Some extension and classification theorems for maps of movable spaces

by

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Abstract. Suppose that Y is an $(n-1)$-connected CW complex. We establish, cohomological criteria for extendability of a map $f\colon A \to Y$ onto X, where A is a closed subset of X, in the cases where X, A or (X, A) are movable.

We classify also the homotopy classes from a movable space X to Y if X satisfies certain cohomological conditions.

In the last section we prove that there exists a CE map $f\colon X \to Y$ (where X is a compactum) which raises the topological dimension iff there are a natural number n and a CE map $f\colon I^n \to Z$ such that the shape of Z is non-trivial.

H. Hopf proved two theorems about maps of finite dimensional compacta into the n-dimensional sphere S^n. The first theorem (the Hopf classification theorem) permits to classify the homotopy classes of maps of an n-dimensional compactum X into S^n and the second (the Hopf extension theorem) reduces the problem of extending a map $f\colon A \to S^n$ from a closed subset A of an $(n+1)$-dimensional compactum X to an algebraic problem.

These results have been extended to larger classes of spaces and polyhedra. In particular, it is known that S^n may be replaced by any $(n-1)$-connected polyhedron Y and that whenever Y is an Eilenberg–MacLane space $K(G, n)$, the hypothesis concerning the dimension of X is redundand in both theorems.

The main purpose of the present note is to generalize these classical theorems to the case of paracompact spaces satisfying some special conditions concerning the shape only.

As an immediate consequence we get in particular that, in the case of movable paracompact spaces, one may replace the hypothesis of finite dimensionality by that of acyclity of X and A in almost all dimensions (in the sense of the Čech cohomology theory).

We constructed an example which shows that the assumptions of movability and acyclicity are essential.

Since one can characterize the dimension in terms of extension of maps into spheres, our theorems give us a pretext for proving some facts concerning CE maps which raise the dimension.

Among other things, we prove that if there exists a CE map $f\colon I^n \to Y$ with $\dim Y > n$, then there exists a CE map $g\colon I^n \to Z$ such that the shape of Z is non-trivial.

Numereous papers in shape theory ([$B_{1,2}$], [B–H], [Ko], [L]) deal with the problem of extension of a fundamental sequence and the problem of classification of fundamental classes to $(n-1)$-shape connected FANR-set Y. Since every FANR set Y has the shape of a polyhedron, we infer that these questions reduce themselves to our investigations.

In order to prove our theorem we have to develop some special techniques. Among other things we introduce the notion of the *deformation dimension with respect to a class of* CW *complexes* (compare [N–S]) and we study the properties of this notion.

If $\mathscr{C}$ is a category, we denote by $\mathscr{C}$ also the class of objects of $\mathscr{C}$. This will not cause confusion because the meaning of the symbols will be clear from the context.

In this paper we denote by Pol (resp. HPol) the category whose objects are polyhedra (resp. all spaces having the homotopy type of a polyhedron) and whose morphisms are maps (resp. homotopy classes of maps).

We shall consider the categories Pol^2 and HPol^2. The objects of Pol^2 are polyhedral pairs and the morphisms are maps of pairs. The objects of HPol^2 are pairs which have the homotopy type of a polyhedral pair and the morphisms are homotopy classes of maps of pairs.

$\mathscr{W}$ and $\mathscr{S}$ will denote the classes of all CW complexes and all connected and simply connected polyhedra, respectively.

If $\mathscr{C}$ is a subclass of $\mathscr{W}$, we denote by $\mathscr{C}^2$ the class of all pairs (P, Q), where Q is a subcomplex of P and both are members of $\mathscr{C}$.

We shall only use the Čech cohomology groups (see [M–S]).

1. The deformation dimension of a pair with respect to a class of pairs of CW complexes. Suppose that $\mathscr{C}$ is a subclass of $\mathscr{W}^2$ consisting of at least one element (P, Q) with $P \neq \emptyset$. To every pair (X, A) with $X \neq \emptyset$ we assign the *deformation dimension of* (X, A) *with respect to* $\mathscr{C}$ (denoted by $\mathrm{def}((X, A); \mathscr{C})$), which is an integer $\geqslant -1$ or ∞ and which equals to the minimum n such that any map $f\colon (X, A) \to (P, Q) \in \mathscr{C}$ is homotopic to one whose image is in the set $Q \cup P^{(n)}$ (we assume $P^{(-1)} = \emptyset$).

Similarly one can introduce the notion of the *deformation dimension of a topological space* $X \neq \emptyset$ with respect to a subclass $\mathscr{C}$ of $\mathscr{W}$ (denoted by $\mathrm{def}(X; \mathscr{C})$) which contains at least one CW complex $P \neq \emptyset$. Obviously, $\mathrm{def}(X; \mathscr{C}) = \mathrm{def}((X, \emptyset); \mathscr{C}')$, where $\mathscr{C}'$ denotes the class of all pairs $(P, \emptyset)$ with $P \in \mathscr{C}$.

(1.1) Remark. It is known ([Sp] p. 57) that if A is a closed subset of X and B is a subset of Y such that (X, A) has the HEP (Homotopy Extension Property) with respect to B and $(X \times [0, 1], X \times \{0, 1\} \cup A \times [0, 1])$ has the HEP with respect to Y, then a map $f\colon (X, A) \to (Y, B)$ is homotopic (as a map of pairs) to a map which sends all of X to B iff it is homotopic relative to A to such a map. It follows that if A is a closed subset of X and if X and $\mathscr{C}$ satisfy one of the following conditions

(a) X is a metrizable space and $\mathscr{C} \subset \mathrm{Pol}^2$;

(b) X is a compact Hausdorff space and $\mathscr{C} \subset \mathscr{W}^2$;

(c) X is a paracompact space and $\mathscr{C}$ contains only pairs of compact CW complexes;

then in the definition of $\mathrm{def}(X, A; \mathscr{C})$ we may additionally assume that f is homotopic relative to A to a map $g\colon X \to P$ with $g(X) \subset Q \cup P^{(n)}$.

(1.2) Theorem. *Suppose that* (X, A) *and* (Y, B) *are pairs of topological spaces with* $X \neq \emptyset \neq Y$ *and* $\mathscr{C}$ *is a subclass of* $\mathscr{W}^2$ *and* $\mathrm{Sh}(X, A) \leqslant \mathrm{Sh}(Y, B)$. *Then* $\mathrm{def}(X, A; \mathscr{C}) \leqslant \mathrm{def}(Y, B; \mathscr{C})$.

Proof. Let $(P, Q) \in \mathscr{C}$ and $\mathrm{def}(Y, B; \mathscr{C}) \leqslant n$. Assume also that $\underline{p}\colon (X, A) \to (Y, B)$ and $\underline{q}\colon (Y, B) \to (X, A)$ are shape morphisms such that $\underline{q} \circ \underline{p} = \mathrm{Sh}(\mathrm{id}_{(X, A)})$ and that $f\colon (X, A) \to (P, Q) \in \mathscr{C}$ is a map. Then there exists a map $g\colon (Y, B) \to (P^{(n)} \cup Q, Q)$ such that $\mathrm{Sh}(f) \circ \underline{q} = \mathrm{Sh}(j) \circ \mathrm{Sh}(g)$, where $j\colon (P^{(n)} \cup Q, Q) \to (P, Q)$ is the inclusion.

Observe that $\mathrm{Sh}(f) = \mathrm{Sh}(f) \circ \underline{q} \circ \underline{p} = \mathrm{Sh}(j) \circ \mathrm{Sh}(g) \circ \underline{p}$ and that there is a map $f'\colon (X, A) \to (P^{(n)} \cup Q, Q)$ satisfying the equality $\mathrm{Sh}(f') = \mathrm{Sh}(g) \circ \underline{p}$.

Hence $jf' \simeq f$ and the proof is finished.

2. Auxiliary theorems. Now we prove the following

(2.1) Proposition. *Let* $A \neq \emptyset$ *be a closed subset of a paracompact space* X *and let* $(P, Q) \in \mathscr{S}^2$ *be a pair with* $Q \neq \emptyset$. *If* (X, A) *has the* HEP *with respect to* $\mathscr{S}$, $\mathrm{def}(A; \mathscr{S}) < \infty$ *and* $\mathrm{def}(X; \{P\}) < \infty$, *then* $\mathrm{def}((X, A); \{(P, Q)\}) < \infty$. *If we assume additionally that* $\mathrm{def}(X; \mathscr{S}) < \infty$, *then* $\mathrm{def}((X, A); \mathscr{S}^2) < \infty$.

Proof. Let $n \geqslant \max(2, \mathrm{def}(X; \{P\}), \mathrm{def}(A; \mathscr{S})+1)$ and let $f\colon (X, A) \to (P, Q)$ be a map. Using the properties of a resolution and an expansion of (X, A) (see [M-S]), we can find a map $g\colon (X, A) \to (W, V) \in \mathscr{W}^2$ and cellular maps $f'\colon (W, V) \to (P, Q)$ and $f''\colon W \to P_0 = Q \cup P^{(n)}$ such that $f \simeq f'g$ as maps from (X, A) to (P, Q) and $f' \simeq i f''$ as maps from W to P, where $i\colon P_0 \to P$ is the inclusion.

Since f' and if'' are cellular, we may assume that $f' \simeq if''$ as maps from $(W, W^{(n-1)})$ to $(P, P_0) \in \mathscr{S}^2$ and we can find a map $f'''\colon W \to P_0$ satisfying the condition

$$if''' \simeq f' \ \mathrm{rel}\, W^{(n-1)}. \tag{2.2}$$

We may also assume that $W, V \in \mathscr{S}$. Indeed, if W and V are not members of $\mathscr{S}$, then we can replace W and V by CW complexes obtained from W and V by ad joining 1-cells which join all components of W and all components of V and attaching 2-dimensional simplicial cells to the 1-skeletons of W and V in such a way that the resulting spaces are simply connected. It is clear that f', f''' and the homotopy joining relative to $W^{(n-1)}$ the maps f' and if''' are extendable over W and V thus modified.

Since $(W, V) \in \mathcal{S}^2$ and $\operatorname{def}(A; \mathcal{S}) \leqslant n-1$, we see that there exists $g \simeq g'$: $(X, A) \to (W, V)$ with

$$g'(A) \subset V^{(n-1)} \subset W^{(n-1)}. \tag{2.3}$$

From (2.2) and (2.3) we get that $f'g \simeq f'g'$ and $f'g' \simeq if'''g'$ as maps from (X, A) to (P, Q) and $if'''g'(X) \subset f'''(W) \subset P_0$. The proof is finished.

In [N–S] S. Spież and the author of this paper have proved the following theorem.

(2.4) THEOREM. *Suppose that* $(P, Q) \in \mathrm{HPol}^2$ *and* Q *is connected and simply connected. If* A *is a closed subset of a paracompact space* X *such that the pair* (X, A) *satisfies one of the following conditions*:

(a) $\operatorname{def}(X, A; \mathcal{S}^2) < \infty$,

(b) $\operatorname{def}(A; \mathcal{S}) < \infty$, X *is movable and* (X, A) *has the* HEP *with respect to* P,

(c) (X, A) *is a movable pair*,

and that $H^n(X, A; \pi_n(P, Q)) = 0$ *for every* $n = 2, 3, \ldots$, *then any map* $f\colon (X, A) \to (P, Q)$ *is homotopic to a map* $g\colon (X, A) \to (P, Q)$ *with* $g(X) \subset Q$.

We denote by $c(X, A)$ the maximum (finite or infinite) of all integers n such that there is an abelian group G with $H^n(X, A; G) \neq 0$.

As an immediate consequence of (2.1) and (2.4) we get the following

(2.5) COROLLARY. *Suppose that* A *is a closed subset of* X, *where* X *is a compact Hausdorff space or* X *is metrizable. Suppose also that the pair* (X, A) *satisfies one of the following conditions*:

(a) X *and* A *are movable with* $c(A) < \infty$,

(b) $\operatorname{def}(X; \mathcal{S}) < \infty$ *and* $\operatorname{def}(A; \mathcal{S}) < \infty$.

If $H^n(X, A; \pi_n(P, Q)) = 0$ *for every* $n \geqslant 2$, *where* $(P, Q) \in \mathrm{HPol}^2$ *and* Q *is simply connected and connected, then any map* $f\colon (X, A) \to (P, Q)$ *is homotopic to a map* $g\colon (X, A) \to (P, Q)$ *with* $g(X) \subset Q$.

3. The Hopf Extension Theorem. We begin with the following simple lemma.

(3.1) LEMMA. *Suppose that* $Y \in \mathcal{W}$ *is* $(n-1)$*-connected, where* $n \geqslant 2$. *Suppose also that* A *is a closed subset of* X *and* X *and* Y *satisfy one of the following conditions*

(a) X *is metrizable and* $Y \in \mathrm{Pol}$,

(b) X *is a compact Hausdorff space*,

(c) X *is a paracompact space and* Y *is compact*.

Then Y *is a subspace of the Eilenberg–MacLane space* $\hat{Y} = K(\pi_n(Y), n) \in \mathcal{W}$ *such that the inclusion* $i Y \to \hat{Y}$ *is an* $(n+1)$*-equivalence, the groups* $\pi_m(Y)$ *and* $\pi_{m+1}(\hat{Y}, Y)$ *are isomorphic for* $m \geqslant n+1$, *the pair* (X, A) *has the* HEP *with respect to* Y *and* $\hat{Y}$ *and the pair* $(X \times [0, 1], X \times \{0, 1\} \cup A \times [0, 1])$ *has the* HEP *with respect to* $\hat{Y}$.

For every $(n-1)$-connected CW complex Y, we denote by $\varkappa^n(Y)$ the characteristic element of Y ([Hu] p. 193 or [Sp] p. 425).

One checks easily that the inclusion $i\colon Y \to \hat{Y}$ induces a canonical isomorphism $i^*\colon H^n(\hat{Y}; \pi_n(\hat{Y})) = H^n(\hat{Y}; \pi_n(Y)) \to H^n(Y; \pi_n(Y))$ and that $i^*(\varkappa^n(\hat{Y})) = \varkappa^n(Y)$.

Let A be a closed subset of a paracompact space X. An element of the group $H^n(A; G)$ is said to be extendable over X ([Hu] p. 192) iff it is contained in the image of the homomorphism $i^*\colon H^n(X, G) \to H^n(A; G)$ which is induced by the inclusion map $i\colon A \to X$.

If A is a closed subset of X and the pair (X, A) has the HEP with respect to an $(n-1)$-connected CW complex Y with vanishing homotopy groups in dimensions greater than n, then a map $f\colon A \to Y$ is extendable over X iff $f^*(\varkappa^n(Y))$ is extendable over X. The same situation is in the case where (X, A), Y and $\hat{Y}$ satisfy the conditions in the assertion of Lemma (3.1) and every map from (X, A) to $(\hat{Y}, Y)$ is homotopic to a map with values in Y (compare Remark (1.1)).

Combining this fact with Proposition (2.1), Theorem (2.4), Corollary (2.5) and Lemma (3.1) we get

(3.2) THEOREM. *Assume that* $n \geqslant 2$, $Y \in \mathcal{W}$ *is* $(n-1)$*-connected and* A *is a closed subset of a paracompact space* X *such that* $H^m(X, A; \pi_{m-1}(Y)) = 0$ *for every* $m \geqslant n+2$ *and the pair* (X, A) *and* Y *satisfy one of the conditions* (a), (b) *and* (c) *of Lemma* (3.1) *and* (X, A) *satisfies one of the following conditions*:

(i) (X, A) *is movable*,

(ii) X *is movable and* $\operatorname{def}(A; \mathcal{S}) < \infty$,

(iii) $\operatorname{def}(X, A; \mathcal{S}^2) < \infty$,

(iv) X *is metrizable or* X *is a compact Hausdorff space with* $\operatorname{def}(X; \mathcal{S}) < \infty$ *and* $\operatorname{def}(A; \mathcal{S}) < \infty$ (*in particular*, X *and* A *are movable with* $c(X) < \infty$ *and* $c(A) < \infty$).

Then a map $f\colon A \to Y$ *is extendable over* X *iff* $f^*(\varkappa^n(Y))$ *is extendable over* X.

(3.3) EXAMPLE. There are a movable metric continuum X_0, its movable subcontinuum A_0, a map $f\colon A_0 \to S^3$ such that $c(X_0, A_0) \leqslant 0$ and f is not extendable over X_0.

D. S. Kahn described ([Ka]) an inverse sequence $\underline{K} = \{K_n, q_n^{n+1}\}$ of compact members of $\mathcal{S}$ such that $K = \lim \operatorname{inv} \underline{K}$ is a non-movable ([H–S]) acyclic continuum and there exists a map $f_1\colon K_1 \to S^3$ such that the composition $f' = f_1 q_1\colon K \to S^3$ is an essential map ($q_1\colon K \to K_1$ denotes the projection).

Let $\{k_n\}_{n=1}^{\infty} \in K \subset \mathop{\mathrm{P}}\limits_{n=1}^{\infty} K_n$ and $(X_n, x_n) = \bigvee\limits_{i=1}^{n} (K_i, k_i)$. Setting

$$p_n^{n+1}(x) = \begin{cases} x & \text{for} \quad x \in K_i,\ 1 \leqslant i \leqslant n \\ q_n^{n+1} & \text{for} \quad x \in K_{n+1} \end{cases}$$

we get a map $p_n^{n+1}\colon (X_{n+1}, K_{n+1}) \to (X_n, K_n)$ (we identify K_n with the space $\{x_1\} \times \{x_2\} \times \ldots \times \{x_{n-1}\} \times K_n$ for $n = 1, 2, \ldots$).

Let $r_n\colon X_n \to X_{n+1}$ be a map defined by the formula

$$r_n(x) = x \quad \text{for every } x \in K_i,\ 1 \leqslant i \leqslant n.$$

Observe that

$$p_n^{n+1} r_n \simeq \mathrm{id}_{X_n} \tag{3.4}$$

Let $(A_0, K) = \lim\operatorname{inv}\{(X_n, K_n), p_n^{n+1}\}$ and B be the cone over K. Using (3.4) one can verify that A_0 is a movable continuum. Obviously $X_0 = A_0 \cup B$ is movable continuum (X_0 has the same shape as an ANR bouquet X_0/B: [G] p. 171), the map $f = f_1 p_1\colon A_0 \to S^3$ is essential and is an extension of $f'\colon K \to S^3$.

Since f and f' are essential and K is contractible in X_0 to a point, we infer that f is not extensible over X_0. The proof is finished.

4. The Hopf Classification Theorem. In this section for every $(n-1)$-connected CW complex Y we denote by $\hat{Y}$ the Eilenberg–MacLane space $K(\pi_n(Y), n)$, which is obtained by attaching cells of dimension $\geqslant n+2$ to Y. The inclusion map $i\colon Y \to \hat{Y}$ is an $(n+1)$-equivalence, the groups $\pi_{m+1}(\hat{Y}, Y)$ and $\pi_m(Y)$ are isomorphic for $m \geqslant n+1$ and $\hat{Y}^{(m)} = Y^{(m)}$ for $m \leqslant n+1$.

(4.1) LEMMA. *Assume that X is a paracompact space. Then the pair*

$$(X\times[0,1],\ X\times\{0,1\})$$

is movable if X is movable and

$$\operatorname{def}((X\times[0,1],\ X\times\{0,1\});\ \mathscr{S}^2) < \infty \ \textit{if}\ \operatorname{def}(X;\ \mathscr{S}) < \infty.$$

Proof. Suppose that $\underline{p} = \{p_\alpha\}\colon X \to \{X_\alpha, p_\alpha^\beta, A\} = \underline{X} \in$ pro-Pol is a resolution. It is easy to see that the morphism $\underline{p\times\mathrm{id}_{[0,1]}} = \{p_\alpha\times\mathrm{id}_{[0,1]}\}\colon X\times[0,1] \to \underline{X\times[0,1]} = \{X_\alpha\times[0,1], p_\alpha^\beta\times\mathrm{id}_{[0,1]}, A\}$ satisfies the conditions (B. 1) and (B. 2) of [M-S] (see [M-S] p. 76) and is a resolution ([M–S] p. 79, Theorem 5). It follows (see [M-S] p. 87, Theorem 9) that

$$\underline{p\times\mathrm{id}_{[0,1]}}\colon (X\times[0,1],\ X\times\{0,1\}) \to \underline{(X\times[0,1],\ X\times\{0,1\})} \in \mathrm{proPol}^2$$

is a resolution. It is clear that $\underline{(X\times[0,1], X\times\{0,1\})}$ is movable if $\underline{X}$ is movable.

Suppose that $\operatorname{def}(X; \mathscr{S}) = n < \infty$. Let $f\colon (X\times[0,1], X\times\{0,1\}) \to (P, Q \in \mathscr{S}^2$ be a map. Then there exists $\alpha_0 \in A$ and a cellular map $f'\colon (X_{\alpha_0}\times[0,1], X_{\alpha_0}\times\{0,1\}) \to (P, Q)$ such that $f'\circ(p_{\alpha_0}\times\mathrm{id}_{[0,1]})$ is homotopic to f. Since P and Q are simply connected, f' is homotopic to $g\colon (X_{\alpha_0}\times[0,1], X_{\alpha_0}\times\{0,1\}) \to (P, Q)$ with $g(X_{\alpha_0}^{(1)}) = \{q_0\}$, where $q_0 \in Q$.

Let Y be the union of the cone over $X_{\alpha_0}^{(1)}$ and X_{α_0}. Then the composition $f'' = jf'$ of the inclusion map $j\colon X \to Y$ and the map f' is homotopic to a map $h\colon X \to Y$ with $h(X) \subset Y^{(n)}$ and

$$h\times\mathrm{id}_{[0,1]})\colon (X\times[0,1],\ X\times\{0,1\}) \to (Y\times[0,1],\ Y\times\{0,1\})$$

is homotopic to a map

$$r\colon (X\times[0,1],\ X\times\{0,1\}) \to (Y\times[0,1],\ Y\times\{0,1\})$$

with $r(X\times[0,1]) \subset (Y\times[0,1])^{(n+1)}$.

It is clear that there exists a cellular map $\hat{g}\colon (Y\times[0,1], Y\times\{0,1\}) \to (P, Q)$ which is an extension of g.

Then f is homotopic to the composition gr and $gr(X\times[0,1]) \subset Q \cup P^{(n+1)}$.

The proof is finished.

We now prove the following

(4.2) THEOREM. *Suppose that Y is an $(n-1)$-connected* CW *complex (where $n \geqslant 2$) and that X is a paracompact space with $H^m(X; \pi_m(Y)) = 0 = H^{m+1}(X; \pi_m(Y))$ for every $m \geqslant n+1$ satisfying one of the following conditions*

(a) *X is movable,*

(b) $\operatorname{def}(X;\ \mathscr{S}) < \infty$.

Then the set of the homotopy classes of the maps from X to Y is in one-to-one correspondence with the group $H^n(X; \pi_n(Y))$ under the map $[f] \to f^*(\varkappa^n(Y))$.

Proof. It is well known that for every paracompact space X (without any restrictions concerning the deformation dimension) the assertion of Theorem (4.2) holds when we replace Y by $\hat{Y}$. Our assumptions guarantee that every homotopy class from X to $\hat{Y}$ can be represented by a map with values in Y and that every homotopy in Y joining maps with values in Y can be represented by a homotopy in $\hat{Y}$ (see Theorem (2.4) and Lemma (4.1)). The proof is finished.

5. Final remarks and applications. The following two problems are equivalent to the Cell-like Mapping Problem and to the Cohomological Dimension Problem ([P] pp. 294, 297, 301).

(5.1) PROBLEM. *Do there exist a finite-dimensional compactum X and a* CE *map $f\colon X \to Y$ which is not a shape equivalence?*

(5.2) PROBLEM. *Do there exist a natural number n and a* CE *map $f\colon I^n \to Y$ which raises dimension* (i.e. $\dim Y > n$)?

In [P] the following problem is also formulated ([P], p. 295).

(5.3) PROBLEM. *Do there exist a non-contractible compactum Y, a natural number n and a* CE *map $f\colon I^n \to Y$?*

We shall show that (5.3) is also equivalent to the Cell-like Mapping Problem and to the following

(5.4) PROBLEM. *Do there exist a natural number n, a compactum Y with a non-trivial shape and a* CE *map $f\colon I^n \to Y$?*

Precisely, we have the following

(5.5) THEOREM. *For every natural number n the following three statements are equivalent:*

(a_n) *$n \geqslant \dim Y$ for every* CE *map $f\colon I^n \to Y$,*

(b_n) $\mathrm{Sh}(Y) = \mathrm{Sh}(\{p\})$ *for every* CE *map* $f\colon I^n \to Y$.

(c_n) $Y \in \mathrm{AR}$ *for every* CE *map* $f\colon I^n \to Y$.

The implications $(a_n) \Rightarrow (c_n)$, $(c_n) \Rightarrow (a_n)$ and $(c_n) \Rightarrow (b_n)$ are well known (see [P]) or trivial. The implication $(b_n) \Rightarrow (a_n)$ is a consequence of the following

(5.6) Lemma. *Suppose that $f\colon X \to Y$ is a* CE *map, where* $\dim Y > \dim X = n < \infty$. *Then for every $m > n$ there exist a* CE *map $g\colon X \to Z$ and an essential map $h\colon Z \to S^m$.*

Proof. Since $\dim Y = \infty$ ([Sh] p. 382), we infer that for every $m \geqslant n+1$ there exist a closed subset B of Y and a map $h_0\colon B \to S^m$ which cannot be extended over Y.

Consider the decomposition space Z of the uper semicontinuous decomposition of X into the single points of $X \setminus f^{-1}(B)$ and the sets $f^{-1}(y)$ with $y \in B$.

Then the quotient map $g\colon X \to Z$ is a CE map. Since the inequality $\dim(Z \setminus B) \leqslant n$ implies $\mathrm{def}((Z, B)\colon \mathcal{S}^2) \leqslant n$, we get that $h_0\colon B \to S^m$ has an extension $h\colon Z \to S^m$ (see Theorem (3.2)). Since h_0 is an essential map, we infer that h is essential. The proof is finished.

It is known ([Sh] p. 382) that if $Y \in \mathrm{ANR}$ or Y is finite-dimensional, then Y cannot be the image of a CE map $f\colon I^n \to Y$ which raises the dimension.

We say that a compactum Y belongs to $\mathcal{HM}$ iff for every closed subset B of Y and for every open subset $U \supset B$ of Y there exists a movable space (not necessarily compact) $V \subset Y$ such that $B \subset V \subset U$.

If $Y \in \mathrm{ANR}$, then $Y \in \mathcal{HM}$.

(5.7) Proposition. *If $f\colon X \to Y$ is a* CE *map such that* $\dim X < \infty$ *and X and $Y \in \mathcal{HM}$ are compacta, then* $\dim X \geqslant \dim Y$.

Proof. Let $n = \dim X$ and suppose that $\dim Y > n$. This means that there exist a compactum $B \subset Y$ and an essential map $f\colon B \to S^m$, where $m > n$, since $Y \in \mathcal{HM}$, we see that there exist a movable space $V \subset Y$, $B \subset V$ and an extension $f'\colon V \to S^m$ of f which is an essential map. On the other hand, f' cannot be essential, since $c(V) \leqslant \dim X = n < m$ and V is movable (see Theorem (4.2)).

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