Uniquely edge extendible graphs

by

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Abstract. A graph G is *uniquely edge extendible* if $G+e$ is isomorphic with $G+f$ for every pair e, f of edges in its complement $\overline{G}$. It is shown that a graph G is uniquely edge extendible if and only if $\overline{G}$ is edge-symmetric. This result is extended to multiple edge additions and structural characterizations are obtained.

A nonempty graph G is *edge-reconstructible* if G can be uniquely determined (up to isomorphism) by its subgraphs $G-e$, $e \in E(G)$. One of the foremost unsolved problems in graph theory is to settle the following conjecture.

Conjecture. Every graphs of size at least 4 is edge-reconstructible.

Early workers on this problem (see [1]) quickly realized that several well-known classes of graphs including all regular graphs of size at least 3 and all trees of size at least 4 are edge-reconstructible. Our interest was generated by a more obscure class of edge-reconstructible graphs. Consider the graph G and its edge e as in Figure 1. This graph is easily seen to be edge-reconstructible since the graph $H = G-e$ would be available to us in our reconstruction work and $H+e_1 \cong H+e_2 \cong G$ for any two edges e_1, and e_2 in $E(\overline{H})$.

Fig. 1

We fix our attention now on H and study those graphs J for which the set $\{J+e \mid e \in E(\overline{J})\}$ has cardinality one.

More formally, we say a graph J is *uniquely edge extendible* if $J+e_1 \cong J+e_2$ for any two edges e_1 and e_2 in $E(\overline{J})$. In this work, C_n (for integers $n \geqslant 3$) denotes the *cycle* graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set

$$\{v_i v_{i+1} \mid i = 1, 2, \ldots, n \pmod n\}.$$

The *wheel* W_n is that graph obtained from C_n by adding a new vertex w and edges $v_i w$ for $i = 1, 2, \ldots, n$. If G_1 and G_2 are graphs, then $G = G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The symbol nK_m denotes the union of n graphs, each isomorphic with the complete graph K_m. Finally, the *star* $K(1, n)$ is that graph with vertices $\{v_1, v_2, \ldots, v_n, w\}$ and edges wv_i for $i = 1, 2, \ldots, n$. Other notation and terminology follow that in [1]. The graphs W_4, W_5, C_4, C_5, $K_n \cup K_m$, $K(1, n)$, nK_m, as well as the graph G of Figure 1 are uniquely edge extendible. It is immediate that if J is any graph containing an edge e such that $J-e$ is uniquely edge extendible, then J is edge-reconstructible.

A few additional concepts are needed to study uniquely edge extendible graphs. A graph of order p will be called *irreducible* if $p = 1$ or the graph contains no vertices of degree $p-1$. If $\pi\colon V(G) \to V(H)$ is an isomorphism between the non-empty graphs G and H, then $\hat{\pi}\colon E(G) \to E(H)$ will denote the induced edge-isomorphism. We will let $\mathscr{A}(G)$ [respectively $\mathscr{A}^*(G)$] denote the group of automorphisms [respectively induced edge-automorphisms] of a graph G. A graph H is *edge-symmetric* if for any pair of edges e_1 and e_2 of H there exists $\alpha \in \mathscr{A}^*(G)$ such that $\alpha(e_1) = e_2$. Earlier work on edge-symmetric graphs appears in [2]–[6]. Finally, we let $\mathscr{D}_G$ denote the set of degrees of the vertices of G and call a graph G *biregular* if $|\mathscr{D}_G| = 2$. The following lemma will be helpful.

LEMMA 1. *If G is irreducible and uniquely edge extendible, then exactly one of the following is true:*

(i) *G is trivial* (i.e., $G \cong K_1$);

(ii) *G is regular and not complete*;

(iii) *G is biregular and for every edge $xy \in E(\bar{G})$, $\deg_G x \neq \deg_G y$.*

Proof. Let G be both irreducible and uniquely edge extendible. If G is complete, then G must be trivial. Assume then, that G is not complete. Then there exist vertices u and v in G such that $uv \in E(\bar{G})$. We consider two cases.

Case 1. Assume that $\deg_G u = \deg_G v = n$. In this case the number $n+1$ occurs as a degree in $G+uv$ twice more than it does in G. Now let x be an arbitrary vertex of G. Since G is irreducible and nontrivial, there exists a vertex w of G such that $wx \in E(\bar{G})$. Then, since $G+uv \cong G+wx$, the number $n+1$ occurs as a degree twice more in $G+wx$ than it does in G. It follows that $\deg_G x = n$. Thus G is regular and not complete.

Case 2. Assume that $m = \deg_G u < \deg_G v = n$. In this case the numbers $m+1$ and $n+1$ each occur as a degree once more in $G+uv$ than they do in G. Let x be an arbitrary vertex of G. As is Case 1, there exists a vertex w in G such that $wx \in E(\bar{G})$ and, by the same reasoning as in Case 1, $\deg_G x$ must be either m or n and $\deg_G w \neq \deg_G x$. ■

Our main result for uniquely edge extendible graphs may now be presented for the class of irreducible graphs.

THEOREM 1. *Let G be an irreducible graph. Then G is uniquely edge extendible if and only if $\bar{G}$ is edge-symmetric.*

Proof. First note that if G is complete, then $G \cong K_1$ which is uniquely edge extendible, and $\bar{G}$ is edge-symmetric. Assume then, that G is an irreducible graph which is not complete, the complement of which is edge-symmetric. Let e_1 and e_2 be edges of $\bar{G}$. We know that there exists an induced edge-automorphism $\hat{\pi} \in \mathscr{A}^*(\bar{G})$ such that $\hat{\pi}(e_1) = e_2$, and where, say, $\hat{\pi}$ is induced by the automorphism $\pi\colon V(\bar{G}) \to V(\bar{G})$. Since $\mathscr{A}(\bar{G}) = \mathscr{A}(G)$, the map π is also an automorphism of G. Moreover, since $\hat{\pi}(e_1) = e_2$, it follows that π is an isomorphism of $G+e_1$ with $G+e_2$ so that G is uniquely edge extendible.

Conversely, assume that G is an irreducible uniquely edge extendible graph which is not complete. Using Lemma 1 we need only consider two cases; G is regular or G is biregular.

Case 1. Assume that G is regular of degree n. Let e_1 and e_2 be edges of $\bar{G}$ where $e_i = u_i v_i$ for $i = 1, 2$. By assumption, there exists an isomorphism $\alpha\colon V(G+e_1) \to V(G+e_2)$. For $i = 1, 2$ note that u_i and v_i are the only vertices of $G+e_i$ of degree $n+1$. Therefore, α maps $\{u_1, v_1\}$ onto $\{u_2, v_2\}$ and α is an automorphism of G and hence, also of $\bar{G}$. Thus α induces the edge-automorphism $\hat{\alpha} \in \mathscr{A}^*(\bar{G})$ where $\hat{\alpha}(e_1) = e_2$ so that $\bar{G}$ is edge-symmetric.

Case 2. Assume that G is biregular with $\mathscr{D}_G = \{a, b\}$ and $a < b$. Let e_1 and e_2 be edges of $\bar{G}$ where $e_i = u_i v_i$ for $i = 1, 2$. Since G is uniquely edge extendible, there exists an isomorphism $\pi\colon V(G+e_1) \to V(G+e_2)$. Let

$$V_a = \{v \in V(G) \mid \deg_G v = a\} \quad\text{and}\quad V_b = \{v \in V(G) \mid \deg_G v = b\}.$$

Since G is irreducible and nontrivial, there are edges missing at each of its vertices. By Lemma 1, $\langle V_a \rangle$ and $\langle V_b \rangle$ are complete and we may assume that $u_i \in V_a$ and $v_i \in V_b$ for $i = 1, 2$. We now consider two subcases to complete the proof.

Subcase 2A. Assume that $b > a+1$. Here, for $i = 1, 2$, u_i is the only vertex of $G+e_i$ of degree $a+1$ and v_i is the only vertex of $G+e_i$ of degree $b+1$. Hence π maps the set $\{u_1, v_1\}$ onto the set $\{u_2, v_2\}$ and the same reasoning as in Case 1 may be employed to show that $\bar{G}$ is edge-symmetric.

Subcase 2B. Assume that $b = a+1$. In this situation, for $i = 1, 2$ let $H_i = G+e_i$ and note that v_i is the only vertex of H_i with degree $a+2$. Hence $\pi(v_1) = v_2$. Let $\pi(u_1) = w \in V(G+e_2)$. Then $w \in V_b \cup \{u_2\} - \{v_2\}$. Now u_1 is adjacent to every vertex of degree a in H_1 because $\langle V_a \rangle$ is complete; hence w must be adjacent to every vertex in the set $V_a - \{u_2\}$ in the graph H_2.

If $w \neq u_2$, then $w_b \in V_b - \{v_2\}$. Since G is irreducible and $\langle V_b \rangle$ is complete, $\deg_G w = b = p-2$, where p is the order of G. So $a = p-3$, and it follows that $\bar{G} \cong mK(1, 2)$ where $m = p/3$. Thus $\bar{G}$ is edge-symmetric.

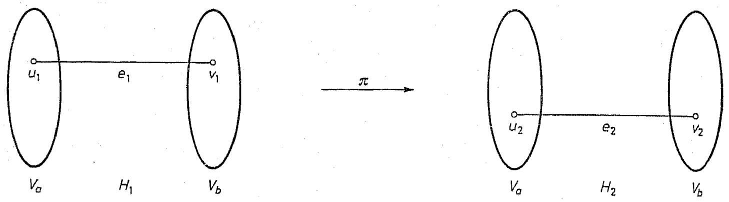

Fig. 2

If $w = u_2$, then π maps $\{u_1, v_1\}$ onto $\{u_2, v_2\}$, and the same reasoning as in Case 1 may be employed again to conclude that $\bar{G}$ is edge-symmetric. ■

Let G and H be graphs where $G = H+K_1$. By this notation we mean that G is obtained from H by adding a new vertex w and edges $\{wv|\ v \in V(G)\}$. It is straight forward to establish the following facts:

(i) G is uniquely edge extendible if and only if H is uniquely edge extendible.
(ii) $\bar{G}$ is edge-symmetric if and only if $\bar{H}$ is edge-symmetric.

Since a graph G fails to be irreducible if and only if $G = H+K_1$ for some graph H, a repeated use of the above facts and Theorem 1 yield the following result:

COROLLARY 1. *A graph G is uniquely edge extendible if and only if $\bar{G}$ is edge-symmetric.*

Our intent now is to generalize uniquely edge extendible and edge-symmetric graphs, establish generalized forms of Theorem 1, and structurally characterize the graphs so defined.

Let k be a positive integer. A graph G is *k-edge-symmetric* if $|E(G)| \geqslant k+1$ and for any pair A, B of sets of edges of G with $|A| = |B| = k$, there exists $\alpha \in \mathscr{A}^*(G)$ such that α maps the set A onto the set B. Note that if G is a graph with at least two edges, then G is 1-edge-symmetric if and only if G is edge-symmetric.

For a positive integer k, a graph G is *uniquely k edge extendible* if $|E(\bar{G})| \geqslant k+1$ and for any pair A, B of sets of edges of $\bar{G}$ with $|A| = |B| = k$, the graphs $G+A$ and $G+B$ are isomorphic. Note that if G is a graph with $|E(\bar{G})| \geqslant 2$, then G is uniquely 1 edge extendible if and only if G is uniquely edge extendible.

For example, the graphs nK_2 and $K(1,n)$ are k-edge-symmetric for each $2 \leqslant k \leqslant n-1$ and the graphs $K_{n(2)}$ and $K_1 \cup K_n$ are uniquely k edge extendible for each $2 \leqslant k \leqslant n-1$. The main thrust of our remaining work here is to show that, except for a couple of isolated graphs of small order, these are the only examples.

We note that if G and H are graphs with $G = H+K_1$, then $E(\bar{G}) = E(\bar{H})$ and, for each $2 \leqslant k \leqslant |E(G)|-1$, G is uniquely k edge extendible if and only if H is uniquely k edge extendible. It follows that the presence in a graph G of one or more vertices of degree $|V(G)|-1$ cannot influence the answer to the question

of whether or not G is uniquely k edge extendible. Therefore, it suffices to study irreducible uniquely k edge extendible graphs, since all others can be obtained from the irreducible ones by joining additional vertices. In a similar fashion, vertices of degree 0 do not affect k-edge-symmetry so that in the study of k-edge-symmetric graphs, we may restrict our attention to graphs which have no isolated vertices.

LEMMA 2. *Let G be a graph of size $q \geqslant 2$ and let k be an integer satisfying $1 \leqslant k \leqslant q-1$. Then G is k-edge-symmetric if and only if G is $(q-k)$-edge-symmetric.*

Proof. Assume that G is a k-edge-symmetric graph of size $q \geqslant 2$ and let k be an integer satisfying $1 \leqslant k \leqslant q-1$. Since $k \geqslant 1$, we have $q \geqslant (q-k)+1$ so that G has enough edges to be $(q-k)$-edge-symmetric. Let A and B be sets of edges of G with $|A| = |B| = q-k$. Let $C = E(G)-A$ and $D = E(G)-B$. Then $|C| = |D| = k$, so there exists $\alpha \in \mathscr{A}^*(G)$ such that α maps C onto D. It follows that α maps $A = E(G)-C$ onto $B = E(G)-D$. Hence G is $(q-k)$-edge-symmetric. The converse is proved similarly. ■

Since a graph of size $q \geqslant 2$ is edge-symmetric if and only if it is $(q-1)$-edge-symmetric we will for the most part limit our attention to k-edge-symmetric graphs of size q where $2 \leqslant k \leqslant q-2$, because a complete classification of edge-symmetric graphs is not yet known (see [3]). Moreover, we are working toward the result that with suitable restrictions on k, a graph G is uniquely k edge extendible if and only if $\bar{G}$ is k-edge-symmetric. Thus we also limit our study of uniquely k edge extendible graphs mainly to those graphs G for which $\bar{G}$ is of size $\bar{q}$ where $2 \leqslant k \leqslant \bar{q}-2$. This does not seriously restrict the scope of our study since the only other case permitted by the definition is that of $k = \bar{q}-1$ and that case is covered completely by the observation that every graph is uniquely $(\bar{q}-1)$ edge extendible.

Our next result shows that, subject to these restrictions, an irreducible graph G which is uniquely k edge extendible must take one of just two possible forms, which surprisingly are independent of the value of k.

LEMMA 3. *Let G be an irreducible graph whose complement is of size $\bar{q} \geqslant 4$ and let k be an integer satisfying $2 \leqslant k \leqslant \bar{q}-2$. If G is uniquely k edge extendible, then for $n = \bar{q}$ either $G \cong K_{n(2)}$ or $G \cong K_1 \cup K_n$.*

Proof. Let G be a graph and k an integer satisfying the hypotheses of the lemma, and assume that G is uniquely k edge extendible. We consider two cases according as $E(\bar{G})$ is or is not an independent set of edges in $\bar{G}$.

Case 1. No two edges of $\bar{G}$ are adjacent. In this case $\bar{G}$ is regular of degree 1 so for $n = \bar{q}$, $\bar{G} \cong nK_2$ and $G \cong K_{n(2)}$.

Case 2. $\bar{G}$ has a pair of adjacent edges. Let e_1, f_1 be edges of $\bar{G}$ that are incident with a common vertex v. Let e and f be two arbitrary edges of $\bar{G}$ and suppose that e and f are independent. Since $\bar{q} \geqslant k+2$ we can find a set $C \subset E(\bar{G})-\{e_1, f_1, e, f\}$ with $|C| = k-2$. Let $A = C \cup \{e_1, f_1\}$ and $B = C \cup \{e, f\}$. Then $G+A$ and $G+B$ have different degree sequences contradicting the hypothesis that G is uniquely k edge extendible. Thus no two edges of $\bar{G}$ are independent. Since $\bar{G}$ is of order $\bar{q} \geqslant 4$ it follows that $\bar{G}$ is the star $K(1, n)$ with $n = \bar{q}$, and that $G \cong K_1 \cup K_n$. ■

The proof of our next result is sufficiently elementary that we omit it.

LEMMA 4. *Let n and k be positive integers with $n \geqslant k+1$. The graphs nK_2 and $K(1, n)$ are k-edge-symmetric.*

The next result ties together our two new classes of graphs.

LEMMA 5. *Let G be a k-edge-symmetric graph for some $k \geqslant 2$. Then $\bar{G}$ is uniquely k edge extendible.*

Proof. Let G be k-edge-symmetris for some $k \geqslant 2$. Then $|E(G)| \geqslant k+1$. Let A, B be subsets of $E(G)$ with $|A| = |B| = k$. Then there exists $\pi \in \mathscr{A}(G)$ such that $\hat{\pi}$ maps A onto B, where $\hat{\pi} \in \mathscr{A}^*(G)$ is induced by π. Since π is also an automorphism of $\bar{G}$, π is an isomorphism of $\bar{G}+A$ wish $\bar{G}+B$. Thus $\bar{G}$ is uniquely k edge extendible. ■

Lemmas 3, 4, and 5 constitute the proof of our main result.

THEOREM 2. *Let G be an irreducible graph whose complement $\bar{G}$ is of size $\bar{q} \geqslant 4$ and let k be an integer satisfying $2 \leqslant k \leqslant \bar{q}-2$. Then the following statements are equivalent:*

(i) *G is uniquely k edge extendible;*
(ii) *$\bar{G}$ is k-edge-symmetric;*
(iii) *For $n = \bar{q}$ either $G \cong K_{n(2)}$ or $G \cong K_1 \cup K_n$.*

Theorem 2 covers only irreducible graphs G for which $\bar{G}$ has size $\bar{q} \geqslant 4$. There are only nine irreducible graphs G such that $\bar{G}$ has size $\bar{q} \leqslant 3$, namely $\bar{K}_3$, $K_1 \cup K_3$, P_4, $\overline{K_2 \cup P_3}$, and $K_{3(2)}$ with $\bar{q} = 3$, and $K_1 \cup K_2$ and C_4 with $\bar{q} = 2$, and $\bar{K}_2$ with $\bar{q} = 1$, and K_1 with $\bar{q} = 0$. These can be individually examined to determine all graphs which are uniquely multiply edge extendible or multiply-edge-symmetric. This leads to the following classifications.

COROLLARY 2a. *Let G be an irreducible graph for which $\bar{G}$ has size $\bar{q} \geqslant 3$. Then following statements are equivalent:*

(i) *G is uniquely 2 edge extendible;*
(ii) *G is uniquely k edge extendible for every $2 \cong k \leqslant \bar{q}-1$;*
(iii) *Either*

(iiia) *$\bar{q} \geqslant 4$ and for $n = \bar{q}$, $G \cong K_{n(2)}$ or $G \cong K_1 \cup K_n$*

or

(iiib) *$\bar{q} = 3$.*

COROLLARY 2b. *Let G be a graph of size $q \geqslant 3$ which has no isolated vertices. Then the following conditions are equivalent:*

(i) *G is 2-edge-symmetric;*
(ii) *G is k-edge-symmetric for every $2 \leqslant k \leqslant q-2$;*
(iii) *Either*

(iiia) *$q \geqslant 4$ and for $n = q$, $G \cong nK_2$ or $G \cong K(1, n)$*

or

(iiib) *$q = 3$ and $G \cong K_3$ or $G \cong K(1, 3)$ or $G \cong 3K_2$.*

Actually the preceding corollaries classify all graphs which are either uniquely multiply edge extendible or multiply-edge-symmetric. If a graphs G is not irreducible, then for all $2 \leqslant k \leqslant \bar{q}-1$, G is uniquely k edge extendible if and only if the graph obtained from G by deleting all vertices of degree $|V(G)|-1$ (all but one such vertex for a complete graph) is uniquely k edge extendible. Also, if G has isolated vertices then for all $2 \leqslant k \leqslant q-1$, G is k-edge-symmetric if and only it the graph obtained from G by deleting all isolated vertices (all but one vertex for an empty graph) is k-edge-symmetric.

From Corollaries 2a and 2b we may deduce the following result:

COROLLARY 2c. *The only graphs which are both uniquely multiply edge extendible and multiply-edge-symmetric are $K_1 \cup K_3$ and $K(1, 3)$.*

References

[1] M. Behzad, G. Chartrand and L. Lesniak-Foster, *Graphs and Digraphs*, Prindle, Weber and Schmidt, Boston 1979.

[2] I. Z. Bouwer, *On edge but not vertex transitive regular Graphs*, J. Combinatorial Theory Ser. B 12 (1972), pp. 32–40.

[3] D. Burns, S. F. Kapoor, and P. A. Ostrand, *On line-symetric graphs*, Fund. Math. 116 (1983), pp. 119–139.

[4] — — — *Edge cosymmetric Graphs*, Discrete Math. 32 (1980), pp. 99–103.

[5] E. Dauber and F. Harary, *On Point-Symmetric and Line-Symmetric Graphs*, (unpublished manuscript). See [7], p. 172.

[6] J. Folkman, *Regular line-symmetric graphs*, J. Combinatorial Theory 3 (1967), 215–232.

[7] F. Harary, *Graph Theory*, Addison-Wesley Co., Reading, Mass., 1969.

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