On the existence of measures on $\sigma$-algebras

by

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Abstract. We investigate the problem whether the existence of measures on a family of sub-
algebras of an algebra $\mathcal{A}$ implies the existence of measures on $\mathcal{A}$ itself. Some relationships between
various methods of constructing regularizing families for ultradisintegrators are also discussed in this
context.

0. Introduction. Our main object is to study the problem whether the existence
of non-trivial measures on a family of subalgebras of an algebra $\mathcal{A}$ implies the
existence of a non-trivial measure on $\mathcal{A}$ itself. The actual questions we consider
are motivated by the following problem of Banach [1]. Are there two countably
generated $\sigma$-algebras on the interval $[0, 1]$ which both carry a non-trivial $\sigma$-additive
measure but the $\sigma$-algebra generated by their union has no such measure.

This problem was answered by Grzegorek who constructed such algebras
in [5]. Galvin suggested to us to consider the following more general problem.
Does there exist a family of $\kappa$ $\sigma$-algebras such that every $\lambda$ of them generate an
algebra carrying a non-trivial $\sigma$-additive measure and every $\varrho$ of them generate
an algebra without such measure. We construct such families of algebras for some
infinite values of the cardinal parameters and some further cases, in particular
those involving finite values, are discussed in a forthcoming paper by the first
author.

There is some similarity between the problems we are considering and the
so called marginal problem (cf. e.g. [7]): given a family of algebras $F_\alpha$: $\alpha < \kappa$ of
subsets of a fixed set $X$ and for each $\alpha < \kappa$ a probability measure $m_\alpha$ defined on $F_\alpha$,
under what conditions will there exist a common extension of all measures $m_\alpha$,
defined on an algebra $F$ containing $\bigcup F_\alpha$. However the difference is that in the
Banach-type problems we are interested in the existence of any measure on the
large algebra, whereas in the marginal problem the object is to extend a particular
family of measures.

Some of our results deal with measures that are only finitely additive. One of
these results also sheds some light on the relationship between several techniques

* The second author received partial support from the NSF Grants MCS 78-01525 and
81-02700.
for establishing the regularity of ultrafilters on $\omega_1$ — see Prikry [8], Benda, Keeton [2] and Laver [6].

The main tool we use are matrices of sets. Hence an entire section of the paper is devoted to this topic. For most of our results we need either the continuum hypothesis or the axiom of constructibility.

1. *Notation and terminology.* We use standard set theoretic notation. Ordinals are identified with the set of their predecessors, cardinals with initial ordinals. $\kappa$ and $\lambda$ always denote infinite cardinals. $\mathcal{P}(X)$ is the set of subsets of $X$, $[X]^\omega$ and $[X]^\kappa$ denote the family of sets of $X$ of cardinality $\kappa$ and $\kappa$ respectively. $\mathcal{F}$ is an ideal on $X$ means that elements of $\mathcal{F}$ are subsets of $X$. An ideal $\mathcal{F}$ is $\kappa$-complete if $\mathcal{F} \preceq 1^\kappa$, $\mathcal{F}$ is uniform on $\kappa$ means that $[\kappa]^\kappa \preceq \mathcal{F}$. A family $\mathcal{A}$ of subsets of $X$ is $\mathcal{F}$-almost disjoint if for distinct $A, B \in \mathcal{A}, A \cap B \in \mathcal{F}$. $\mathcal{F}$ is $\kappa$-almost disjoint if it is $\mathcal{F}$-almost disjoint for $[X]^\kappa$. An ideal $\mathcal{F}$ on $X$ is $\kappa$-saturated if there is no $\mathcal{F}$-almost disjoint family $\mathcal{A} \subseteq \mathcal{P}(X)$ such that $\bigcap \mathcal{A} \neq \emptyset$ and $\mathcal{A} \equiv \mathcal{F}$. If $\mathcal{F}$ is an ideal on $X$ then $\mathcal{F}^+$ denotes the family $\mathcal{F}(X \setminus \mathcal{F}) \setminus \mathcal{F}$ and $\mathcal{F}^*$ denotes the family $\{X \setminus A : A \in \mathcal{F}\}$. $\mathcal{F}^*$ is called the dual filter. An ultrafilter $\mathcal{U}$ on $\kappa$ is regular if there is a family $\mathcal{A} \subseteq \mathcal{P}(\kappa)$ such that for any $\mathcal{B} \subseteq \mathcal{A}$, $\bigcap \mathcal{B} \neq \emptyset$. Such a family $\mathcal{A}$ is then called regularizing for $\mathcal{U}$. It is clear how to relativize these notions to any algebra $\mathcal{A} \subseteq \mathcal{P}(X)$, e.g. an ideal $\mathcal{F} \preceq \mathcal{A}$ is uniform if it contains all sets of cardinality $<|X|$ belonging to $\mathcal{A}$.

If $\mathcal{A}$ is an algebra of subsets of $X$ and $\mu$ is a measure defined on $\mathcal{A}$, it is called a probability measure if $\mu(X) = 1$ and a diffuse measure if $\mu$ vanishes on atoms of $\mathcal{A}$.

2. *Matrices of sets.* In this section we prove the existence of matrices of sets with various combinatorial properties. They will provide an important tool for getting our main results. The first theorem works only for $\omega$ (or for strongly inaccessible cardinals).

**Theorem 2.1.** (a) There exist sets $X_{\alpha, \beta} \subseteq [\alpha]^{\omega}$ ( $\alpha \cdot 2^\omega = \kappa, \kappa \in \omega$), such that:

(i) $\forall \alpha \forall \beta \forall \xi \forall \eta [ \xi \neq \eta \rightarrow X_{\alpha, \beta} \cap X_{\xi, \eta} = \emptyset);$  
(ii) $\forall \alpha \forall \beta \forall \xi \forall \eta [ \xi \subset \eta \rightarrow |X_{\alpha, \beta} \cap X_{\xi, \eta}| < |\alpha|);$  
(iii) $\forall \alpha \forall \beta \forall \xi \forall \eta [ \xi \neq \eta \rightarrow \{X_{\alpha, \beta} \cap X_{\xi, \eta} \subseteq \omega \};$  
(iv) for every $\xi \subseteq \omega$ and every one-to-one $f : S \rightarrow \omega$, $\bigcap S_{\xi, f(S)} \subseteq [\alpha]^{\omega}$.

(b) There exists sets $X_{\alpha, \beta} \subseteq [\alpha]^{\omega}$ ( $\alpha, \beta \cdot 2^\omega = \kappa$) such that:

(i) $\forall \alpha \forall \beta \forall \xi \forall \eta [ \xi \neq \eta \rightarrow X_{\alpha, \beta} \cap X_{\xi, \eta} \neq \emptyset);$  
(ii) $\forall \alpha \forall \beta \forall \xi \forall \eta [ \xi \subset \eta \rightarrow |X_{\alpha, \beta} \cap X_{\xi, \eta}| < |\alpha|);$  
(iii) $\forall \alpha \forall \beta \forall \xi \forall \eta [ \xi \neq \eta \rightarrow \{X_{\alpha, \beta} \cap X_{\xi, \eta} \subseteq \omega \};$  
(iv) for every $\xi \subseteq \omega$ and every one-to-one $f : S \rightarrow \omega$, $\bigcap S_{\xi, f(S)} \subseteq [\alpha]^{\omega}$.

**Proof.** (a) Let $F_\alpha = \{ f : [\alpha]^{\omega} \rightarrow 1 \}, F = \bigcup F_\alpha.$ Since $|F| = \omega$, if suffices to construct the desired matrix on $F$ rather than on $\omega$. Let

$$X_{\alpha, \beta} = \bigcup \{ f \in F_\alpha : \beta \cap \alpha \neq \emptyset \} \quad \text{for} \quad \alpha \subseteq \omega, \beta \subseteq \omega.$$

(i) and (iii) are obvious. To check (ii), let $f \in F_\alpha, a \neq b$ and $f \in X_{\alpha, a} \cap X_{\beta, b}$. Hence $f(a \cap b) = f(b \cap a) = \alpha$ since $f$ is $1 \rightarrow 1$. This can happen only for finitely many $\alpha$ and since all $F_\alpha$ are finite, we get (ii). Finally, to get (iv), let $a \subset f \subseteq \beta$, be distinct subsets of $\alpha$ and $m_1, m_2, f \subseteq 1$ distinct natural numbers. Then for sufficiently large $n \in \omega$, there is a function $f : [\alpha]^{\omega} \rightarrow 2^n$ such that $\forall j \in 1 \{ f(a \cap b) = m_j \}$. This shows that $\bigcap X_{\alpha, m_1}$ is infinite and hence the constructed sets have all required properties.

(b) Let $F_\alpha = \{ f : [\alpha]^{\omega} \rightarrow 2 \}$. Again we use $F$ as the underlying set instead of $\omega$. We define for $\alpha, \beta \subseteq \omega, X_{\alpha, \beta} = \bigcup \{ f \in F_\alpha : f(a \cap b) = b \cap a \}$. It is easy to check as before that the sets $X_{\alpha, \beta}$ have all required properties.

This type of result can be extended to an uncountable cardinal $\lambda$ using $\subseteq$.

**Theorem 2.2.** Let $\lambda$ be an uncountable regular and $\subseteq$ hold. Then there exist stationary sets $X_{\alpha, \beta} \subseteq (\alpha, \beta \cdot 2^\omega = \lambda)$ such that:

(i) for all $a$, the family $\{X_{\alpha, \beta} : \beta \in a\}$ is $\lambda$-almost disjoint,

(ii) for all $\beta$, the family $\{X_{\alpha, \beta} : \alpha \in a\}$ is $\lambda$-almost disjoint,

(iii) for every $S \subseteq [\alpha]^{\omega}$ and every one-to-one $f : S \rightarrow \omega$, $\{X_{\alpha, f(S)} \} \subseteq X_{S, \omega}$.

**Proof.** For $\alpha < \lambda$, let $S_\alpha = \{ (a \in \omega, b \subseteq a) \beta < \alpha, a_\beta \neq a, b_\beta \neq b, \}$ distinct subsets of $a, b_\beta$ distinct subsets of $a$ be a $\subseteq$ sequence, i.e. for every sequence $(a_\beta, b_\beta)$, $\beta < \alpha < \lambda$,

$$\{a_\beta \neq a \beta < \alpha \alpha_\beta \neq a \beta \neq a \} \subseteq X_{a_\beta, b_\beta} \subseteq X_{\alpha, \omega}.$$

is stationary.

Now define, for $a, b \subseteq \lambda$,$X_{ab} = (\{a \in \omega, b \subseteq \alpha \alpha \neq a \beta \neq a \beta \neq b \beta \} \subseteq X_{ab, \omega}.$

(i) and (ii) are straightforward. To check (iii), let $\beta < \alpha < \lambda$ be distinct.

$$\{X_{\alpha, \beta} \cap X_{\alpha_\beta} \neq \emptyset \} \subseteq \{ \beta \beta \} \subseteq X_{\alpha, \omega}.$$ 

hence we get that the set $\bigcap X_{\alpha, \beta}$ is stationary, in view of $\subseteq$.

Using the continuum hypothesis one can get the following $\omega \times \omega$ matrix of subsets of $\omega$:

**Theorem 2.3.** Assume CH. There exists a matrix of subsets $A_{\alpha, \beta} \subseteq \omega$, such that:

(i) $\forall \alpha \forall \beta [ \alpha \neq \beta \rightarrow A_{\alpha, \beta} \neq \emptyset];$

(ii) $\forall \alpha \forall \beta [ \alpha \neq \beta \rightarrow |A_{\alpha, \beta} \cap A_{\alpha, \beta}| < \omega];$

(iii) $\forall \alpha \forall \beta [ \alpha \neq \beta \rightarrow |A_{\alpha, \beta} \cap A_{\alpha, \beta}| < \omega];$

(iv) for all sequences $\{X_{\alpha, \beta} : j \in \omega \}$ such that $j_1 \neq j_2 \neq j_3 \neq j_4 \neq \eta_1 \neq \eta_2$ and such that $\alpha \cap X_{\eta, \beta} = \emptyset$ is infinite, $\bigcap A_{\alpha, \beta}$ is uncountable.
Proof. Let $\{t_k; \xi \in \varnothing_0\}$ be an enumeration of all sequences as in (iv) s.t. every sequence is repeated $\varnothing_0$ times. Fix an infinite $\alpha < \varnothing_0$. Define $\varphi_\alpha: \varnothing_\alpha \to \varnothing_\alpha$ as follows. Let $t_\alpha = \{(a_j, n_j); j \in \varnothing_\alpha\}$. If $\exists \alpha_{a_j, n_j} \in S_{\varnothing_\alpha}$ and $\varphi_\alpha(a_j, n_j) \in S_{\varnothing_\alpha}$. Let $\psi_\alpha: \{a \in \varnothing_\alpha; \psi_\alpha(a_j, n_j) \in S_{\varnothing_\alpha}\}$ be arbitrary, and put

$$\psi_\alpha(a) = \begin{cases} \psi_\alpha(a_j, n_j) & \text{if } a = a_j, \\ \psi_\alpha(b_j, n_j) & \text{if } a \neq \psi_\alpha(a_j, n_j) \land \exists b_j \in \varnothing_\alpha. \end{cases}$$

If $a \in \varnothing_\alpha$ is finite or $\exists a \in \varnothing_\alpha \land \varphi_\alpha$ be arbitrary.

Now define $A_{\varnothing_\alpha} = \{a \in \varnothing_\alpha; \varphi_\alpha(a) \in S_{\varnothing_\alpha}\}$. It is easy to check that (i)-(iv) are satisfied.

In order to get matrices of larger size, we use Kurepa's Hypothesis.

**Theorem 2.4.** Assume Kurepa's Hypothesis. Then there are sets $X_{\varnothing_\alpha}(\xi, \eta \in \varnothing_\alpha)$ such that $X_{\varnothing_\alpha}(\xi, \eta \in \varnothing_\alpha)$ is countable.

(i) for all $\xi$ and all distinct $\eta_1, \eta_2$, there is $\exists \eta_0 \in \varnothing_\alpha$ such that $X_{\varnothing_\alpha}(\xi, \eta_0) \cap X_{\varnothing_\alpha}(\eta_0, \eta_2) = \varnothing$.

(ii) for all $\eta$ and all distinct $\eta_1, \eta_2$, there is $\exists \eta_0 \in \varnothing_\alpha$ such that $X_{\varnothing_\alpha}(\eta_0, \eta_1) \cap X_{\varnothing_\alpha}(\eta_0, \eta_2) = \varnothing$.

(iii) for all $S \in \varnothing_\alpha$, each one-to-one $f: S \to \varnothing_\alpha$ and all sufficiently large $\alpha \leq \varnothing_\alpha$, $|\bigcap X_{\varnothing_\alpha}(\xi, \eta \in \varnothing_\alpha)| = 2^\alpha$.

**Proof.** Let $\mathcal{K}$ be a Kurepa family on $\varnothing_\alpha$. Let $F_\alpha = \{f: \mathcal{K} | a \in \mathcal{K} \}$ and $X_{\varnothing_\alpha} = \bigcup \{f \in F_\alpha; f(a \cap a) = b \cap a, f \in \mathcal{K}\}$. Since $\mathcal{K}$ is countable for every $a$, we can regard $X_{\varnothing_\alpha}$ as subsets of $\mathcal{K} \times \varnothing_\alpha$. It is easy to check that (i)-(iii) are satisfied.

Assuming both Kurepa's Hypothesis and $\varnothing_\alpha$, we can get an $\omega \times \omega_\alpha$ matrix which combines some of the properties of previously constructed matrices.

**Theorem 2.5.** Assume $\varnothing_\alpha$ and let $\mathcal{K} = \{\varnothing_\alpha\}$, $|\mathcal{K}| = \alpha < \varnothing_\alpha$ for every $\alpha \in \varnothing_\alpha$. Then there exists a matrix $A_{\varnothing_\alpha}$ (i.e., $\alpha \in \varnothing_\alpha$, $\alpha \in \varnothing_\alpha$) of subjects of $\alpha$, such that:

(i) $\forall \alpha \in \mathcal{K}, \alpha \in \varnothing_\alpha$. \(\exists \alpha \in \varnothing_\alpha \wedge \alpha \in \varnothing_\alpha \cap \alpha \in \varnothing_\alpha \wedge \alpha \in \varnothing_\alpha \).

(ii) $\forall \alpha \in \mathcal{K}, \alpha \in \varnothing_\alpha$. \(\exists \alpha \in \varnothing_\alpha \wedge \alpha \in \varnothing_\alpha \cap \alpha \in \varnothing_\alpha \wedge \alpha \in \varnothing_\alpha \).

(iii) $\forall \alpha \in \mathcal{K}, \alpha \in \varnothing_\alpha$. \(\exists \alpha \in \varnothing_\alpha \wedge \alpha \in \varnothing_\alpha \cap \alpha \in \varnothing_\alpha \wedge \alpha \in \varnothing_\alpha \).

(iv) for every sequence $\{(a_j, n_j); j \in \omega, a_j \in \mathcal{K}, e_j \in \omega\}$ s.t. $j_1 \neq j_2 \neq a_j \neq a_j, n_j \neq n_j$ and $\omega \in \{e_j; j \in \omega\}$ is infinite, the set $\bigcap \mathcal{K}_{\omega, \alpha}$ is stationary.

**Proof.** For $\alpha \in \varnothing_\alpha$, let $S_\alpha = \{\{a_j, n_j\}; j \in \omega, a_j \in \mathcal{K}, e_j \in \omega\}$ such that $a_j \neq a_j, n_j \neq n_j$ is distinct for $j$ varies and $\omega \in \{n_j; j \in \omega\}$ is infinite. Let $\psi_\alpha: \{a \in \mathcal{K}; \psi_\alpha(a_j, n_j) \in \mathcal{K}\}$ be arbitrary. Let $\psi_\alpha: \{a \in \mathcal{K}; \psi_\alpha(a_j, n_j) \in \mathcal{K}\}$ be arbitrary. Put $S_\alpha = S_{\varnothing_\alpha} \cup \psi_\alpha$, now we define the matrix $A_{\varnothing_\alpha}(e \in \mathcal{K}, \alpha \in \varnothing_\alpha)$:

$$A_{\varnothing_\alpha}(e \in \mathcal{K}, \alpha \in \varnothing_\alpha) = \{(a \in \varnothing_\alpha; \{a \in \alpha, n \in \omega) \}.$$ (i)-(iv) are easy to check.

Various matrices with further properties of the type we discussed so far can be considered. We conclude this section by pointing out a related problem which seems to be open: Do there exist sets $X_{\varnothing_\alpha} = \{\varnothing_\alpha; \{e \in \mathcal{K}, \alpha \in \varnothing_\alpha\}$ s.t.

(i) for all $e \in \mathcal{K}, \{X_{\varnothing_\alpha}; \beta \in \varnothing_\alpha\}$ is a disjoint family.

(ii) for all $e \in \mathcal{K}, \{X_{\varnothing_\alpha}; \beta \in \varnothing_\alpha\}$ is disjoint.

(iii) for all $S \in \{\varnothing_\alpha\}^\varnothing_\alpha$ and for all one-to-one $f: S \to \varnothing_\alpha$, $\bigcap \mathcal{K}_{\varnothing_\alpha}(\xi, \eta \in \varnothing_\alpha) = 2^\varnothing_\alpha$.

**3. Measures and filters on $\sigma$-algebras.** The following theorem provides an answer to a question of Woodin.

**Theorem 3.1.** There is a compact $F \subseteq \omega^\varnothing_\alpha$ such that

(i) for every measurable $E \subseteq F$ there is a $0, 1$-valued finitely additive diffuse measure $\mu$ on $\omega$ such that for every $f \subseteq E, f$ is constant on a set of measure $1$.

(ii) if $H \subseteq F, \|H\| = \omega_\alpha$ and $\mu$ is a $0, 1$-valued additive diffuse probability measure on $\omega$, then there is some $f \subseteq H$ such that $f$ is not constant on any set of positive measure.

**Remark.** In fact, the theorem remains valid if (ii) is replaced by the following statement:

(iii) If $H \subseteq F, \|H\| = \omega_\alpha$ and $\mu$ is a $0, 1$-valued uniform ideal on $\omega$ then there is some $f \subseteq H$ such that $f$ is not constant on any set on $\omega$.

**Other results of this type also follow.**

**Proof.** We use the matrix constructed in Theorem 2.1. a). Since for every $e \in \mathcal{K}, \alpha \in \varnothing_\alpha$, the family $\{X_{\varnothing_\alpha}; \alpha \in \varnothing_\alpha\}$ is a partition of $\mathcal{K}$, we can associate with it a function $\psi_\alpha$ defined by $\psi_\alpha(m) = n$ if $m \in X_{\varnothing_\alpha}$.

Let $\mathcal{F} = \{\varnothing_\alpha; \alpha \in \mathcal{K}\}$. It follows from the definition of the sets $X_{\varnothing_\alpha}, \mathcal{F}$ is a compact subset of $\omega^\varnothing_\alpha$.

Let $\mathcal{E} = \{\varnothing_\alpha; \alpha \in \varnothing_\alpha\}$ be a countable subset of $\mathcal{F}$.

In view of property (iv) in Theorem 2.1, the family $\{X_{\varnothing_\alpha}; \alpha \in \varnothing_\alpha\}$ can be extended to a uniform ultrafilter on $\varnothing_\alpha$, hence (i) of Theorem 3.1. follows. To get (ii) which implies (i), let $H = \{\varnothing_\alpha; \alpha \in \varnothing_\alpha\}$ be an uncountable subset of $\mathcal{F}$. Suppose (ii) is false. Let $\mathcal{F}$ be an $\omega_\alpha$-saturated uniform ideal on $\omega$, such that every $\mathcal{F}$ is constant on a set from $\mathcal{F}$. This means that for every $\alpha \in \varnothing_\alpha$, there is $\eta \in \omega_\alpha$ such that $X_{\varnothing_\alpha}(\xi, \eta) \subseteq \mathcal{F}$.

$\varnothing_\alpha$ is the same for uncountably many $\alpha$, hence we get a contradiction with $\omega_\alpha$-saturation. This finishes the proof.

A similar result can be obtained for uncountable cardinals using Theorem 2.2.
THEOREM 3.2. Assume $\Diamond$. There exists a family $F=\omega^\omega$ such that $|F| = \omega_2$ and

(i) for every $E \in [F]^\omega$ there is a countably complete filter on $\omega_1$ such that each $f \in E$ is constant on a set from the filter.

(ii) for every $H \in [F]^\omega$ and every $\omega_2$-saturated uniform ideal $\mathcal{I}$, there is an $f \in H$, such that $f$ is not constant on any set in $\mathcal{I}$.

Proof. We take the matrix $X_{\alpha,\beta}$ $(\alpha, \beta \in 2^n)$ which exists in view of Theorem 2.2, (for $\lambda = \omega_1$). Consider only the sets $X_{\alpha,\beta}$ $(\alpha \in 2^n, \beta \in \omega_1)$. Since every family $\{X_{\alpha,\beta}: \beta \in \omega_1\}$ is $\omega_1$-almost disjoint and $X_{\alpha,\beta}$ are all stationary, we can disjoint them and then proceed exactly as in the proof of the proceeding theorem.

We now turn our attention to Banach's problem mentioned in the introduction.

Problem of Banach: Do there exist two countably generated $\sigma$-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ of subsets of $[0, 1]$ such that both of them carry $\sigma$-additive probability measures vanishing on atoms, but the $\sigma$-algebra generated by $\mathcal{A}_1 \cup \mathcal{A}_2$ does not carry any such measure?

It is not obvious whether every countably generated $\sigma$-algebra which does not carry $\sigma$-additive diffuse probability measures cannot be generated by $\sigma$-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ as above. The next theorem provides an example of an algebra which strongly fails to have this property.

THEOREM 3.3. Assume that the union of $<2^n$ meager subsets of $\omega_1$ is not $\omega_1$. Then there exists a countably generated $\sigma$-algebra $\mathcal{A}_1$ of subsets of $[0, 1]$ such that no countably generated $\sigma$-algebra $\mathcal{A}_1 \subset \mathcal{A}_2$ carries a $\sigma$-additive diffuse probability measure.

Proof. The following lemma follows from a result of Darst [3].

LEMMA 3.4. For every sequence $(B_n: n \in \omega)$ of subsets of $[0, 1]$ and every $\sigma$-additive probability Borel measure $\mu$, there exists a set $S \subset 2^n$ s.t. $\mu(S) = 1$ and $\bigcup \mu(B_n)$ is meager. (For $B_n = B_n \in [0, 1]$, $B$).

The following lemma is due to Rothberger [9].

LEMMA 3.5. Assume that the union of $<2^n$ meager subsets of $\omega_1$ is not $\omega_1$. Then every set of cardinality $<2^n$ has strong measure 0.

Proof. Let $X = \{x_n: \alpha < \eta, \alpha < 2^n, \text{ and let } s_\eta > 0, n \in \omega \}$. Let $I_{\alpha,\beta} (\eta \in \omega)$ enumerate rational intervals s.t. the length of $I_{\alpha,\beta} (\eta \in \omega)$ is $s_\eta$. Put $U_f = \bigcup \bigcap I_{\alpha,\beta}$ for every $f \in \omega_1$. The set $D_n = \{x: x \in U_f\}$ is dense open. Hence $\bigcap D_n \neq \emptyset$. It follows that $X \cap \bigcup I_{\alpha,\beta}$, hence $X$ has strong measure 0.

Under the assumption of the theorem there exists a set $\mathcal{A} = [0, 1]$ of cardinality $2^n$ such that for every meager set $M$, $||M \cap \mathcal{A}|| < 2^n$. It suffices to show that the algebra of Borel subsets of $\mathcal{A}$ has the required properties.

Suppose not. Hence there are Borel sets $B_n (n \in \omega)$ s.t. $\{B_n \cap \mathcal{A}: n \in \omega\}$ generates a $\sigma$-algebra carrying a $\sigma$-additive diffuse probability measure $\nu$. The measure $\nu$ defined by $\nu(S) = \nu \bigcap f(n)(B_n \cap \mathcal{A})$ is clearly a Borel measure on $2^n$. Take $S$ as in Lemma 3.4. Then $C = \bigcup \bigcap f(n)(B_n \cap \mathcal{A})$ is a meager subset of $\mathcal{A}$, hence $|C| < 2^n$ and $\nu(C) = 1$. This, in view of Lemma 3.5, gives a contradiction, hence our theorem is proved. Let us remark that another proof of Theorem 3.3 can be obtained using Theorem 1, §3, Chapter II of Sierpiński [10].

The next two theorems give positive answers to a generalization of Banach's problem.

THEOREM 3.6. Assume CH. There exists an increasing sequence $\mathcal{A}_n (\alpha < \omega_1)$ of countably generated $\sigma$-algebras of subsets of $[0, 1]$ such that every $\mathcal{A}_n$ carries a $\sigma$-additive diffuse probability measure, but the $\sigma$-algebra $\bigcup \mathcal{A}_n$ does not carry any such measure.

Proof. We use Theorem 2.3. Fix $\alpha \in \omega_1$. For each $\beta < \alpha$ pick a different even $n_\beta \in \omega$. Then $\{\beta, n_\beta: \beta < \alpha\}$ is in (iv) from Theorem 2.3. Hence $\mathcal{A}_n (\alpha < \omega_1)$ are $\sigma$-independent, which shows that there is a $\sigma$-additive diffuse probability measure on the $\sigma$-algebra $\mathcal{A}_n$ generated by $\{A_\alpha: \beta < \alpha, n \in \omega\}$. A well known argument of Ulam [11] shows that there is no $\sigma$-additive diffuse probability measure on the $\sigma$-algebra generated by $\{A_\alpha: \alpha \in \omega, n \in \omega\}$. This finishes the proof.

Remark. After having read the above proof J. Cichoń (unpublished) eliminated the assumption of CH via a different argument.

THEOREM 3.7. Assume Kurepa's Hypothesis and $\Diamond$. Then there exists a sequence $\mathcal{A}_n (\alpha < \omega_1)$ of countably generated $\sigma$-algebras of subsets of $[0, 1]$, s.t. the union of every countable subcollection generates a $\sigma$-algebra carrying a $\sigma$-additive diffuse probability measure, but the union of every uncountable subcollection generates a $\sigma$-algebra which does not carry any such measure.

Proof. In view of Kurepa's Hypothesis we can take the family $\mathcal{X}$ from Theorem 2.5. with cardinality $\omega_2$. Define $\mathcal{A}_n$ as being generated by $\{A_{\alpha, n}: \alpha \in \omega_1, n \in \omega\}$, where $C_\alpha$ countable and $\alpha \in C_\alpha$. The family $\{\mathcal{A}_n: \alpha < \omega_1\}$ is as required.

The last part of this section deals with the regularity of ultralifters, and is of the same type as above. A large collection of algebras is shown to have the property that algebras generated by small unions carry non-uniform uniform ultralifters, but algebras generated by large unions contain regularizing families for every uniform ultralifter.

Regularity of ultralifters was previously discussed e.g. by Priky [8], Benda, Ketonen [2] and Laver [6]. In all these papers matrices of sets were constructed under various additional assumptions, providing regularizing families for ultralifters e.g. on $\omega_1$. In [8] and [6] these matrices had size $\omega_1$, in [2] under the assumption of Kurepa's Hypothesis a matrix of size $\omega_2$ is constructed and for every uniform ultralifter some initial segment of it provides a regularizing family.
Our result shows (under the assumption of KH and C) that this method cannot be improved by considering any initial segment of a Kurepa matrix (used in [2]), for the algebra generated by every initial segment carries a non-regular uniform ultrafilter.

**Theorem 3.8.** Assume Kurepa's Hypothesis and C. There are algebras $A_{\alpha_i}$: $\alpha < \omega_2$ of subsets $\omega_1$, such that the algebra generated by the union of any $\alpha_i$ of them carries a non-regular uniform ultrafilter, but every uniform ultrafilter on the algebra generated by the union of $\omega_2$ of them is regular.

**Proof.** We use the matrix from Theorem 2.5, with the family $\mathcal{F}$ of cardinality $\omega_1$ given by Kurepa's Hypothesis. The algebras $A_{\alpha_i}$ ($\alpha \in \omega_\omega \cap \omega_2$) are defined using sets $A_{\alpha_i}$ ($\beta < \alpha_i, n \in \omega_1$). It is enough to consider $\alpha_i = \omega_1$. Let $\mathcal{A}_\alpha$ be the $\alpha$-algebra generated by the sets $A_{\alpha_i}$ ($\alpha \in \omega_1$). Define $\mathcal{A}_\alpha = \bigcup \{ \mathcal{A}_\alpha^i : i \in [\omega_1]^\omega \}$. $\mathcal{A}_\alpha$ is an algebra since the family of $\mathcal{A}_\alpha^i$ is a directed system of algebras. By the argument of Benda, Ketenci [2] every ultrafilter is regular on the algebra generated by $\{A_{\alpha_i} : \alpha < \omega_1, n \in \omega_1 \}$ hence the last statement of the theorem follows. It is enough to show the existence of a non-regular ultrafilter on $A_{\omega_1}$. (for any union of $\omega_1$ algebras $A_{\alpha_i}$ the proof is similar.)

Let $\mathcal{F}$ be a uniform ultrafilter on $\omega$. We define by induction ultrafilters $\mathcal{F}_n$ on $\omega^n$ ($n \in \omega$). For $X \in \omega^{n+1}$, $X \in \mathcal{F}_{n+1}$ iff $\exists n \in \omega_1 : \{ k : k < n, X \in \mathcal{F}_k \} \in \mathcal{F}_n$. Now for each $f \in [\omega_1]^\omega$, let $\mathcal{F}_f$ be the ultrafilter on $\omega$ obtained as follows: let $a_0 < \cdots < a_{n+1}$ enumerate $f$. Define $\mathcal{F}_f : \omega^n \rightarrow \omega \rightarrow \omega$. Now $\forall e \in [\omega_1]^\omega$, let $\mathcal{F}_e$ be the ultrafilter on $\omega$ obtained as follows: let $a_0 < \cdots < a_{n+1}$ enumerate $e$. Define $\mathcal{F}_e : \omega^n \rightarrow \omega \rightarrow \omega$. Now $\mathcal{F}_e$ is an ultrafilter on $\omega$. Clearly $\mathcal{F}_f$ is uniform. We show that $\mathcal{F}_f$ is non-regular. Claim: If $Y_\omega \in \mathcal{F} (\alpha \in \omega_1)$ then there is an infinite $S \subseteq \omega_1$ such that $\bigcap_{e \in S} Y_e$ is stationary.

**Proof of the claim.** For each $e \in \omega_1$, let $e \in [\omega_1]^\omega$ be such that $Y_e \in \mathcal{F}_e$. Without loss of generality we may assume that $|\alpha| = n$ for all $\alpha < \omega_1$ and that the sets $e$ form an $A$-system with kernel $\omega$. Let $X_\omega \in \mathcal{F}_e$ be such that $Y_\omega = \bigcup_{e \in \omega_1} \mathcal{F}_e$ is the initial segment of all $e$. Let $X_\omega^\omega = \{ g \in \omega^\omega : \exists h \in \omega^\omega : g \cup h \in X_\omega \} \in \mathcal{F}_e$.

Since $X_\omega \in \mathcal{F}_e$, $X_\omega^\omega$ belongs to $\mathcal{F}_e$. Hence there is $f \in X_\omega^\omega$ such that $f$ is one-to-one. The same $f$ must occur as $f_e$ for an uncountable set $I$ of $e$'s. For $e \in I$, let $X_e^\omega = \{ h \in \omega^\omega : f \cup h \in X_e \}$. Hence (by the definition of $f$) $X_e^\omega \in \mathcal{F}_e$. Let $Z_e = \bigcap_{e \in I} A_{e_e} ; Z_e \in \mathcal{F}_e$. Clearly $\bigcap Z_e = Y_\omega$.

Let $a_n : \alpha \in \omega$ be the least $\alpha$ elements of $I$. Since $Z_e \in \mathcal{F}_e$, we can pick inductively functions $f_e : s_{e_n} \rightarrow \omega_1$ so that $\bigcap \{ A_{e_{m+n}} : m \in \omega_1 \} \subseteq Z_{e_n}$ and $\text{rng}(f) = \omega \cup \bigcup_{e \in I} \text{rng}(f) + 1$. Let $g = f + \bigcup_{e \in I} f_e$, then $g$ is one-to-one and $\alpha \in \omega$.