On the existence of measures on σ-algebras

by

Andrzej Pelc (Warszawa) and **Karel Prikry** * (Minneapolis)

Abstract. We investigate the problem whether the existence of measures on a family of subalgebras of an algebra $\mathscr{A}$ implies the existence of measures on $\mathscr{A}$ itself. Some relationships between various methods of constructing regularizing families for ultrafilters are also discussed in this context.

0. Introduction. Our main object is to study the problem whether the existence of non-trivial measures on a family of subalgebras of an algebra $\mathscr{A}$ implies the existence of a non-trivial measure on $\mathscr{A}$ itself. The actual questions we consider are motivated by the following problem of Banach [1]. Are there two countably generated σ-algebras on the interval $[0, 1]$ which both carry a non-trivial σ-additive measure but the σ-algebra generated by their union has no such measure.

This problem was answered by Grzegorek who constructed such algebras in [5]. Galvin suggested to us to consider the following more general problem. Does there exist a family of $\varkappa$ σ-algebras such that every λ of them generate an algebra carrying a non-trivial σ-additive measure and every ϱ of them generate an algebra without such measure. We construct such families of algebras for some infinite values of the cardinal parameters and some further cases, in particular those involving finite values, are discussed in a forthcoming paper by the first author.

There is some similarity between the problems we are considering and the so called marginal problem (cf. e.g. [7]): given a family of algebras F_α: $\alpha < \varkappa$ of subsets of a fixed set X and for each $\alpha < \varkappa$ a probability measure m_α defined on F_α, under what conditions will there exist a common extension of all measures m_α, defined on an algebra F containing $\bigcup_{\alpha<\varkappa} F_\alpha$. However the difference is that in the Banach-type problems we are interested in the existence of any measure on the large algebra, whereas in the marginal problem the object is to extend a particular family of measures.

Some of our results deal with measures that are only finitely additive. One of these results also sheds some light on the relationship between several techniques

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for establishing the regularity of ultrafilters on ω_1 — see Prikry [8], Benda, Ketonen [2] and Laver [6].

The main tool we use are matrices of sets. Hence an entire section of the paper is devoted to this topic. For most of our results we need either the continuum hypothesis or the axiom of constructibility.

1. Notation and terminology. We use standard set theoretic notation. Ordinals are identified with the set of their predecessors, cardinals with initial ordinals. $\varkappa$ and λ always denote infinite cardinals. $\mathscr{P}(X)$ is the set of subsets of X, $[X]^{\varkappa}$ and $[X]^{<\varkappa}$ denote the family of subsets of X of cardinality $\varkappa$ and $<\varkappa$ respectively. $\mathscr{I}$ is an ideal on X means that elements of $\mathscr{I}$ are subsets of X. An ideal $\mathscr{I}$ is $\varkappa$-complete if $\mathscr{A}\subset\mathscr{I}$, $|\mathscr{A}|<\varkappa \to \bigcup\mathscr{A}\in\mathscr{I}$. $\mathscr{I}$ is uniform on $\varkappa$ means that $[\varkappa]^{<\varkappa}\subset\mathscr{I}$. A family $\mathscr{A}$ of subsets of X is $\mathscr{I}$-almost disjoint if for distinct $A, B\in\mathscr{A}$, $A\cap B\in\mathscr{I}$. $\mathscr{A}\subset\mathscr{P}(X)$ is $\varkappa$-almost disjoint if it is $\mathscr{I}$-almost disjoint for $\mathscr{I}=[X]^{<\varkappa}$. An ideal $\mathscr{I}$ on X is $\varkappa$-saturated if there is no $\mathscr{I}$-almost disjoint family $\mathscr{A}\subset\mathscr{P}(X)$ such that $\mathscr{A}\cap\mathscr{I}=\emptyset$ and $|\mathscr{A}|=\varkappa$. If $\mathscr{I}$ is an ideal on X then $\mathscr{I}^+$ denotes the family $\mathscr{P}(X)\setminus\mathscr{I}$ and $\mathscr{I}^*$ denotes the family $\{X\setminus A: A\in\mathscr{I}\}$. $\mathscr{I}^*$ is also called the *dual filter*. An ultrafilter $\mathscr{U}$ on $\varkappa$ is regular if there is a family $\mathscr{A}\subset\mathscr{U}$, $|\mathscr{A}|=\varkappa$ such that for any $\mathscr{B}\in[\mathscr{A}]^{\omega}$, $\bigcap\mathscr{B}=\emptyset$. Such a family $\mathscr{A}$ is then called *regularizing* for $\mathscr{U}$. It is clear how to relativize these notions to any algebra $\mathscr{A}\subset\mathscr{P}(X)$, e.g. an ideal $\mathscr{I}\subset\mathscr{A}$ is uniform if it contains all sets of cardinality $<|X|$ belonging to $\mathscr{A}$.

If $\mathscr{A}$ is an algebra of subsets of X and μ is a measure defined on $\mathscr{A}$, μ is called a *probability measure* if $\mu(X)=1$ and a *diffuse measure* if μ vanishes on atoms of $\mathscr{A}$.

2. Matrices of sets. In this section we prove the existence of matrices of sets with various combinatorial properties. They will provide an important tool for getting our main results. The first theorem works only for ω (or for strongly inaccessible cardinals).

THEOREM 2.1. (a) *There exist sets* $X_{\alpha,n}\in[\omega]^{\omega}$ $(\alpha\in 2^{\omega}=\varkappa,\ n\in\omega)$, *such that*:

(i) $\forall\alpha\,\forall n_1,n_2[n_1\neq n_2\to X_{\alpha,n_1}\cap X_{\alpha,n_2}=\emptyset]$;

(ii) $\forall n\,\forall\alpha,\beta\ [\alpha\neq\beta\to|X_{\alpha,n}\cap X_{\beta,n}|<\omega]$;

(iii) $\bigcup_{n\in\omega}X_{\alpha,n}=\omega$, *for all* α;

(iv) *for every* $S\in[\varkappa]^{<\omega}$ *and every one-to-one* $f\colon S\to\omega$, $\bigcap_{\alpha\in S}X_{\alpha,f(\alpha)}\in[\omega]^{\omega}$.

(b) *There exist sets* $X_{\alpha,\beta}\in[\omega]^{\omega}$, $(\alpha,\beta\in 2^{\omega}=\varkappa)$ *such that*:

(i) *for all* α, *the family* $\{X_{\alpha,\beta}\colon\beta\in\varkappa\}$ *is almost disjoint*,

(ii) *for all* β, *the family* $\{X_{\alpha,\beta}\colon\alpha\in\varkappa\}$ *is almost disjoint*,

(iii) *for every* $S\in[\varkappa]^{<\omega}$ *and every one-to-one* $f\colon S\to\varkappa$, $\bigcap_{\alpha\in S}X_{\alpha,f(\alpha)}\in[\omega]^{\omega}$.

Proof. (a) Let $F_n=\{f\colon f\colon\mathscr{P}(n)\xrightarrow{1-1}2^n\}$, $F=\bigcup_{n\in\omega}F_n$. Since $|F|=\omega$, it suffices to construct the desired matrix on F rather than on ω. Let

$$X_{a,m}=\bigcup_{n\in\omega}\{f\in F_n\colon f(a\cap n)=m\}\quad\text{for}\quad a\subset\omega,\ m\in\omega.$$

(i) and (iii) are obvious. To check (ii), let $f\in F_n$, $a\neq b$ and $f\in X_{a,m}\cap X_{b,m}$, hence $f(a\cap n)=f(b\cap n)=m$. Since f is $1-1$, this can happen only for finitely many n, and since all F_n are finite, we get (ii). Finally, to get (iv), let $a_j\colon j\leqslant 1$ be distinct subsets of ω and $m_j\colon j\leqslant 1$ distinct natural numbers. Then for sufficiently large $n\in\omega$, there is a function $f\colon\mathscr{P}(n)\xrightarrow{1-1}2^n$ such that $\forall j\leqslant 1\, f(a_j\cap n)=m_j$. This shows that $\bigcap_{j\leqslant 1}X_{a_j,m_j}$ is infinite and hence the constructed sets have all required properties.

(b) Let $F_n=\{f\colon f\colon\mathscr{P}(n)\xrightarrow{1-1}\mathscr{P}(n)\}$, $F=\bigcup_{n\in\omega}F_n$. Again we use F as the underlying set instead of ω. We define for $a,b\subset\omega$, $X_{a,b}=\bigcup_{n\in\omega}\{f\in F_n\colon f(a\cap n)=b\cap n\}$.

It is easy to check as before that the sets $X_{a,b}$ have all required properties.

This type of result can be extended to an uncountable cardinal λ using $\diamondsuit_\lambda$.

THEOREM 2.2. *Let λ be uncountable regular and $\diamondsuit_\lambda$ hold. Then there exist stationary sets* $X_{\alpha,\beta}\subset\lambda$ $(\alpha,\beta\in 2^{\lambda}=\varkappa)$ *such that*:

(i) *for all* α, *the family* $\{X_{\alpha,\beta}\colon\beta\in\varkappa\}$ *is* λ-*almost disjoint*,

(ii) *for all* β, *the family* $\{X_{\alpha,\beta}\colon\alpha\in\varkappa\}$ *is* λ-*almost disjoint*,

(iii) *for every* $S\in[\varkappa]^{<\lambda}$ *and every one-to-one* $f\colon S\to\varkappa$, $\bigcap_{\alpha\in S}X_{\alpha,f(\alpha)}$ *is stationary*.

Proof. For $\alpha<\lambda$, let $S_\alpha=\{\langle a_{\alpha,\beta},b_{\alpha,\beta}\rangle,\ \beta<\alpha,\ a_{\alpha,\beta}$ distinct subsets of α, $b_{\alpha,\beta}$ distinct subsets of $\alpha\}$ be a $\diamondsuit_\lambda$ sequence, i.e. for every sequence $\{\langle a_\beta,b_\beta\rangle,\ \beta<\theta<\lambda$, a_β distinct subsets of λ, b_β distinct subsets of $\lambda\}$, the set

$$\{\alpha<\lambda\colon\forall\beta<\alpha\cap\theta\langle a_\beta\cap\alpha,b_\beta\cap\alpha\rangle=\langle a_{\alpha,\beta},b_{\alpha,\beta}\rangle\}$$

is stationary.

Now define, for $a,b\subset\lambda$,

$$X_{a,b}=\{\alpha<\lambda\colon\exists\beta<\alpha\ \langle a\cap\alpha,b\cap\alpha\rangle=\langle a_{\alpha,\beta},b_{\alpha,\beta}\rangle\}.$$

(i) and (ii) are straightforward. To check (iii), let $\theta<\lambda$ and a_ξ $(\xi<\theta)$ distinct b_ξ $(\xi<\theta)$ distinct. $\alpha\in\bigcap_{\xi<\theta}X_{a_\xi,b_\xi}$ iff $\forall\xi<\theta\,\exists\beta_\xi<\alpha\,\langle a_\xi\cap\alpha,b_\xi\cap\alpha\rangle=\langle a_{\alpha,\beta_\xi},b_{\alpha,\beta_\xi}\rangle$, hence we get that the set $\bigcap_{\xi<\theta}X_{a_\xi,b_\xi}$ is stationary, in view of $\diamondsuit_\lambda$.

Using the continuum hypothesis one can get the following $\omega\times\omega_1$ matrix of subsets of ω_1:

THEOREM 2.3. *Assume* CH. *There exists a matrix of subsets* $A_{\alpha,n}$ $(\alpha\in\omega_1,n\in\omega)$ *of* ω_1 *such that*:

(i) $\forall\alpha\in\omega_1\ \bigcup_{n\in\omega}A_{\alpha,n}=\omega_1$,

(ii) $\forall\alpha\in\omega_1\ \forall n,m\in\omega\ [n\neq m\to|A_{\alpha,n}\cap A_{\alpha,m}|\leqslant\omega]$,

(iii) $\forall n\in\omega\ \forall\alpha,\beta\in\omega_1\ [\alpha\neq\beta\to|A_{\alpha,n}\cap A_{\beta,n}|\leqslant\omega]$,

(iv) *for all sequences* $\{\langle\alpha_j,n_j\rangle\colon j\in\omega\}$ *such that* $j_1\neq j_2\to\alpha_{j_1}\neq\alpha_{j_2}$, $n_{j_1}\neq n_{j_2}$ *and such that* $\omega\setminus\{n_j\colon j\in\omega\}$ *is infinite*, $\bigcap_{j\in\omega}A_{\alpha_j,n_j}$ *is uncountable*.

Proof. Let $\{t_\xi: \xi \in \omega_1\}$ be an enumeration of all sequences as in (iv) s.t. every sequence is repeated ω_1 times. Fix an infinite $\alpha < \omega_1$. Define $\varphi_\alpha: \alpha \xrightarrow[\text{onto}]{1-1} \omega$ as follows. Let $t_\alpha = \{\langle \alpha_j, n_j\rangle: j \in \omega\}$. If $\{\alpha_j: j \in \omega\} \subset \alpha$ and $\alpha \setminus \{\alpha_j: j \in \omega\}$ is infinite let

$$\psi_\alpha: \alpha \setminus \{\alpha_j: j \in \omega\} \xrightarrow[\text{onto}]{1-1} \omega \setminus \{n_j: j \in \omega\}$$

be arbitrary, and put

$$\varphi_\alpha(\beta) = \begin{cases} n_j & \text{if} \quad \beta = \alpha_j \\ \psi_\alpha(\beta) & \text{if} \quad \beta \in \alpha \setminus \{\alpha_j: j \in \omega\}. \end{cases}$$

If $\alpha \setminus \{\alpha_j: j \in \omega\}$ is finite or $\{\alpha_j: j \in \omega\} \not\subset \alpha$ then let $\varphi_\alpha: \alpha \xrightarrow[\text{onto}]{1-1} \omega$ be arbitrary.

Now define $A_{\beta,n} = \{\alpha: n = \varphi_\alpha(\beta)\} \cup (\beta+1)$.

It is easy to check than (i)–(iv) are satisfied.

In order to get matrices of larger size, we use Kurepa's Hypothesis.

THEOREM 2.4. *Assume Kurepa's Hypothesis. Then there are sets* $X_{\xi,\eta} (\xi, \eta \in \omega_2)$ *such that* $X_{\xi,\eta} \subset \omega_1 \times 2^\omega$ *and*

(i) *for all* ξ *and all distinct* η_1, η_2 *there is* $\alpha \in \omega_1$ *s.t.* $X_{\xi,\eta_1} \cap X_{\xi,\eta_2} \subset \alpha \times 2^\omega$,

(ii) *for all* η *and all distinct* ξ_1, ξ_2 *there is* $\alpha \in \omega_1$ s.t. $X_{\xi_1,\eta} \cap X_{\xi_2,\eta} \subset \alpha \times 2^\omega$,

(iii) *for all* $S \in [\omega_2]^\omega$, *all one-to-one* $f: S \to \omega_2$ *and all sufficiently large* $\alpha \in \omega_1$, $|\bigcap_{\xi \in S} X_{\xi, f(\xi)} \cap (\{\alpha\} \times 2^\omega)| = 2^\omega$.

Proof. Let $\mathscr{K}$ be a Kurepa family on ω_1. Let $F_\alpha = \{f: \mathscr{K} \restriction \alpha \xrightarrow{1-1} \mathscr{K} \restriction \alpha\}$ and $X_{a,b} = \bigcup_{\alpha \in \omega_1} \{f \in F_\alpha: f(a \cap \alpha) = b \cap \alpha\}$, $a, b \in \mathscr{K}$. Since $\mathscr{K} \restriction \alpha$ is countable for every α, we can regard $X_{a,b}$ as subsets of $\omega_1 \times 2^\omega$. It is easy to check that (i)–(iii) are satisfied.

Assuming both Kurepa's Hypothesis and $\diamondsuit$, we can get an $\omega \times \omega_2$ matrix which combines some of the properties of previously constructed matrices.

THEOREM 2.5. *Assume* $\diamondsuit$ *and let* $\mathscr{K} \subset [\omega_1]^{\omega_1}$, $|\mathscr{K} \restriction \alpha| \leqslant \omega$ *for every* $\alpha \in \omega_1$. *Then there exists a matrix* $\{A_{a,n}: a \in \mathscr{K}, n \in \omega\}$ *of subsets of* ω_1, *such that*:

(i) $\forall a \in \mathscr{K}\ \forall m, n \in \omega\, [m \neq n \to A_{a,m} \cap A_{a,n} = \emptyset]$,

(ii) $\forall m \in \omega\ \forall a, b \in \mathscr{K}\, [a \neq b \to |A_{a,m} \cap A_{b,m}| \leqslant \omega]$,

(iii) $\forall a \in \mathscr{K}\ \bigcup_{n \in \omega} A_{a,n} = \omega_1$,

(iv) *for every sequence* $\{\langle a_j, n_j\rangle: j \in \omega,\ a_j \in \mathscr{K}, n_j \in \omega\}$ *s.t.* $j_1 \neq j_2 \to a_{j_1} \neq a_{j_2}$, $n_{j_1} \neq n_{j_2}$ *and* $\omega \setminus \{n_j: j \in \omega\}$ *is infinite, the set* $\bigcap_{j \in \omega} A_{a_j, n_j}$ *is stationary.*

Proof. For $\alpha \in \omega_1$, let $S_\alpha = \{\langle a_{\alpha,j}, n_{\alpha,j}\rangle: j \in \omega,\ a_{\alpha,j} \subset \alpha,\ n_{\alpha,j} \in \omega,\ a_{\alpha,j}, n_{\alpha,j}$ distinct as j varies and $\omega \setminus \{n_{\alpha,j}: j \in \omega\}$ infinite$\}$ be a $\diamondsuit$-sequence, i.e. for every sequence $\{\langle a_j, n_j\rangle: j \in \omega\}$ as in (iv), the set $\{\alpha \in \omega_1: \forall j \in \omega\, \langle a_j \cap \alpha, n_j\rangle = \langle a_{\alpha,j}, n_{\alpha,j}\rangle\}$ is stationary. Let $\varphi_\alpha: \mathscr{K} \restriction \alpha \setminus \{a_{\alpha,j}: j \in \omega\} \xrightarrow{1-1} \omega \setminus \{n_{\alpha,j}: j \in \omega\}$ be arbitrary. Put $t_\alpha = S_\alpha \cup \varphi_\alpha$. Now we define the matrix $A_{a,n} (a \in \mathscr{K}, n \in \omega)$:

$$A_{a,n} = \{\alpha \in \omega_1: \langle a \cap \alpha, n\rangle \in t_\alpha\}.$$

(i)–(iv) are easy to check.

Various matrices with further properties of the type we discussed so far can be considered. We conclude this section by pointing out a related problem which seems to be open: Do there exist sets $X_{\alpha,\beta} \subset 2^\omega$ $(\alpha \in (2^\omega)^+, \beta \in 2^\omega)$ s.t.

(i) for all α, $\{X_{\alpha,\beta}: \beta \in 2^\omega\}$ is a disjoint family.

(ii) for all β, $\{X_{\alpha,\beta}: \alpha \in (2^\omega)^+\}$ is 2^ω-almost disjoint.

(iii) for all $S \in [(2^\omega)^+]^{\leqslant 2^\omega}$ and for all one-to-one $f: S \to 2^\omega$, $|\bigcap_{\alpha \in S} X_{\alpha, f(\alpha)}| = 2^\omega$.

3. Measures and filters on σ-algebras. The following theorem provides an answer to a question of Woodin.

THEOREM 3.1. *There is a compact* $F \subset \omega^\omega$ *such that*

(i) *for every enumerable* $E \subset F$ *there is a* 0, 1*-valued finitely additive diffuse measure on* ω *such that for every* $f \in E$, f *is constant on a set of measure* 1.

(ii) *if* $H \subset F$, $|H| = \omega_1$ *and* μ *is any finitely additive diffuse probability measure on* ω, *then there is some* $f \in H$ *such that* f *is not constant on any set of positive measure.*

Remark. In fact, the theorem remains valid if (ii) is replaced by the following statement

(ii′) *If* $H \subset F$, $|H| = \omega_1$ *and* $\mathscr{I}$ *is an* ω_1*-saturated uniform ideal on* ω *then there is some* $f \in H$ *such that* f *is not constant on any set in* $\mathscr{I}^+$.

Other results of this type also follow.

Proof. We use the matrix constructed in Theorem 2.1. a). Since for every $\alpha \in 2^\omega$, the family $\{X_{\alpha,n}: n \in \omega\}$ is a partition of ω, we can associate with it a function g_α defined by $g_\alpha(m) = n$ iff $m \in X_{\alpha,n}$.

Let $F = \{g_\alpha: \alpha \in 2^\omega\}$. It follows from the definition of the sets $X_{\alpha,n}$, that F is a compact subset of ω^ω.

Let $E = \{g_{\alpha_n}: n \in \omega\}$ be a countable subset of F.

In view of property (iv) in Theorem 2.1, the family $\{X_{\alpha_n,n}: n \in \omega\}$ can be extended to a uniform ultrafilter on ω, hence (i) of Theorem 3.1. follows. To get (ii)′ which implies (ii), let $H = \{g_{\alpha_\xi}: \xi \in \omega_1\}$ be an uncountable subset of F. Suppose (ii)′ is false. Let $\mathscr{I}$ be an ω_1-saturated uniform ideal on ω, such that every g_{α_ξ} is constant on a set from $\mathscr{I}^+$. This means that for every $\xi \in \omega_1$ there is $n_\xi \in \omega_1$ such that $X_{\alpha_\xi n_\xi} \in \mathscr{I}^+$.

n_ξ is the same for uncountably many ξ, hence we get a contradiction with ω_1-saturation. This finishes the proof.

A similar result can be obtained for uncountable cardinals using Theorem 2.2.

THEOREM 3.2. *Assume $\diamondsuit$. There exists a family $F \subset \omega_1^{\omega_1}$ such that $|F| = \omega_2$ and*

(i) *for every $E \in [F]^{\omega_1}$ there is a countably complete filter on ω_1 such that each $f \in E$ is constant on a set from the filter.*

(ii) *for every $H \in [F]^{\omega_2}$ and every ω_2-saturated uniform ideal $\mathcal{I}$, there is an $f \in H$, such that f is not constant on any set in $\mathcal{I}^+$.*

Proof. We take the matrix $X_{\alpha,\beta}$ $(\alpha, \beta \in 2^{\omega_1})$ which exists in view of Theorem 2.2. (for $\lambda = \omega_1$). Consider only the sets $X_{\alpha,\beta}$ $(\alpha \in 2^{\omega_1}, \beta \in \omega_1)$. Since every family $\{X_{\alpha,\beta}: \beta \in \omega_1\}$ is ω_1-almost disjoint and $X_{\alpha,\beta}$ are all stationary, we can disjointise them and then proceed exactly as in the proof of the proceeding theorem.

We now turn our attention to Banach's problem mentioned in the introduction.

Problem of Banach: Do there exist two countably generated σ-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ of subsets of $[0, 1]$ such that both of them carry σ-additive probability measures vanishing on atoms, but the σ-algebra generated by $\mathcal{A}_1 \cup \mathcal{A}_2$ does not carry any such measure?

It is not obvious whether every countably generated σ-algebra which does not carry σ-additive diffuse probability measures cannot be generated by σ-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ as above. The next theorem provides an example of an algebra which strongly fails to have this property.

THEOREM 3.3. *Assume that the union of $<2^\omega$ meager subsets of ω^ω is not ω^ω. Then there exists a countably generated σ-algebra $\mathcal{A}$ of subsets of $[0, 1]$ such that no countably generated σ-algebra $\mathcal{A}_1 \subset \mathcal{A}$ carries a σ-additive diffuse probability measure.*

Proof. The following lemma follows from a result of Darst [3].

LEMMA 3.4. *For every sequence $\{B_n: n \in \omega\}$ of Borel subsets of $[0, 1]$ and every σ-additive probability Borel measure μ, there exists a set $S \subset 2^\omega$ s.t. $\mu(S) = 1$ and $\bigcup_{f \in S} \bigcap_{n \in \omega} f(n) B_n$ is meager.* $(0B = B,\ 1B = [0, 1] \setminus B)$.

The following lemma is due to Rothberger [9].

LEMMA 3.5. *Assume that the union of $<2^\omega$ meager subsets of ω^ω is not ω^ω. Then every set of cardinality $<2^\omega$ has strong measure* 0.

Proof. Let $X = \{x_\alpha: \alpha < \varkappa\}$, $\varkappa < 2^\omega$, and let $\varepsilon_n > 0$, $n \in \omega$. Let $I_{n,j}$ $(j \in \omega)$ enumerate rational intervals s.t. the length of $I_{n,j}$ is $<\varepsilon_n$. Put $U_f = \bigcup_{n \in \omega} I_{n,f(n)}$ for every $f \in \omega^\omega$. The set $D_x = \{f: x \in U_f\}$ is dense open. Hence $\bigcap_{\alpha < \varkappa} D_{x_\alpha} \neq \emptyset$. It follows that $X \subset \bigcup_{n \in \omega} I_{n,f(n)}$, hence X has strong measure 0.

Under the assumption of the theorem there exists a set $\mathcal{L} \subset [0, 1]$ of cardinality 2^ω such that for every meager set M, $|\mathcal{L} \cap M| < 2^\omega$. It suffices to show that the algebra of Borel subsets of $\mathcal{L}$ has the required properties.

Suppose not. Hence there are Borel sets B_n $(n \in \omega)$ s.t. $\{B_n \cap \mathcal{L}: n \in \omega\}$ generates a σ-algebra carrying a σ-additive diffuse probability measure ν. The measure μ defined by $\mu(S) = \nu\big(\bigcup_{f \in S} \bigcap_{n \in \omega} f(n)(B_n \cap \mathcal{L})\big)$ is clearly a Borel measure on 2^ω. Take S as in Lemma 3.4. Then $C = \bigcup_{f \in S} \bigcap_{n \in \omega} f(n)(B_n \cap \mathcal{L})$ is a meager subset of $\mathcal{L}$, hence $|C| < 2^\omega$ and $\nu(C) = 1$. This, in view of Lemma 3.5. gives a contradiction, hence our theorem is proved. Let us remark that another proof of Theorem 3.3 can be obtained using Theorem 1, § 3, Chapter II of Sierpiński [10].

The next two theorems give positive answers to a generalization of Banach's problem.

THEOREM 3.6. *Assume* CH. *There exists an increasing sequence $\mathcal{A}_\alpha$ $(\alpha < \omega_1)$ of countably generated σ-algebras of subsets of $[0, 1]$ such that every $\mathcal{A}_\alpha$ carries a σ-additive diffuse probability measure, but the σ-algebra $\bigcup_{\alpha \in \omega_1} \mathcal{A}_\alpha$ does not carry any such measure.*

Proof. We use Theorem 2.3. Fix $\alpha \in \omega_1$. For each $\beta < \alpha$ pick a different even $n_\beta \in \omega$. Then $\{\langle \beta, n_\beta \rangle: \beta < \alpha\}$ is as in (iv) from Theorem 2.3., hence $A_{\beta,n_\beta} (\beta < \alpha)$ are σ-independent, which shows that there is a σ-additive diffuse probability measure on the σ-algebra $\mathcal{A}_\alpha$ generated by $\{A_{\beta,n}: \beta < \alpha, n \in \omega\}$. A well known argument of Ulam [11] shows that there is no σ-additive diffuse probability measure on the σ-algebra generated by $\{A_{\alpha,n}: \alpha \in \omega_1, n \in \omega\}$. This finishes the proof.

Remark. After having read the above proof J. Cichoń (unpublished) eliminated the assumption of CH via a different argument.

THEOREM 3.7. *Assume Kurepa's Hypothesis and $\diamondsuit$. Then there exists a sequence $\mathcal{A}_\alpha$ $(\alpha \in \omega_2)$ of countably generated σ-algebras of subsets of $[0, 1]$, s.t. the union of every countable subcollection generates a σ-algebra carrying a σ-additive diffuse probability measure, but the union of every uncountable subcollection generates a σ-algebra which does not carry any such measure.*

Proof. In view of Kurepa's Hypothesis we can take the family $\mathcal{K}$ from Theorem 2.5. with cardinality ω_2. Define $\mathcal{A}_\alpha$ as being generated by

$$\{A_{a_k,n}: a_k \in C_\alpha, n \in \omega\},$$

where C_α countable and $\alpha \in C_\alpha$. The family $\{\mathcal{A}_\alpha: \alpha < \omega_2\}$ is as required.

The last result of this section deals with the regularity of ultrafilters, and is of the same type as above. A large collection of algebras is shown to have the property that algebras generated by small unions carry non-regular uniform ultrafilters, but algebras generated by large unions contain regularizing families for every uniform ultrafilter.

Regularity of ultrafilters was previously discussed e.g. by Prikry [8], Benda, Ketonen [2] and Laver [6]. In all these papers matrices of sets were constructed under various additional assumptions, providing regularizing families for ultrafilters e.g. on ω_1. In [8] and [6] these matrices had size ω_1, in [2] under the assumption of Kurepa's Hypothesis a matrix of size ω_2 is constructed and for every uniform ultrafilter some initial segment of it provides a regularizing family.

Our result shows (under the assumption of KH and $\diamondsuit$) that this method cannot be improved by considering any initial segment of a Kurepa matrix (used in [2]), for the algebra generated by every initial segment carries a non-regular uniform ultrafilter.

THEOREM 3.8. *Assume Kurepa's Hypothesis and* $\diamondsuit$. *There are algebras* $\mathcal{A}_\alpha$: $\alpha < \omega_2$ *of subsets of* ω_1 *such that the algebra generated by the union of any* ω_1 *of them carries a non-regular uniform ultrafilter, but every uniform ultrafilter on the algebra generated by the union of* ω_2 *of them is regular.*

Proof. We use the matrix from Theorem 2.5. with the family $\mathcal{K}$ of cardinality ω_2 given by Kurepa's Hypothesis. The algebras $\mathcal{A}_\alpha$ $(\alpha \in \omega_2 \setminus \omega_1)$ are defined using sets $A_{\beta,n}$ $(\beta < \alpha, n \in \omega)$. It is enough to consider $\alpha = \omega_1$. Let $\mathcal{A}_t$ be the σ-algebra generated by the sets $A_{\alpha,n}$ $(\alpha \in t, n \in \omega)$. Define $\mathcal{A}_{\omega_1} = \bigcup \{\mathcal{A}_t: t \in [\omega_1]^{<\omega}\}$. $\mathcal{A}_{\omega_1}$ is an algebra since the family of $\mathcal{A}_t$ is a directed system of algebras. By the argument of Benda, Ketonen [2] every ultrafilter is regular on the algebra generated by $\{A_{\alpha,n}: \alpha < \omega_2, n \in \omega\}$ hence the last statement of the theorem follows. It is enough to show the existence of a non-regular ultrafilter on $\mathcal{A}_{\omega_1}$. (for any union of ω_1 algebras $\mathcal{A}_\alpha$ the proof is similar.)

Let $\mathcal{U}$ be a uniform ultrafilter on ω. We define by induction ultrafilters $\mathcal{U}^n$ on ω^n $(n \in \omega)$. For $X \subset \omega^{n+1}$, $X \in \mathcal{U}^{n+1}$ iff $\{s \in \omega^n: \{k: s^\frown k \in X\} \in \mathcal{U}\} \in \mathcal{U}^n$. Now for each $t \in [\omega_1]^{<\omega}$, let $\mathcal{U}^t$ be the ultrafilter on ω^t obtained as follows: let $\alpha_0 < \ldots < \alpha_{|t|-1}$ enumerate t. Define $\varphi: \omega^{|t|} \to \omega^t$ by $(\varphi(s))(\alpha_j) = s_j$, $s \in \omega^{|t|}$, $j < |t|$ and $\mathcal{U}^t = \varphi(\mathcal{U}^{|t|})$. Let $\mathcal{U}_t$ be the ultrafilter on $\mathcal{A}_t$ consisting of sets of the form $\bigcup_{s \in X} \bigcap_{\alpha \in t} A_{\alpha, s(\alpha)}$, $(X \in \mathcal{U}^t)$. It is not hard to see that if $t \subset v$ then $\mathcal{U}_t \subset \mathcal{U}_v$. Hence $\tilde{\mathcal{U}} = \bigcup \{\mathcal{U}_t: t \in [\omega_1]^{<\omega}\}$ is an ultrafilter on $\mathcal{A}$. Clearly $\tilde{\mathcal{U}}$ is uniform. We show that $\tilde{\mathcal{U}}$ is non-regular. Claim: If $Y_\alpha \in \tilde{\mathcal{U}}$ $(\alpha \in \omega_1)$ then there is an infinite $S \subset \omega_1$ such that $\bigcap \{Y_\alpha: \alpha \in S\}$ is stationary.

Proof of the claim. For each $\alpha \in \omega_1$ let $s_\alpha \in [\omega_1]^{<\omega}$ be such that $Y_\alpha \in \mathcal{A}_{s_\alpha}$. Without loss of generality we may assume that $|s_\alpha| = n$ for all $\alpha < \omega_1$ and that the sets s_α form a Δ-system with kernel s. Let $X_\alpha \in \mathcal{U}^{s_\alpha}$ be such that $Y_\alpha = \bigcup_{g \in X_\alpha} \bigcap_{\eta \in s_\alpha} A_{\eta, g(\eta)}$. We may assume that s is the initial segment of all s_α. Let

$$X'_\alpha = \{g \in \omega^s: \{h \in \omega^{s_\alpha \setminus s}: g \cup h \in X_\alpha\} \in \mathcal{U}^{s_\alpha \setminus s}\}.$$

Since $X_\alpha \in \mathcal{U}^{s_\alpha}$, X'_α belongs to $\mathcal{U}_s$. Hence there is $f_\alpha \in X'_\alpha$ such that f_α is one-to-one. The same f must occur as f_α for an uncountable set I of α's. For $\alpha \in I$, let $X''_\alpha = \{h \in \omega^{s_\alpha \setminus s}: f \cup h \in X_\alpha\}$. Hence (by the definition of f), $X''_\alpha \in \mathcal{U}^{s_\alpha \setminus s}$. Let $Z_\alpha = \bigcup_{h \in X''_\alpha} \bigcap_{\eta \in s_\alpha \setminus s} A_{\eta, h(\eta)}$. $Z_\alpha \in \mathcal{U}_{s_\alpha \setminus s}$. Clearly $\bigcap_{\eta \in s} A_{\eta, f(\eta)} \cap Z_\alpha \subset Y_\alpha$.

Let α_n: $n \in \omega$ be the least ω elements of I. Since $Z_{\alpha_n} \in \mathcal{U}_{s_{\alpha_n} \setminus s}$, we can pick inductively functions $f_n: s_{\alpha_n} \setminus s \xrightarrow{1-1} \omega$, so that $\bigcap \{A_{\beta, f_n(\beta)}: \beta \in s_{\alpha_n} \setminus s\} \subset Z_{\alpha_n}$ and $\operatorname{rng}(f_n) \subset \omega \setminus (\max \operatorname{rng}(f) \cup \bigcup_{j<n} \operatorname{rng}(f_j) + 1)$. Let $g = f \cup \bigcup_{n \in \omega} f_n$, then g is one-to-one and $\omega \setminus \operatorname{rng} g$ is infinite. It follows that $\bigcap_{\beta \in \operatorname{dom} g} A_{\beta, g(\beta)} \subset \bigcap_{n \in \omega} Y_{\alpha_n}$. Hence by property (iv) of the matrix, the set $\bigcap_{n \in \omega} Y_{\alpha_n}$ is stationary. This proves the claim and finishes the proof of the theorem.

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