On complexity of metric spaces

by

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Abstract. In this paper we study the question whether the αth "slowed down" Ginsburg–Isbell derivative of a given metric uniformity is fine. We characterize the metric spaces with this property and generalize the work begun in [5]. We state related results for infinite products and hyperspaces of complete metric spaces.

1. Introduction. In their fundamental 1959 paper [1], S. Ginsburg and J. Isbell proved that the locally fine coreflection of a complete metric uniformity is fine. The locally fine coreflection of a uniformity μ is constructed combinatorially by forming the successive Ginsburg–Isbell derivatives $\mu^{(\alpha)}$ of μ. Hence, it is natural to try to characterize those complete spaces whose αth derivatives for some given ordinal α is fine. This and some other questions ([4) connected with the Ginsburg–Isbell derivative are facilitated by slightly changing the definition of the original derivatives (as was done in [5]) to make them more suitable for inductive purposes. In [5] the first author characterized the metric spaces for which there is a countable ordinal α such that their αth derivative is fine; here we will extend this result to the uncountable case. In the proof of the main result the technique of special trees from [12] and [13] is used. The main theorem is then used to give estimates for the ranks of infinite products and hyperspaces. It should be noted that the main result (Theorem 3.1) is a sort of a reduction theorem since it refers to the notion of derivative which seems impossible to dispose with. We also remark that our result is related to earlier investigations, see for example [10], [2], [1], [14] and [3].

2. Preliminaries. First we shall give some preliminary definitions. The reader is referred to [6] and [7] for information on uniform spaces. Let X be a set and let μ and ν be filters of coverings of X, ordered by the relation of refinement. Then ν/μ denotes the collection of all covers of X that have a refinement of the form $\{U_i \cap V_j^i\}$, where $\{U_i\} \in \mu$ and for each i, $\{V_j^i\}$ belongs to ν. Then ν/μ is a filter

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of coverings of X. Following [5], we define the derivatives $\mu^{(\alpha)}$ of a uniformity μ on X by setting

$$\mu^{(0)} = \mu; \quad \mu^{(\alpha+1)} = \mu^{(\alpha)}/\mu \quad \text{and} \quad \mu^{(\beta)} = \bigcup\{\mu^{(\alpha)}: \alpha < \beta\}$$

if β is a limit ordinal. (Note that these derivatives are different from the Ginsburg–Isbell derivatives: the expression $\mu^{(\alpha)}/\mu^{(\alpha)}$ is replaced by $\mu^{(\alpha)}/\mu$.) It was proved in [5] that if τ is a regular infinite ordinal, $\alpha < \tau$ and the αth Ginsburg–Isbell derivative of μ is fine, then there is a $\beta < \tau$ such that $\mu^{(\beta)}$ is fine.

There is an $\alpha \in \mathrm{Ord}$ such that $\mu^{(\alpha)} = \mu^{(\alpha+1)} = \lambda\mu$, where $\lambda\mu$ denotes the Ginsburg–Isbell locally fine coreflection [2]. If $\lambda\mu X = FX$ (= the fine uniform space associated with X), then μX is called ranked and we define rank

$$\operatorname{rank}(\mu X) = \inf\{\alpha \in \mathrm{Ord}: \mu^{(\alpha)}X = FX\}$$

provided that $|X| > 1$. If $|X| \leqslant 1$, let $\operatorname{rank}(\mu X) = -1$. For the purpose of this paper, the symbol ϱX refers to a metric space and ϱ denotes both the metric and the associated metric uniformity.

A special tree is a partially ordered set $\mathscr{T} = (T, \leqslant)$ with a (unique) minimal element such that branch (i.e. a subset linearly ordered by $\leqslant$) is finite and for any $x \in T$, the set of $\leqslant$-predecessors of x is well-ordered. Given an element $x \in T$, let $\tilde{S}(x) = \{y \in T: y > x\}$ and let $S(x) = \tilde{S}(x) - \bigcup\{\mathscr{T}(y): y \in T(x)\}$. Thus, $S(x)$ is the set of all immediate successors of x in $\mathscr{T}$. Define $\operatorname{End}(\mathscr{T}) = \{x \in T: S(x) = \emptyset\}$. If X is a subset of T, then $\mathscr{T}|X$ denotes the restriction $(X, \leqslant|X^2)$ of $\mathscr{T}$ to X. In particular, let $\mathscr{T}(x) = \mathscr{T}|(\mathscr{T}(x) \cup \{x\})$ for all $x \in T$. Given a special tree $\mathscr{T}$, define

$$T^{(0)} = T, \quad \mathscr{T}^{(0)} = \mathscr{T};$$
$$T^{(\alpha+1)} = T^{(\alpha)} - \operatorname{End}(\mathscr{T}^{(\alpha)}), \quad \mathscr{T}^{(\alpha+1)} = \mathscr{T}|T^{(\alpha+1)}$$

and

$$\mathscr{T}^{(\beta)} = \mathscr{T}|\bigcap\{T^{(\alpha)}: \alpha < \beta\}$$

if β is a limit ordinal. Given in addition a uniform space μX, a $\langle\mathscr{T}, \mu\rangle$-map is any mapping $\varphi: T \to P(X)$ such that $\{\varphi(y): y \in S(x)\}$ is a uniform cover of $\varphi(x)$ for each $x \in T^{(1)}$. Special trees were earlier applied in [12] and [13]. The following notion yields a connection between ranks and special trees.

Definition. If $\mathscr{T}$ is a special tree, then the *length complexity* $l(\mathscr{T})$ of $\mathscr{T}$ is the unique ordinal α such that $|\mathscr{T}^{(\alpha)}| = 1$.

Remark. For each ordinal $\alpha \geqslant 1$, let $P(\alpha)$ denote the following statement: if μX is a uniform space, $\mathscr{T}$ is a special tree with $l(\mathscr{T}) < \alpha$, $\varphi: T \to P(X)$ is a $\langle\mathscr{T}, \mu\rangle$-map and $\varphi[\operatorname{End}(\mathscr{T})]$ is a cover of X, then $\varphi[\operatorname{End}(\mathscr{T})] \in \mu^{(\alpha)}$. Clearly $P(\alpha)$ is valid for $\alpha = 1$. Suppose that $P(\alpha)$ is valid for $1 \leqslant \alpha \leqslant \beta$. Let μX, $\mathscr{T}$, and φ be as above and let $\tau = l(\mathscr{T}) < \beta$. Denote by p the unique minimal element of $\mathscr{T}$. Note that $l(\mathscr{T}(x)) < \tau$ for all $x \in S(p)$. Thus, by the inductive hypothesis (applied to the subspaces $\varphi(x)$ of μX) there exist covers $\mathscr{V}_x \in \mu^{(\tau)}$, $x \in S(p)$, such that $\mathscr{V}_x|\varphi(x) < \varphi[\operatorname{End}(\mathscr{T}(x))]$. On the other hand, by the definition of φ, $\{\varphi(x): x \in S(p)\}$ is a uniform cover of μX. Therefore,

$$\{V \cap \varphi(x): x \in S(p), V \in \mathscr{V}_x\} < \varphi[\operatorname{End}(\mathscr{T})]$$

belongs to $\mu^{(\tau)}/\mu = \mu^{(\tau+1)} \subset \mu^{(\beta)}$, as required. This proves that $P(\alpha)$ is valid for all $\alpha \geqslant 1$. Hence, in order to show that $\operatorname{rank}(\mu X) \leqslant \alpha$, it is enough to prove that for each open cover $\mathscr{V}$ of X there is a special tree $\mathscr{T}$ and a $\langle\mathscr{T}, \mu\rangle$-map φ such that $l(\mathscr{T}) < \alpha$ and $\varphi[\operatorname{End}(\mathscr{T})] < \mathscr{V}$.

Let α be a cardinal number. A tree $\mathscr{T}$ is called an α-tree provided that each element of $\mathscr{T}$ has at most α immediate successors. The following result was stated as Proposition 5(v) in [13] without proof—the proof will be given here. (We assume that α is an infinite cardinal.)

Proposition 2.1. *If $\mathscr{T}$ is a special α-tree and α is an infinite cardinal, then $l(\mathscr{T}) < \alpha^+$.*

Proof. First note that $|T| < \alpha^+$ because each element of $\mathscr{T}$ has less than α^+ successors, the branches of $\mathscr{T}$ are finite and α^+ is an infinite regular cardinal. Define a map $f: T \to \mathrm{Ord}$ so that for each $p \in T$, $f(p)$ is the unique ordinal β for which $p \in \operatorname{End}(\mathscr{T}^{(\beta)})$. We claim that $f(p) < \alpha^+$ for all $p \in T$. Indeed, if the claim were not true, then there would exist a maximal $p \in T$ such that $f(p) \geqslant \alpha^+$. Now $p \in \operatorname{End}(\bigcap\{\mathscr{T}^{(f(q))}: q > p\})$ and hence $f(p) = \sup\{f(q)+1: q > p\} < \alpha^+$, since $f(q) < \alpha^+$ for every $q > p$ and $|\tilde{S}(p)| < \alpha^+$: a contradiction. In particular, $l(\mathscr{T}) = f(p) < \alpha^+$ for the unique minimal element p of $\mathscr{T}$.

3. The main result. Let α be an ordinal number. In case α is a successor ordinal, let $\operatorname{cf}\alpha = 0$. Otherwise define $\operatorname{cf}\alpha = \inf\{|A|: A \subset \mathrm{Ord},\ \alpha \notin A \text{ and } \alpha = \sup A\}$. If X is a topological space and $\varkappa$ is an infinite cardinal, then X is $\varkappa$-compact provided that every open cover of X has a subcover of cardinality $< \varkappa$. The following theorem is the main result in our paper.

Theorem 3.1. *Let ϱX be a complete metric space, let $\alpha \in \mathrm{Ord}$ and let $\varkappa = \max(\operatorname{cf}\alpha, \omega)$. The following statements are equivalent:*

(i) $\operatorname{rank}(\varrho X) \leqslant \alpha$;

(ii) *there is a closed $\varkappa$-compact subspace E of X such that for each $\varepsilon > 0$ there exist an $r(\varepsilon) > 0$ and a $\beta(\varepsilon) < \alpha$ and with $\operatorname{rank}(\bar{B}_{r(\varepsilon)}(x)) \leqslant \beta(\varepsilon)$ for all $x \in X - B_\varepsilon(E)$.*

Proof. To show that (i) implies (ii), let $\operatorname{rank}(\varrho X) \leqslant \alpha$. We shall assume that α is a limit ordinal, since the case of a successor ordinal is essentially the same as the case of countable α in [5]. Consider the cover $\mathscr{B}(1/(n+1))$ of X consisting of all closed balls $\bar{B}_{1/(n+1)}(x)$ with ϱ-radius $1/(n+1)$.

Observation. *If $\mathscr{M} \subset \overline{\mathscr{B}}(1/(n+1))$ is uniformly discrete and*

$$\mathscr{M}' = \{\bar{B} \in \mathscr{M}: \operatorname{rank}\bar{B} = \alpha\},$$

then $|\mathscr{M}'| < \varkappa$ and $\sup\{\operatorname{rank}\bar{B}: \bar{B} \in \mathscr{M} - \mathscr{M}'\} < \alpha$.

Proof of Observation. Suppose that $|\mathscr{M}'| \geqslant \varkappa$. Then there is a map $f: \mathscr{M}' \to \alpha$ such that $\alpha = \sup f[\mathscr{M}']$. For each $\bar{B} \in \mathscr{M}'$, $\operatorname{rank}\bar{B} = \alpha$ and hence there is an open cover $\mathscr{V}_B$ of $\bar{B}$ such that $\mathscr{V}_B$ is not refined by any cover $\mathscr{W}|\bar{B}$, where belongs to $\varrho^{(f(\bar{B}))}$. Since $\mathscr{M}'$ is uniformly discrete it is easily seen that we can find a disjoint collection $\{\mathscr{V}'_B\colon \bar{B}\in\mathscr{M}'\}$ of open families $\mathscr{V}'_B$ of X such that $\mathscr{V}_B=\mathscr{V}'_{\mathscr{B}}|\bar{B}$. Let

$$\mathscr{W}=\bigcup\{\mathscr{V}'_B\colon \bar{B}\in\mathscr{M}'\}\cup\{X-\bigcup\{\bar{B}\colon \bar{B}\in\mathscr{M}'\}\}.$$

Then $\mathscr{W}$ is an open cover of X and thus $\mathscr{W}\in\varrho^{(\beta)}$ for some $\beta<\alpha$ since $\operatorname{rank}(\varrho X)\leqslant\alpha$ and α is a limit ordinal. Choose a $\bar{B}\in\mathscr{M}'$ for which $f(\bar{B})>\beta$; we obtain a contradiction since $\mathscr{W}|\bar{B}$ refines $\mathscr{V}_B$ and $\mathscr{W}\in\varrho^{(f(\bar{B}))}$. The proof of the latter statement is similar but simpler.

It follows from Observation that for each $n\in\omega$ there is an $\mathscr{M}_n\subset\bar{\mathscr{B}}(2^{-n-1})$ and a $\beta(n)<\alpha$ such that

(1) $|\mathscr{M}_n|<\varkappa$,

(2) $\mathscr{M}_n$ is uniformly discrete,

(3) $\operatorname{rank}\bar{B}\leqslant\beta(n)$ whenever $\bar{B}\in\bar{\mathscr{B}}(2^{-n-1})$ and $\bar{B}\cap B_{2^{-n-1}}(\bigcup(\mathscr{M}_n))=\emptyset$.

The desired collections $\mathscr{M}_n$ will be constructed by induction as follows. Let $f\colon\varkappa\to\alpha$ be a map with $\alpha=\sup\{f(\beta)\colon\beta<\varkappa\}$. We first construct $\mathscr{M}_0$. Let $\bar{B}_{0,0}$ be an arbitrary element of $\bar{\mathscr{B}}(2^{-1})$. Suppose that $\bar{B}_{\tau,0}$ is defined for all $\tau<\sigma$, where $\sigma<\varkappa$, and let $\mathscr{B}_{\sigma,0}=\{\bar{B}\in\bar{\mathscr{B}}(2^{-1})\colon \varrho(\bar{B},\bigcup_{\tau<\sigma}\bar{B}_{\tau,0})\geqslant 2^{-1}\}$. If $\operatorname{rank}|\mathscr{B}_{0,0}$ has no upper bound $<\alpha$, choose a $\bar{B}_{\sigma,0}\in\mathscr{B}_{\sigma,0}$ such that $\operatorname{rank}\bar{B}_{\sigma,0}>f(\sigma)$. Otherwise, stop here and let $\mathscr{M}_0=\{\bar{B}_{\tau,0}\colon\tau<\sigma\}$. Suppose that $\mathscr{M}_n$ is defined and let Z_n denote the set of all centers of balls from $\mathscr{M}_n$. Let $\bar{B}_{0,n+1}$ be an arbitrary element of $\bar{B}(2^{-n-2})$ such that $\bar{B}_{0,n+1}\subset\bar{B}_{2^{-n}}(Z_n)$. If $\bar{B}_{\tau,n+1}$ is defined for all $\tau<\sigma$, $\sigma<\varkappa$, let $\mathscr{B}_{\sigma,n+1}=\{\bar{B}\in\bar{\mathscr{B}}(2^{-n-2})\colon \bar{B}\subset\bar{B}_{2^{-n}}(Z_n)$ and $\varrho(\bar{B},\bigcup_{\tau<\sigma}\bar{B}_{\tau,n+1})\geqslant 2^{-n-2}\}$. Then proceed as in the case $n=0$.

Finally, let

$$E=\overline{\bigcup\{Z_n\colon n\in\omega\}}.$$

Then E is $\varkappa$-precompact. For, if $\mathscr{U}$ is a uniform cover of E, there is an $n\in\omega$ such that the balls $\bar{B}_{2^{-n+2}}(z)$, $z\in Z_n$, partially refine $\mathscr{U}$. The relation $Z_{n+1}\subset\bar{B}_{2^{-n+1}}(Z_n)$ implies that $\mathscr{U}$ has a subcover of cardinality $\leqslant|Z_0|+\ldots+|Z_n|<\varkappa$. Thus, E is $\varkappa$-compact, since it is complete. (Completeness is really needed for the case $\varkappa=\omega$ only.) Clearly E witnesses that (ii) is satisfied.

To prove that (ii) implies (i), we consider two cases.

Case 1. $\varkappa=\omega$. Then E is compact. If $\mathscr{V}$ is any open cover of X, then $\mathscr{V}$ is uniform over some uniform neighbourhood of E and it is not difficult to see that $\mathscr{V}\in\varrho^{(\beta)}$ for some $\beta<\alpha$.

Case 2. $\varkappa>\omega$. Put $\beta=\sup\{\beta(1/(n+1))\colon n\in\omega\}$. Then $\beta<\alpha$. As E is $\varkappa$-compact and $\varkappa>\omega$, there is a dense subset D of E with $|D|<\varkappa$. Let $\mathscr{V}$ be an open cover of X. We shall show that there is a special tree $\mathscr{T}^*$ with a $\langle\mathscr{T}^*,\varrho\rangle$-map φ such that $l(\mathscr{T}^*)<\alpha$ and $\varphi[\operatorname{End}(\mathscr{T}^*)]<\mathscr{V}$. Then the Remark of Section 2 applies to prove that $\operatorname{rank}(\varrho X)\leqslant\alpha$. We will first construct a tree $\mathscr{T}$ by induction.

To start with, let $T_0=\{X\}$ and let $\leqslant_0$ be the trivial (linear) ordering of T_0 and let $\mathscr{T}_0=(T_0,\leqslant_0)$.

For $n>0$, we shall construct trees $\mathscr{T}_n=(\mathscr{T}_n,\leqslant_n)$ whose points are $(m+1)$-tuples $\langle\bar{B}_m,\ldots,\bar{B}_1,X\rangle$ $(1\leqslant m\leqslant n)$ of subsets of X that satisfy the following properties:

(i) $\langle\bar{B}_m,\ldots,\bar{B}_1,X\rangle<_n\langle\bar{B}_{m+1},\bar{B}_m,\ldots,\bar{B}_1,X\rangle$ for $1\leqslant m<n$;

(ii) if $1\leqslant m<n$ and $\langle\bar{B}_m,\ldots,\bar{B}_1,X\rangle$ is an end point of $\mathscr{T}_n$, then the restriction of $\mathscr{V}$ to $\bar{B}_m$ belongs to the βth derivative of $\varrho|\bar{B}_m$;

(iii) $\bar{B}_{m+1}\subset\bar{B}_m$ and $\operatorname{diam}\bar{B}_{m+1}<1/m$ for $1\leqslant m<n$;

(iv) each element of $\mathscr{T}_n^{(1)}$ has only $|D|$ immediate successors in $\mathscr{T}_n^{(2)}$.

Step 1. $\varepsilon=1/2$. By assumption there is an $r(1)\in\,]0,\varepsilon[$ such that (ii) is satisfied for $\varepsilon=1/2$. Define

$$M_1=\{\bar{B}_{r(1)}(x)\colon x\in X-B_{1/2}(E)\}$$
$$N_1=\{\bar{B}_{1/2}(d)\colon d\in D\}.$$

Let T_1 be the union of T_0 and $\{\langle\bar{B},X\rangle\colon \bar{B}\in M_1\cup N_1\}$. Extend $\leqslant_0$ to $\leqslant_1$ on T_1 by setting $A>_1 A'$ iff A is a 2-tuple and $A'=X$. Let $\mathscr{T}_1=(T_1,\leqslant_1)$.

Step $n+1$. $\varepsilon=1/(n+1)$. Suppose that $\mathscr{T}_n=(T_n,\leqslant_n)$ is defined. Let

$$\mathscr{B}_n=\{\langle\bar{B}_n,\ldots,\bar{B}_1,X\rangle\in T_n\colon \mathscr{V}|\bar{B}_n\notin(\varrho|\bar{B}_n)^{(\beta)}\}.$$

Choose an $r(n+1)\in\,]0,1/(n+1)[$ satisfying (ii). Given an $(n+1)$-tuple $S=\langle\bar{B}_n,\ldots,\bar{B}_1,X\rangle\in\mathscr{B}_n$, put

$$M_{n+1}(S)=\{\bar{B}_{r(n+1)}(x)\cap\bar{B}_n\colon x\in\bar{B}_n-B_{1/(n+1)}(E)\}$$
$$N_{n+1}(S)=\{\bar{B}_{1/(n+1)}(d)\cap\bar{B}_n\colon d\in D\}.$$

For each such an S, define

$$T_{n+1}(S)=\{\langle\bar{B},S\rangle\colon \bar{B}\in M_{m+1}(S)\cup N_{n+1}(S),\ \bar{B}\neq\emptyset\}$$

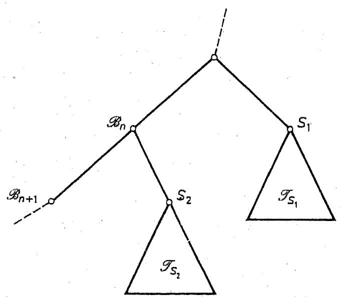
and let T_{n+1} be the union of T_n and $\bigcup\{T_{n+1}(S)\colon S\in\mathscr{B}_n\}$. We extend $\leqslant_n$ to $\leqslant_{n+1}$ by setting $A>_{n+1}S$ iff $S\in\mathscr{B}_n$ and $A\in T_{n+1}(S)$. Let

$$T=\bigcup\{T_n\colon n\in\omega\},\ \leqslant\ =\bigcup\{\leqslant_n\colon n\in\omega\}$$

and put $\mathscr{T}=(T,\leqslant)$. As ϱX is complete and $\mathscr{V}$ is open, the method used to prove VII 9. in [7] can be applied here to show that $\mathscr{T}$ is special. Indeed, if $\mathscr{T}$ were not special, then it would contain an infinite branch and hence there would be a sequence $\{\langle\bar{B}_n,\ldots,\bar{B}_1,X\rangle\colon n\in\omega\}$ from $\mathscr{T}$. Now $\bar{B}_n\supset\bar{B}_{n+1}\neq\emptyset$ for every $n\in\omega$ and $\operatorname{diam}\bar{B}_n\to 0$, whence there would be a point $p\in X$ such that $\{p\}=\bigcap\{\bar{B}_n\colon n\in\omega\}$ and hence there would exist an $n\in\omega$ and a $V\in\mathscr{V}$ with $\bar{B}_n\subset V$, which would imply $\langle\bar{B}_n,\ldots,\bar{B}_1,X\rangle\notin\mathscr{B}_n$. Furthermore, if $S=\langle\bar{B}_n,\ldots,\bar{B}_1,X\rangle\in\operatorname{End}(\mathscr{T})$, then there is a special tree $\mathscr{T}_S=(T_S,\leqslant_S)$ and a $\langle T_S,\varrho|\bar{B}_n\rangle$-map $\varphi_S\colon T_S\to P(\bar{B}_n)$ such that $l(T_S)\leqslant\beta$ and $\varphi_S[\operatorname{End}(\mathscr{T}_S)]<\mathscr{V}|\bar{B}_n$. Let

$$T^*=T\cup\{\langle x,S\rangle\colon S\in\operatorname{End}(\mathscr{T})\ \text{and}\ x\in T_S\}$$

and extend $\leqslant$ to $\leqslant^*$ on T^* by setting $\langle x, S\rangle^* > \langle x', S'\rangle$ iff $S = S'$, $x, x' \in T_S$ and $x >_S x'$; and let $\langle x, S\rangle^* > S$ for each $S \in \mathrm{End}(\mathscr{T})$ and every $x \in T_S$. Then $\mathscr{T}^* = (T^*, \leqslant^*)$ is a special tree. (See Figure.) Define $\varphi\colon T^* \to P(X)$ in an obvious



way by setting $\varphi(\langle x, S\rangle) = \varphi_S(x)$ if $S \in \mathrm{End}(\mathscr{T})$ and $x \in T_S$, and if $S \in T$, let $\varphi(S)$ be the first member of S. Since $\mathscr{T}^{(1)}$ is a $|D|$-tree (each element has at most $|D|$ immediate successors), Proposition 2.1. Implies that $l(\mathscr{T}) < |D|^+$. Recall that $|D| < \varkappa$ and $\beta < \alpha$. Hence $l(\mathscr{T}^*)] \leqslant \beta + l(\mathscr{T}) < \alpha$, because $|D| < \mathrm{cf}(\alpha)$. Clearly $\varphi[\mathrm{End}(\mathscr{T}^*)] < \mathscr{V}$ and our proof is complete.

4. Some applications of Theorem 3.1. Call a topological space X $C(\varkappa)$-scattered if every nonempty closed subset F of X contains a point with a $\varkappa$-compact neighbourhood in F. The following corollary can be proved by transfinite induction.

PROPOSITION 4.1. *Let ϱX be a metric space such that* $\mathrm{rank}(\varrho X) \leqslant \alpha$ *and let* $\varkappa = \max(\omega, |\alpha|)$. *Then X is $C(\varkappa)$-scattered.*

Let ϱX be a metric space with $\mathrm{rank}(\varrho X) = 0$. It was shown in [5] that $\mathrm{rank}(\varrho X)^n \leqslant n-1$ for each $n \in \omega - \{0\}$. One might be tempted to conjecture that $\mathrm{rank}(\varrho X)^\omega \leqslant \omega_1$ or at least $\mathrm{rank}(\varrho X)^\omega \leqslant 2^\omega$. However, the actual situation is much worse. In fact, there is no upper bound for the ranks of countable powers of metric spaces of the rank zero. Indeed, let $\varkappa \geqslant \omega$ and let $D_\varkappa$ be a uniformly discrete space of power $\varkappa$. Then $\mathrm{rank}(D_\varkappa^\omega) \geqslant \varkappa^+$. For, if $\alpha = \mathrm{rank}(D_\varkappa^\omega) < \varkappa^+$, then $|\alpha| \leqslant \varkappa$ and thus $D_\varkappa$ would be $C(\varkappa)$-scattered, by Proposition 4.1. But then some point of $D_\varkappa^\omega$ would have a $\varkappa$-compact neighbourhood and consequently $D_\varkappa$ would be $\varkappa$-compact, which is impossible. On the other hand, $\mathrm{rank}(D_\varkappa^\omega) \leqslant \varkappa^+$, since $D_\varkappa^\omega$ is $\varkappa^+$-compact. Hence, $\mathrm{rank}(D_\varkappa^\omega) = \varkappa^+$.

These remarks enable us to determine the rank of the countably infinite power of any complete metric space. If μX is a uniform space, let $\delta(\mu X)$ denote the least cardinal $\varkappa$ such that $|D| < \varkappa$ for each uniformly discrete subsets D of μX. Let us first state two lemmas.

LEMMA 4.2. *Let ϱX be a complete metric space with* $\mathrm{cf}(\delta(\varrho X)) > \omega$. *Then* $\mathrm{rank}(\varrho X) \leqslant \delta(\varrho X)$.

Proof. We must show that for any open cover $\mathscr{V}$ of X there is a special tree $\mathscr{T}$ and a $\langle \mathscr{T}, \varrho\rangle$-map $\varphi\colon T \to P(X)$ such that $\varphi[\mathrm{End}(\mathscr{T})] < \mathscr{V}$ and $l(\mathscr{T}) < \delta(\varrho X)$. To this end, let $\mathscr{V}$ be an open cover of X. For each $n \in \omega$, let $\mathscr{B}_n$ be an open cover of ϱX by balls of radius 2^{-n}. As $\mathscr{B}_n$ is a uniform cover, we can choose a uniform subcover $\mathscr{B}'_n$ with $\varkappa_n = |\mathscr{B}'_n| < \delta(\varrho X)$. The points of $\mathscr{T}$ are finite ordered $(n+1)$-tuples $\langle B_n, \ldots, B_1, X\rangle$, where $B_i = B'_1 \cap \ldots \cap B'_i \neq \emptyset$ and $B'_i \in \mathscr{B}'_i$ for $1 \leqslant i \leqslant n$, and $B'_{n-1} \subset V$ for no element V of $\mathscr{V}$. The tree $\mathscr{T}$ is ordered in the same way as in the proof of 3.1. Thus, we set $\langle B_{n+1}, \ldots, B_1, X\rangle \geqslant \langle B_n, \ldots, B_1, X\rangle \geqslant \langle X\rangle$ for all elements $\langle B_{n+1}, \ldots, B_1, X\rangle$ of $\mathscr{T}$. Since ϱX is complete, we can see in the proof of 3.1. that $\mathscr{T}$ is special and that by defining $\varphi(\langle B_n, \ldots, B_1, X\rangle) = B_n$ we obtain a $\langle \mathscr{T}, \varrho\rangle$-map φ such that $\varphi[\mathrm{End}(\mathscr{T})] < \mathscr{V}$. Let $\varkappa = \sup\{\varkappa_n\colon n \in \omega\}$. Then $\varkappa < \delta(\varrho X)$. Now $\mathscr{T}$ is a $\varkappa$-tree and thus by Proposition 2.1., the length complexity of $\mathscr{T}$ is less than $\varkappa^+ \leqslant \delta(\varrho X)$.

LEMMA 4.3. *Let ϱX be a noncompact complete metric space with* $\mathrm{cf}(\delta(\varrho X)) = \omega$. *Then X^ω is not $C(\delta(\varrho X))$-scattered.*

Proof. Since $\mathrm{cf}(\delta(\varrho X)) = \omega$, there exist uniformly discrete subsets X_n of X such that $\sup\{|X_n|\colon n \in \omega\} = \delta(\varrho X)$. Since ϱX is noncompact and complete, there is an infinite uniformly discrete subset $Y = \{y_n\colon n \in \omega\}$ of ϱX. We shall show that X^ω is nowhere locally $\delta(\varrho X)$-compact. Let $p \in X^\omega$ and let U be a neighbourhood of p in X^ω. Then there exists an embedding of X^ω onto a closed subspace of U. Thus, it is enough to show that X^ω is not $\delta(\varrho X)$-compact. To show this, let $\pi_n\colon X^\omega \to X$ denote the nth projection and define a subset S of X^ω as the set of all points $x \in X^\omega$ such that

(i) $\pi_0(x) \in Y$;
(ii) if $\pi_0(x) = y_n$, then $\pi_1(x) \in X_n$;
(iii) $\pi_n(x) = \pi_0(x)$ for $n \geqslant 2$.

It is easy to see that S is a discrete closed subset of X^ω with $|S| = \delta(\varrho X)$ and hence X^ω is not $\delta(\varrho X)$-compact, as desired.

PROPOSITION. 4.4. *Let ϱX be a noncompact complete metric space. If* $\mathrm{cf}(\delta(\varrho X)) < \omega$, *then* $\mathrm{rank}(\varrho X)^\omega = \delta(\varrho X)$, *if* $\mathrm{cf}(\delta(\varrho X)) = \omega$, *then* $\mathrm{rank}(\varrho X)^\omega = \delta(\varrho X)^+$.

Proof. For each $\varkappa < \delta(\varrho X)$ there is a uniformly discrete subset $D_\varkappa$ of ϱX and consequently a uniform embedding $D_\varkappa^\omega \to (\varrho X)^\omega$. Thus,

$$\mathrm{rank}(\varrho X)^\omega \geqslant \sup\{\mathrm{rank}(D_\varkappa^\omega)\colon \varkappa < \delta(\varrho X)\} = \sup\{\varkappa^+\colon \varkappa < \delta(\varrho X)\} = \delta(\varrho X).$$

If $\mathrm{cf}(\delta(\varrho X)) > \omega$, then $\delta((\varrho X)^\omega) = \delta(\varrho X)$ implies by Lemma 4.2. that $\mathrm{rank}(\varrho X)^\omega \leqslant \delta(\varrho X)$. Consequently $\mathrm{rank}(\varrho X)^\omega = \delta(\varrho X)$. On the other hand, suppose that $\mathrm{cf}(\delta(\varrho X)) = \omega$. By Lemma 4.3, X^ω is not $C(\delta(\varrho X))$-scattered and hence by Proposition 4.1, $\mathrm{rank}(\varrho X)^\omega \geqslant \delta(\varrho X)^+$. Now $(\varrho X)^\omega$ is $\delta(\varrho X)^+$-compact and it easily follows from Lemma 4.2. that $\mathrm{rank}(\varrho X)^\omega \leqslant \delta(\varrho X)^+$. Hence, $\mathrm{rank}(\varrho X)^\omega = \delta(\varrho X)^+$.

Now we shall consider hyperspaces of metric spaces. For a metric space ϱX, the hyperspace $K(X)$ of all compact subsets of X can be metrized by the Hausdorff metric $\hat{\varrho}$ given by setting $\hat{\varrho}(A_1, A_2) < \varepsilon$ iff $A_1 \subset B_{\varrho,\varepsilon}(A_2)$ and $A_2 \subset B_{\varrho,\varepsilon}(A_1)$ for each $\varepsilon > 0$. For an element A of $K(X)$, let $\bar{B}^K_\varepsilon(A)$ denote the closed $\hat{\varrho}$-ball of radius ε and center A. E. Michael proved in [9] that if $C \in K(X)$ is compact, then $\bigcup\{A: A \in C\}$ is a compact subset of X. His proof can be modified to prove the following generalization.

LEMMA 4.5. *If $C \in K(X)$ is $\varkappa$-compact, then $\bigcup\{A: A \in C\}$ is a $\varkappa$-compact subset of X.*

On the other hand, if $\varkappa = \omega$ or $\mathrm{cf}(\varkappa) > \omega$ and X is a $\varkappa$-compact metric space, then so is $K(\varrho X)$. (This is far from being true for more general spaces. Indeed, A. Okuyama gives in [11] an example of a cosmic space X for which $K(X)$ is not even paracompact.) The following proposition can in some cases be used to estimate the rank of the hyperspace of compact subsets.

PROPOSITION 4.6. *Let ϱX be a complete metric space and let $\alpha \in \mathrm{Ord}$. Put $\varkappa = \max(\omega, \mathrm{cf}\,\alpha)$. The following statements are equivalent:*

(i) $\mathrm{rank}(K(\varrho X)) \leqslant \alpha$;

(ii) *either X is $\varkappa$-compact or there is an $\varepsilon > 0$ and a $\beta < \alpha$ such that* $\mathrm{rank}(\bar{B}^K_\varepsilon(A)) \leqslant \beta$ *for each* $A \in K(X)$.

Proof. Note that (ii) $\to$ (i) is obvious. To prove that (i) $\to$ (ii), let $\mathrm{rank}(K(\varrho X)) \leqslant \alpha$. By Theorem 3.1 there is a $\varkappa$-compact $C \subset K(X)$ that satisfies the following condition: given an $\varepsilon > 0$, there exist an $r(\varepsilon) > 0$ and a $\beta(\varepsilon) < \alpha$ such that the rank of $\bar{B}^K_{r(\varepsilon)}(A) \subset K(X) - C$ is at most $\beta(\varepsilon)$ for each $A \in Y_\varepsilon = K(X) - B^K_\varepsilon(C)$. Put $E = \bigcup\{A: A \in C\}$. Then E is $\varkappa$-compact, by Lemma 4.3. If X is not $\varkappa$-compact, then $X - \bar{E} \neq \emptyset$. In that case choose an $x_0 \in X - \bar{E}$ and an $\varepsilon > 0$ with $\varrho(E, x_0) > \varepsilon$. If $A \subset B_{\varepsilon/2}(E)$ is compact, then $\hat{\varrho}(A \cup \{x_0\}, C) \geqslant \hat{\varrho}(A \cup \{x_0\}, K(E)) \geqslant \varrho(x_0, E) > \varepsilon$. Hence, rank $\bar{B}^K_{r(\varepsilon)}(A \cup \{x_0\}) \leqslant \beta(\varepsilon)$ for each such an A. Put

$$r = 1/8 \min\{\varepsilon, r(\varepsilon), r(\varepsilon/2)\}.$$

Then $A \subset B_{\varepsilon/2}(E)$ implies $\varrho(x_0, A) > 4r$ and thus map

$$\varphi\colon \bar{B}^K_r(A) \to \bar{B}^K_r(A \cup \{x_0\})$$

given by $\varphi(L) = L \cup \{x_0\}$ is an isometric embedding. Indeed, clearly $\hat{\varrho}(\varphi(L_1), \varphi(L_2)) \leqslant \hat{\varrho}(L_1, L_2)$ for any $L_1, L_2 \in \bar{B}^K_r(A)$, and the conditions $\varphi(L_1) \subset B(\varphi(L_2))$, $\varphi(L_2) \subset B_\eta(\varphi(L_1))$ imply that $L_1 \subset B_\eta(L_2)$ and $L_2 \subset B_\eta(L_1)$ for any $\eta \in [0, 2r]$. Moreover, φ is a closed map since its domain is complete. It follows that $\mathrm{rank}(\bar{B}^K_r(A)) \leqslant \mathrm{rank}(\bar{B}^K_r(A \cup \{x_0\})) \leqslant \beta(\varepsilon)$. On the other hand, if $A \cap (X - B_{\varepsilon/2}(E)) \neq \emptyset$, then $\mathrm{rank}(\bar{B}^K_r(A)) \leqslant \beta(\varepsilon/2)$ by assumption.

COROLLARY 4.7. *The rank of the hyperspace $K(\varrho X)$ is never a nonregular limit ordinal. If $\varkappa$ is a regular ordinal, then $\mathrm{rank}(K(\varrho X)) = \varkappa$ iff X is $\varkappa$-compact.*

Remark. Let $H(\varkappa)$ be the usual hedgehog with $\varkappa$ spines equipped with its standard geodesic metric. Let $H^*(\varkappa)$ be otherwise the same as $H(\varkappa)$ but replace each spine, a copy of the unit interval $[0, 1]$, by $\{1/n: n \in \omega - \{0\}\} \cup \{0\}$. Let $\varkappa \geqslant \omega$. Then $\mathrm{rank}(H^*(\varkappa)) = 0$ but $\mathrm{rank}(K(H^*(\varkappa)) = \varkappa^+$. In general, $\mathrm{rank}(K(\varrho X)) \leqslant \delta(\varrho X)^+$ for any complete metric space ϱX, because $\delta(K(\varrho X)) = \delta(\varrho X)$ for any metric space with $|X| \geqslant \omega$.

Let ϱX be a metric space. Then ϱX is uniformly isomorphic to the bounded metric space σX, where $\sigma(x, y) = \min\{1, \varrho(x, y)\}$ for all $x, y \in X$. It can be shown that the uniform hyperspace $H(\varrho X)$ of all closed subsets of X can be metrized by the Hausdorff metric $\hat{\sigma}$ defined as for $K(X)$. If ϱX is complete, then so is σX and hence by [7], II.48, $H(\sigma X)$ is complete. Thus, σX is ranked if, and only if, $H(\sigma X)$ is ranked.

PROPOSITION 4.8. *Let ϱX be a complete metric space, let $\varkappa$ be an infinite cardinal and let $\{Z^n_\alpha: \alpha < \varkappa, n \in \omega\}$ be a collection of subsets of ϱX for which there is a sequence $\{\delta_n\}$ of real numbers such that*

(i) *the collection $\{Z^n_\alpha: \alpha < \varkappa, n \in \omega\}$ is δ_0-discrete;*

(ii) $0 < \delta_{n+1} < \mathrm{diam} Z^n_\alpha < \delta_n$ *for all $\alpha < \varkappa$ and $n \in \omega$;*

(iii) $\lim_{n\to\infty} \delta_n = 0$.

Then rank $(H(\varrho X)) \geqslant (2^\varkappa)^+$.

Proof. Note that we can assume that ϱX is bounded and that $|Z^n_\alpha| = 2$ for all $n \in \omega$. We shall apply Proposition 4.1. For each $n \in \omega$, let D_n be the set of all $p \in H(\bigcup\{Z^n_\alpha: \alpha < \varkappa\})$ such that for each $\alpha < \varkappa$, $|p \cap Z^n_\alpha| = 1$. Then $|D_n| = 2^\varkappa$. The collection $\{D_n: n \in \omega\}$ is δ_0-discrete and $\mathrm{diam}\, D_n \leqslant \delta_n$ for all n. For each n, let $\varphi_n\colon D_{2^\varkappa} \to D_n$ be a one-to-one map. Define a map $f\colon (D_{2^\varkappa})^\omega \to H(\varrho X)$ by setting $f(p_0, p_1, p_3, \ldots) = \bigcup\{\varphi_n(p_n): n \in \omega\}$, where $p_n \in D_{2^\varkappa}$ for all n. (Note that by (i) of 4.8., $f(p_0, p_1, p_3, \ldots)$ is a closed subset of X.) The conditions (ii) and (iii) of 4.8. ensure that f is a uniform embedding. Thus, it follows from 4.1. that

$$\mathrm{rank}(H(\varrho X)) \geqslant \mathrm{rank}(D_{2^\varkappa})^\omega = (2^\varkappa)^+.$$

Proposition 4.8 enables us to give an example of a complete metric space whose rank exceeds the cardinality of the space. Let X be the subset

$$\{2^m 3^n: m, n \in \omega\} \cup \{2^m 3^n + 1/n+1: m, n \in \omega\}$$

of the real line. For all $x, y \in X$, let $\varrho(x, y) = \min\{1, |x-y|\}$. Then ϱX is a bounded complete metric space with $|X| = \omega$. By defining $Z^n_m = \{2^m 3^n, 2^m 3^n + 1/n+1\}$, we see that the conditions of 4.8. are satisfied for ϱX. Hence, $\mathrm{rank}(H(\varrho X)) \geqslant (2^\omega)^+$. However, $|H(\varrho X)| = |P(X)| = 2^\omega$. Let $\mathrm{dens}(X)$ denote the minimum cardinality of a dense subset of X, where X is a topological space. Then for any complete metric space ϱX, $\mathrm{rank}(\varrho X) \leqslant (\mathrm{dens}(X))^+$. The above example shows that this upper bound can be achieved.

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Fake topological Hilbert spaces and characterizations of dimension in terms of negligibility

by

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Abstract. For every $k \in \{-1, 0, 1, \ldots\}$ we construct a topologically complete separable metric AR space X_k which is *not* homeomorphic to the Hilbert space l_2, but which has the following properties:

(1) X_k embeds as a linearly convex subset of l_2.

(2) every compact subset of X_k is a Z-set and homeomorphisms between compact subsets of X_k can be extended (with control),

(3) $X_k \times X_k \approx l_2$,

(4) if $A \subseteq X_k$ is σ-compact, then A is strongly negligible iff $\dim A \leqslant k$ (in particular, $X_k \not\approx X_{k'}$ if $k \neq k'$).

(5) if $A \subseteq X_k$ is any compactum of fundamental dimension at most k, then A is negligible in X_k.

1. Introduction. *All topological spaces under discussion are separable metric.*

Toruńczyk [15] has obtained the following topological characterization of the separable Hilbert space l_2:

1.1. THEOREM. *A topologically complete* AR *space X is homeomorphic to l_2 if and only if every map $f\colon \bigoplus_{i=1}^{\infty} Q_i \to X$ of the countable free union of Hilbert cubes is strongly approximable by maps $g\colon \bigoplus_{i=1}^{\infty} Q_i \to X$ for which the collection $\{g(Q_i)\}_{i=1}^{\infty}$ is discrete.*

This extremely useful characterization has now become the standard method for recognizing topological Hilbert spaces. The above approximation property, referred to as the *strong discrete approximation property*, can be stated in the following way:

1.2. With respect to every admissible metric d on X, for each map $f\colon \bigoplus_{i=1}^{\infty} Q_i \to X$ and each $\varepsilon > 0$, there exists a map $g\colon \bigoplus_{i=1}^{\infty} Q_i \to X$ such that $d(f(y), g(y)) < \varepsilon$ for each y and $\{g(Q_i)\}_{i=1}^{\infty}$ is discrete.

In Anderson, Curtis and van Mill [3] it was shown that the strong discrete approximation property cannot be relaxed by considering only positive constants