigger than this cardinal. Then no $U \in \mathrm{Nuf}_\varkappa(\lambda)$ is η-extendible. But, since $\varkappa$ is supercompact, we can pick some $W \in \mathrm{Nuf}_\varkappa(\eta)$. Then $W \restriction \lambda \in \mathrm{Nuf}_\varkappa(\lambda)$ is η-extendible. This is a contradiction, and we have established the claim.

Proof of Theorem 4a. By the claim, suppose $U \in \mathrm{Nuf}_\varkappa(\lambda)$ is extendible, and assume by way of contradiction that the collection of $(\varkappa-\lambda)$ specifiable cardinals has order type $(2^{\lambda^{\underset{\smile}{\varkappa}}})^+$. By the discussion following the proof of Corollary 2n, it follows that $\{[(2^{\lambda^{\underset{\smile}{\varkappa}}})^+]_{M_{U(\varkappa-\lambda,2^{\gamma^{\underset{\smile}{\varkappa}}})}}: \gamma$ is $(\varkappa-\lambda)$ specifiable$\}$ is cofinal below $(2^{\lambda^{\underset{\smile}{\varkappa}}})^+$. Since $i_U(\varkappa) < (2^{\lambda^{\underset{\smile}{\varkappa}}})^+$, we can pick some $(\varkappa-\lambda)$ specifiable cardinal γ such that $i_U(\varkappa) < [(2^{\lambda^{\underset{\smile}{\varkappa}}})^+]_{M_{U(\varkappa-\lambda,2^{\gamma^{\underset{\smile}{\varkappa}}})}}$. By the inaccessibility of $i_{U(\varkappa-\lambda,2^{\gamma^{\underset{\smile}{\varkappa}}})}(\varkappa)$ in $M_{U(\varkappa-\lambda,2^{\gamma^{\underset{\smile}{\varkappa}}})}$, it follows that $i_U(\varkappa) < i_{U(\varkappa-\lambda,2^{\gamma^{\underset{\smile}{\varkappa}}})}(\varkappa)$. Then, by the definition of $U(\varkappa-\lambda, 2^{\gamma^{\underset{\smile}{\varkappa}}})$, we conclude that U is not $2^{\gamma^{\underset{\smile}{\varkappa}}}$-extendible. Contradiction. ■

We next show that $\{U \in \mathrm{Nuf}_\varkappa(\lambda)$: U is extendible$\}$ is cofinal in the $<_0$ ordering or, equivalently, that $I\{U \in \mathrm{Nuf}_\varkappa(\lambda)$: U is extendible$\}$ is cofinal below $(2^{\lambda^{\underset{\smile}{\varkappa}}})^+$.

First, in order to avoid what appears to be a possible source of confusion, we point out that Theorem 4a does not tell us that if $U \in \mathrm{Nuf}_\varkappa(\lambda)$ is above (in the $<_0$ ordering) every element of every A_γ for γ a $(\varkappa-\lambda)$ specifiable cardinal, then U is extendible. For a fixed $(\varkappa-\lambda)$ specifiable cardinal γ, it is not hard to see, using the definition of $U(\varkappa-\lambda, 2^{\gamma^{\underset{\smile}{\varkappa}}})$ and reasoning as in the proof of Theorem 4a, that for any $U \in \mathrm{Nuf}_\varkappa(\lambda)$, if $i_U(\varkappa) < [(2^{\lambda^{\varkappa}})^+]_{M_{U(\varkappa-\lambda,2^{\gamma^{\underset{\smile}{\varkappa}}})}}$, then U is not $2^{\lambda^{\underset{\smile}{\varkappa}}}$-extendible. On the other hand, we do not know whether there exist $U \in \mathrm{Nuf}_\varkappa(\lambda)$ such that $i_U(\varkappa) > [(2^{\lambda^{\underset{\smile}{\varkappa}}})^+]_{M_{U(\varkappa-\lambda,2^{\lambda^{\underset{\smile}{\varkappa}}})}}$ and U is not $2^{\gamma^{\underset{\smile}{\varkappa}}}$-extendible. This is related to Open Questions 2o and 2r.

THEOREM 4b. *$I\{U \in \mathrm{Nuf}_\varkappa(\lambda)$: U is extendible$\}$ is cofinal below $(2^{\lambda^{\underset{\smile}{\varkappa}}})^+\}$.*

Proof. As in the proof of the claim in Theorem 4a, for each $U \in \mathrm{Nuf}_\varkappa(\lambda)$ which is not extendible, let δ_U be the least cardinal δ such that U is not δ-extendible. Let η be any strong limit cardinal which is bigger than $\sup\{\delta_U$: $U \in \mathrm{Nuf}_\varkappa(\lambda)$ and U is not extendible$\}$. Pick a minimal $W \in \mathrm{Nuf}_\varkappa(2^{\eta^{\underset{\smile}{\varkappa}}})$. I.e., pick $W \in \mathrm{Nuf}_\varkappa(2^{\eta^{\underset{\smile}{\varkappa}}})$ such that for any $U \in \mathrm{Nuf}_\varkappa(2^{\eta^{\underset{\smile}{\varkappa}}})$, $W \leqslant_0 U$. Since M_W is closed under $2^{\eta^{\underset{\smile}{\varkappa}}}$ sequences $M_W \models \varkappa$ is η-supercompact. By the minimality assumption on W, it follows (by Lemma 2m) that $M_W \models \varkappa$ is not $2^{\eta^{\underset{\smile}{\varkappa}}}$-supercompact. In addition, we note that for any cardinal δ with $\varkappa \leqslant \delta < \eta$, $M_W \models \varkappa$ is $2^{\delta^{\underset{\smile}{\varkappa}}}$-supercompact. This is true since η is a strong limit cardinal and thus $2^{\delta^{\underset{\smile}{\varkappa}}} < \eta$. It follows that $M_W \models \eta$ is the least cardinal δ such that $\varkappa$ is δ-supercompact but not $2^{\delta^{\underset{\smile}{\varkappa}}}$-supercompact.

Next, consider the elementary embedding k: $M_{W\upharpoonright\lambda} \to M_W$. Since $M_W \models \varkappa$ is not supercompact, it follows that $M_{W\upharpoonright\lambda} \models \varkappa$ is not supercompact. Thus, for some ordinal α, $M_{W\upharpoonright\lambda} \models \alpha$ is the least cardinal such that $\varkappa$ is α-supercompact but not $2^{\alpha^{\underset{\smile}{\varkappa}}}$-supercompact. By elementarity, we must have $k(\alpha) = \eta$.

We have shown that the hypotheses of Lemma 2c have been satisfied. By part a of the lemma, $I\{U \in \mathrm{Nuf}_\varkappa(\lambda)$: U is η-extendible$\}$ is cofinal below $(2^{\lambda^{\underset{\smile}{\varkappa}}})^+$. But, by the choice of η, if $U \in \mathrm{Nuf}_\varkappa(\lambda)$ is η-extendible, then U is extendible. Thus, $I\{U \in \mathrm{Nuf}_\varkappa(\lambda)$: U is extendible$\}$ is cofinal below $(2^{\lambda^{\underset{\smile}{\varkappa}}})^+$. ■

We point out that minimality was used in the proof of Theorem 4b on a different "level" than previously. In particular, it need not be true that for W as in the theorem, $W \upharpoonright \lambda = U(\varkappa-\lambda, 2^{\eta^{\underset{\smile}{\varkappa}}})$.

As a corollary, we get an alternate proof of a theorem due to Magidor ([3], p. 191).

COROLLARY 4c. *There are (at least) $(2^{\lambda^{\underset{\smile}{\varkappa}}})^+$ many elements of $\mathrm{Nuf}_\varkappa(\lambda)$ which are extendible.*

Proof. It follows immediately from the theorem and the regularity of $(2^{\lambda^{\underset{\smile}{\varkappa}}})^+$ that $|I\{U \in \mathrm{Nuf}_\varkappa(\lambda)$: U is extendible$\}| = (2^{\lambda^{\underset{\smile}{\varkappa}}})^+$. Thus $|\{U \in \mathrm{Nuf}_\varkappa(\lambda)$: U is extendible$\}| \geqslant (2^{\lambda^{\underset{\smile}{\varkappa}}})^+$. ■

We conclude by asking whether Corollary 4c can be improved.

OPEN QUESTION 4d. Can it be shown that there exist $2^{2^{\lambda^{\underset{\smile}{\varkappa}}}}$ many elements of $\mathrm{Nuf}_\varkappa(\lambda)$ which are extendible?

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