Remarks on deformability

by

Sławomir Nowak and **Stanisław Spież** (Warszawa)

Abstract. Suppose that (X, A) and (P, Q) are pairs of topological spaces and (P, Q) is homotopically equivalent to a polyhedral pair. We establish shape and cohomological conditions under which a map $f: (X, A) \to (P, Q)$ is homotopic to a map $g: (X, A) \to (P, Q)$ with $g(X) \subset Q$.

Using these results, we prove several theorems in the shape theory and the theory of cohomotopy groups.

The *deformation dimension* of a topological space X is not greater than n ($\text{def-dim} X \leqslant n$) iff every map $f: X \to P$, where P is a polyhedron, is homotopic to a map $g: X \to P$ with $g(X) \subset P^{(n)}$. This is one of the most important shape invariants and appears in numerous theorems in shape theory.

We generalize this notion to the case of pairs of topological spaces and the situation where P is replaced by a polyhedral pair (P, Q) belonging to a fixed class $\underline{C}$ of polyhedral pairs.

The main purpose of the present paper is to establish relationships between the deformation dimension of a pair (X, A) with respect to a class $\underline{C}$ and some homological properties of (X, A).

In the case of the classical deformation dimension of a topological space these results were mentioned in [N–S].

Our theorems can be applied in several problems of topology as e.g. the problem of classification of homotopy classes and the problem of extension of maps with values in an $(n-1)$-connected CW complex (see [N]).

This technique can also be applied to the theory of cohomology groups.

We shall use the terminology of the book [M–S].

1. Deformability and n-deformability. Suppose that $\underline{(X, A)} = \{(X_\alpha, A_\alpha), p_\alpha^\beta, A\} \in \text{pro-Pol}^2$ and that A_0 is the set of all $\alpha \in A$ with $\alpha \geqslant \alpha_0$, where $\alpha_0 \in A$. Suppose that $\mathscr{L}_\alpha$ is the local system of abelian groups on X_α induced by $p_{\alpha_0}^\alpha$ and a local system $\mathscr{L}_{\alpha_0}$ of abelian groups on X_{α_0}, for every $\alpha \in A_0$. The family $\underline{\mathscr{L}} = \{\mathscr{L}_\alpha\}_{\alpha \in A_0}$ will be called a *generalized Čech local system* on $\underline{(X, A)}$ and the direct limit $H^n(\underline{(X, A)}, \underline{\mathscr{L}})$ of the direct system $\{H^n((X_\alpha, A_\alpha), \mathscr{L}_\alpha), (p_\alpha^\beta)^*, A_0\}$ will be called a *Čech cohomology group* of $\underline{(X, A)}$ with coefficients in $\underline{\mathscr{L}}$ (compare [A–M] p. 22 and [N] p. 21).

We denote by $c[(\underline{X,A})]$ the maximum (finite or infinite) of all integers n such that there is a Čech local system $\underline{\mathscr{L}}$ on $(\underline{X,A})$ with $H^n((\underline{X,A}),\underline{\mathscr{L}})\neq 0$.

(1.1) PROPOSITION. *Let* $(\underline{X,A})$ *and* $(\underline{Y,B})$ *be objects in* pro-Pol^2. *If* $(\underline{X,A})$ *and* $(\underline{Y,B})$ *are isomorphic in* pro-H Pol^2 *then* $c[(\underline{X,A})]=c[(\underline{Y,B})]$.

Suppose that $p\colon (X,A)\to(\underline{X,A})$ is a polyhedral resolution. Then $H^n((X,A),\underline{\mathscr{L}}) = H^n((\underline{X,A}),\underline{\mathscr{L}})$ is called a *Čech cohomology group of* (X,A) *with coefficients in* $\underline{\mathscr{L}}$. We define $c[(X,A)]=c[(\underline{X,A})]$. Any two polyhedral resolutions of (X,A) are isomorphic in pro-HPol^2, thus by Proposition (1.1) the last definition does not depend on the choice of resolution.

Suppose that $\underline{C}$ is a subclass of Pol^2 and that (X,A) is a pair of topological spaces. To the pair (X,A) we assign (compare [N]) the *deformation dimension of* (X,A) *with respect to* $\underline{C}$ which is an integer $\geqslant -1$ or ∞ and which equals to the minimum of n such that any map $f\colon (X,A)\to(P,Q)\in\underline{C}$ is homotopic to one, whose image lies in the set $Q\cup P^{(n)}$. To the set A we assign the *deformation dimension of A in the pair* (X,A) *with respect to* $\underline{C}$ which equals to the minimum n such that any map $f\colon (X,A)\to(P,Q)\in\underline{C}$ is homotopic to a map $g\colon (X,A)\to(P,Q)$ with $g(A)\subset P^{(n)}$.

One can also introduce the notion of the *deformation dimension* of a topological space X with respect to a subclass $\underline{C}$ of Pol.

Remark. Suppose that $\underline{C}$ is a subclass of Pol^2 consisting of at least one element (P,Q) with $P\neq\emptyset$ and that (X,A) is a pair of topological spaces. Let $\underline{C}_1\subset$Pol (respectively $\underline{C}_2\subset$ Pol) consist of all polyhedra P such that $(P,Q)\in\underline{C}$ for some $Q\in$ Pol (respectively $(Q,P)\in C$ for some $Q\in$Pol). If the pair (X,A) has the Homotopy Extension Property for the class $\underline{C}_1$, then the deformation dimension of A in (X,A) with respect to $\underline{C}$ is not greater than the deformation dimension of A with respect to $\underline{C}_2$.

A map $f\colon (K,L)\to(P,Q)\in$ Top^2, where $(K,L)\in$ Pol^2, is said to be *n-normal* iff $f(K^{(n-1)})\subset Q$.

A map $f\colon (X,A)\to(P,Q)$, where (X,A) and (P,Q) are pairs of topological spaces, is called *deformable into* Q (*n-deformable into* Q) iff f is homotopic as a map of pairs to a map $g\colon (X,A)\to(P,Q)$ with $g(X)\subset Q$ (iff there exist a pair $(K,L)\in$ Pol^2, an $(n+1)$-normal map $f'\colon (K,L)\to(P,Q)$ and a map $g\colon (X,A)\to(K,L)$ such that the composition $f'g$ is homotopic to f as a map from (X,A) to (P,Q)).

Suppose that $\{[p_\alpha]\}_{\alpha\in A}\colon (X,A)\to(\underline{X,A})=\{(X_\alpha,A_\alpha),[p_\alpha^\beta],A\}\in$ pro-Pol^2 is an HPol^2-expansion of (X,A) and $f\colon (X,A)\to(P,Q)\in$ HPol^2 is a map. Clearly, f is n-deformable into Q iff there exist $\alpha\in A$ and a map $f'\colon (X_\alpha,A_\alpha)\to(P,Q)$ with $f'p_\alpha\simeq f$ which is $(n+1)$-normal.

The notion of deformability and n-deformability were introduced by S. T. Hu (see [Hu$_1$] p. 197 and [Hu$_2$] p. 202).

(1.2) PROPOSITION. *Suppose that* (X,A) *is a movable pair of compact Hausdorff spaces and* $f\colon (X,A)\to(P,Q)\in$ HPol^2 *is a map which is n-deformable into* Q *for every* $n=0,1,2,\ldots$ *Then* f *is deformable into* Q.

Proof. We may assume that $(X,A)=\operatorname{lim\,inv}(\underline{X,A})$, where $(\underline{X,A}) = \{(X_\alpha,A_\alpha),p_\alpha^\beta,A\}$ is an inverse system of pairs of compact polyhedra. Then there exist $\alpha,\beta,\gamma\in A$ with $\gamma\geqslant\beta\geqslant\alpha$ and maps $f'\colon (X_\alpha,A_\alpha)\to(P,Q)$, $r\colon (X_\beta,A_\beta)\to(X_\gamma,A_\gamma)$ such that r is a cellular map satisfying $p_\alpha^\gamma r\simeq p_\alpha^\beta$ and $f'p_\alpha\simeq f$ and $f'p_\alpha^\gamma\simeq g$, where $p_\alpha\colon (X,A)\to(X_\alpha,A_\alpha)$ is the canonical projection, $g(X_\gamma^{(m)})\subset Q$ and $m=\dim X_\beta$. Then $f\simeq f'p_\alpha^\gamma rp_\beta\simeq grp_\beta$ and $grp_\beta(X)\subset gr(X_\beta)\subset g(X_\gamma^{(m)})\subset Q$. The proof is finished.

In Proposition (1.2) the hypothesis of movability and compactness of (X,A) is essential.

(1.3) EXAMPLE. Let X be an acyclic continuum with an essential map $f\colon X\to S^3$ (see [Ka]). Then f is n-deformable into $\{s_0\}$ for every $n=1,2,\ldots$ and f is not deformable into $\{s_0\}$, where $s_0\in S^3$.

(1.4) EXAMPLE. B. I. Gray has proved ([Gr] p. 242) that there exists an essential map $f\colon PC^\infty\to S^3$ such that the restriction $f|_{P_n}\colon P_n\to S^3$ is an inessential map for every $n=1,2,\ldots$, where P_n is the n-skeleton of the infinite dimensional complex projective space PC^∞.

Suppose that $f\colon (K,L)\to(P,Q)\in$ Top^2 is n-normal, where $(K,L)\in$ Pol^2, and Q is connected and $n\geqslant 3$. By $\mathscr{L}_k$ we denote for $k\geqslant n$ a local system of abelian groups on $K^{(n-1)}\cup L$ which is induced by the local system $\Pi_k=\{\pi_k(P,Q,x)\}_{x\in Q}$ on Q and the map $g\colon K^{(n-1)}\cup L\to Q$ defined by the formula

$$f(x)=g(x)\quad\text{for}\quad x\in K^{(n-1)}\cup L.$$

Since $n\geqslant 3$, we see that there exist a local system $\Pi_k(f)$ on K such that

$$\Pi_k(f)|_{K^{(n-1)}\cup L}=\mathscr{L}_k.$$

If $\pi_1(Q)=0$, we may assume that $n\geqslant 2$. In this case $\Pi_k(f)$ is a simple local system of abelian groups on K.

If we assign to every oriented n-simplex σ of K the element $(f,\sigma)\in\Pi_n(f)$ determined by

$$f|_{|\sigma|}\colon (\sigma,\partial\sigma,a)\to(P,Q,f(a)),$$

where a is a vertex of σ, we get an n-cochain $c^n(f)$ of the pair (K,L) with coefficients in $\Pi_n(f)$.

It is known ([Hu$_1$] p. 197 and [Hu$_2$] p. 203) that $c^n(f)$ is a cocycle and its cohomology class $\gamma^n(f)=[c^n(f)]\in H^n(K,L;\Pi_n(f))$ is equal to 0 iff there exists a homotopy $\varphi\colon K\times[0,1]\to P$ such that $\varphi(x,t)=f(x)$ for $(x,t)\in K\times\{0\}\cup K^{(n-2)}\times[0,1]$ and $\varphi(K^{(n)}\times\{1\})\subset Q$. If $g\colon (K',L')\to(K,L)$ is a simplicial map, then $\gamma^n(fg)=g^*(\gamma^n(f))$.

If $\{p_\alpha\}_{\alpha\in A}\colon (X,A)\to(\underline{X,A})=\{(X_\alpha,A_\alpha),p_\alpha^\beta,A\}$ is a polyhedral resolution of (X,A) and $f\colon (X,A)\to(P,Q)\in$ HPol^2 is a map, where Q is connected and $n\geqslant 3$ or Q is connected and simply connected and $n\geqslant 2$, then one can find a map

$f'\colon (X_{\alpha_0}, A_{\alpha_0}) \to (P, Q)$ such that $f'p_{\alpha_0} \simeq f$ and one can define a local system of abelian groups $\Pi_k(f'p^{\alpha}_{\alpha_0})$ on X_α for every $\alpha \geqslant \alpha_0$. It is clear that

$$\underline{\Pi_k(f)} = \{\Pi_k(f'p^{\alpha}_{\alpha_0})\}_{\alpha \geqslant \alpha_0}$$

is a generalized Čech local system on $(\underline{X}, \underline{A})$.

(1.5) PROPOSITION. *Suppose that $(P, Q) \in \mathrm{HPol}^2$ is a pair such that Q is connected (respectively Q is connected and simply connected) and that $f\colon (X, A) \to (P, Q)$ is m-deformable into Q, where $m \geqslant 2$ (respectively $m \geqslant 1$). If $H^n(X, A; \Pi_n(f)) = 0$ for every $m < n \leqslant m'$, then f is n-deformable into Q for every $n = 0, 1, 2, \ldots, m'$.*

2. Sufficient conditions for deformability. Suppose that $\underline{C} \subset \mathrm{Pol}^2$. We say that $(P, Q) \in \mathrm{HPol}^2$ is a $(\underline{C}, n)$-pair if for every maps $f\colon (Y, B) \to (P, Q)$, where $(Y, B) \in \mathrm{Pol}^2$, and $g\colon Y \to P$ such that $f \simeq g$ rel $Y^{(n)}$ (as maps from Y to P) and $g(Y) \subset Q$ there exist maps $\alpha\colon (Y, B) \to (K, L) \in \underline{C}$ and $f'\colon (K, L) \to (P, Q)$ such that $f'\alpha \simeq f$ as maps from (Y, B) to (P, Q) and $f' \simeq g'$ as maps from $(K, L^{(n)})$ to (P, Q), where $g'(K) \subset Q$. We say that (P, Q) is a $\underline{C}$-pair if for every integer m there is an integer $n \geqslant m$ such that (P, Q) is a $(\underline{C}, n)$-pair.

(2.1) THEOREM. *Suppose that $(P, Q) \in \mathrm{HPol}^2$ is a pair such that Q is connected (respectively, Q is connected and simply connected) and that $f\colon (X, A) \to (P, Q)$ is map a m-deformable into Q, where $m \geqslant 2$ (respectively $m \geqslant 1$). If $H^n(X, A; \underline{\Pi_n(f)}) = 0$ for every $n > m$ and the pair (X, A) satisfies one of the following conditions:*

(a) (X, A) is movable,

(b) X is movable and there exists a subclass $\underline{C}$ of Pol^2 such that (P, Q) is a $\underline{C}$-pair and the deformation dimension of A in (X, A) with respect to $\underline{C}$ is finite,

then f is deformable into Q.

For the proof of Theorem (2.1) we prepare the following lemma.

(2.2) LEMMA. *Let W_2 and $B \subset W_2$ be subcomplexes of a CW complex W_1. Suppose that maps $h_1\colon W_1 \to P$, $h_2\colon (W_1, B) \to (P, Q) \in \mathrm{Top}^2$, $h_3\colon (W_1, W_2) \to (P, Q)$ and homotopy $\varphi\colon W_1 \times [0, 1] \to P$ satisfy the conditions*

(a) $\varphi(x, 0) = h_1(x)$, $\varphi(x, 1) = h_2(x)$ for $x \in W_1$

(b) h_2 and h_3 are homotopic as maps from (W_1, B) to (P, Q). Then there exist a homotopy $\psi\colon W_1 \times [0, 1] \to P$ and a map $h_4\colon (W_1, W_2) \to (P, Q)$ with $h_4 \simeq h_3$ (as maps from (W_1, W_2) to (P, Q)) such that $\psi(x, t) = \varphi(x, t)$ for $(x, t) \in B \times [0, 1]$ and $\psi(x, 0) = h_1(x)$, $\psi(x, 1) = h_4(x)$ for $x \in W_1$.

Proof. Consider a homotopy $\hat{\chi}\colon (W_1, B) \times [0, 1] \to (P, Q)$ such that $\hat{\chi}(x, 0) = h_3(x)$, $\hat{\chi}(x, 1) = h_2(x)$ for $x \in W_1$. It follows from the homotopy extension theorem that there exist maps $h_4\colon (W_1, W_2) \to (P, Q)$ and $\chi\colon W_1 \times [0, 1] \to P$ which satisfy the conditions

$$\chi(W_2 \times [0, 1]) \subset Q, \quad \chi|_{B \times [0,1]} = \hat{\chi}|_{B \times [0,1]}$$

and

$$\chi(x, 0) = h_3(x), \quad \chi(x, 1) = h_4(x) \quad \text{for } x \in W_1.$$

Consider the space $Y = W_1 \times \{0\} \cup B \times [0, 1] \cup W_1 \times \{1\}$ and the homotopy $\lambda\colon Y \times [0, 1] \to P$ defined by the conditions: if $s \in [0, 1]$ and $y = (x, t) \in B \times [0, 1]$, then

$$\lambda(y, s) = \begin{cases} \varphi\left(x, \dfrac{2}{s+1}\, t\right) & \text{for} \quad 0 \leqslant t \leqslant \dfrac{s+1}{2}, \\ \hat{\chi}(x, 2+s-2t) & \text{for} \quad \dfrac{s+1}{2} \leqslant t \leqslant 1 \end{cases}$$

and

$$\lambda(y, s) = h_1(x) \quad \text{if} \quad s \in [0, 1] \text{ and } y = (x, 0) \in W_1 \times \{0\},$$
$$\lambda(y, s) = \chi(x, s) \quad \text{if} \quad s \in [0, 1] \text{ and } y = (x, 1) \in W_1 \times \{1\}.$$

Observe that $\lambda((x, t), 0) = \lambda_0(x, t)$ for $(x, t) \in Y$, where $\lambda_0\colon W_1 \times [0, 1] \to P$ is defined by the formula

$$\lambda_0(x, t) = \begin{cases} \varphi(x, 2t) & \text{for} \quad (x, t) \in W_1 \times [0, \tfrac{1}{2}], \\ \hat{\chi}(x, 2-2t) & \text{for} \quad (x, t) \in W_1 \times [\tfrac{1}{2}, 1]. \end{cases}$$

The homotopy extension theorem implies that there exist a homotopy $\hat{\psi}\colon (W_1 \times [0, 1]) \times [0, 1) \to P$ such that $\hat{\psi}((x, t), s) = \lambda((x, t), s)$ for $(x, t) \in Y$ and $0 \leqslant s \leqslant 1$. The homotopy $\psi\colon W_1 \times [0, 1] \to P$ defined by the formula

$$\psi(x, t) = \hat{\psi}((x, t), 1) \quad \text{for} \quad (x, t) \in W_1 \times [0, 1]$$

satisfies the required conditions.

Proof of Theorem (2.1). Let $p\colon (X, A) \to (\underline{X}, \underline{A})$ be a polyhedral resolution. Since $H^n(X, A; \underline{\Pi_n(f)}) = \lim\operatorname{dir} H^n((\underline{X}, \underline{A}); \Pi_n(f)) = 0$ for $n > m$ and f is m-deformable, we can deduce that for $k = 0, 1, 2, \ldots$ there exist a pair $(X_k, A_k) \in \mathrm{Pol}^2$, a cellular map $f_k\colon (X_k, A_k) \to (P, Q)$, a simplicial map $p_k^{k+1}\colon (X_{k+1}, A_{k+1}) \to (X_k, A_k)$ and a map $p_k\colon (X, A) \to (X_k, A_k)$ such that $f \simeq f_0 p_0$, $f_1 = f_0 \cdot p_0^1$ and for $k = 0, 1, 2, \ldots$ the following conditions are satisfied:

$$p_k \simeq p_k^{k+1} p_{k+1}, \quad f_k(X_k^{(m+k)}) \subset Q, \quad (p_k^{k+1})^*(\gamma^{(m+k+1)}(f_k)) = 0 \quad \text{and}$$
$$f_{k+1} \simeq f_k p_k^{k+1} \ \text{rel}\ A_{k+1} \cup X_{k+1}^{(m+k-1)}. \tag{2.3}$$

If follows that f is n-deformable into Q for every n. Hence, without loss of generality, we may assume that in the case when (X, A) satisfies (b) the deformation dimension of A in (X, A) with respect to $\underline{C}$ is less than m and (P, Q) is a $(C, m-1)$-pair.

Since (X, A) is movable (or, respectively, X is movable), we may assume that for every $k = 2, 3, \ldots$ there is a cellular map $r_k\colon (X_{k-1}, A_{k-1}) \to (X_k, A_k)$ (a cellular map $r_k\colon X_{k-1} \to X_k$) such that $p_{k-1}^k p_k^{k+1} r_{k+1} \simeq p_{k-1}^k$ as maps from (X_k, A_k) to (X_{k-1}, A_{k-1}) (as maps from X_k to X_{k-1}).

Let $s_k = p_k^{k+1} r_{k+1} r_k \ldots r_2$ for $k = 1, 2, \ldots$ and $s_0 = p_0^1$. For every $k = -$

$-1, 0, 1, \ldots$ we denote by E_k the set $X_1^{(m+k)} \cup A_1$ in the case where (X, A) satisfies (a) and the set $X_1^{(m+k)}$ in the case when (X, A) satisfies (b).

We denote also by F_k for every $k = 0, 1, 2, \ldots$ the set $X_k^{(m+k)} \cup A_k$ in the case where (X, A) satisfies (a) and the set $X_k^{(m+k)}$ in the case where (X, A) satisfies (b).

Since $p_k^{k+1} s_{k+1}$ and s_k are cellular, it follows that

(2.4) $p_k^{k+1} s_{k+1}$ and s_k are homotopic as maps from (X_1, E_{k-1}) to (X_k, F_k).

Applying induction, we shall construct for $k = 0, 1, \ldots$ a cellular map $g_k\colon X_1 \to P$ and a homotopy $\chi_k\colon X_1 \times [0, 1] \to P$ such that

(2.5) $\chi_k(x, 0) = f_0 s_0(x)$, $\chi_k(x, 1) = g_k(x)$ for $x \in X_1$ and $g_k(E_k) \subset Q$ and $g_k \simeq f_k s_k$ as maps from (X_1, E_k) to (P, Q), $\chi_{k-1}(x, t) = \chi_k(x, t)$ for $(x, t) \in E_{k-2} \times [0, 1]$.

Let $g_0 = f_0 s_0$ and $\chi_0\colon X_1 \times [0, 1] \to P$ be defined by the formula

$$\chi_0(x, t) = f_0 s_0(x) \quad \text{for} \quad x \in X_1 \quad \text{and} \quad t \in [0, 1].$$

Suppose that g_k and χ_k are constructed for $k \leqslant n$. From (2.3), (2.4) and (2.5) we deduce that $g_n \simeq f_n s_n \simeq f_n p_n^{n+1} s_{n+1} \simeq f_{n+1} s_{n+1}$ as maps from (X_1, E_{n-1}) to (P, Q).

Using Lemma (2.2) in the case where $W_1 = W_1$, $W_2 = E_{n+1}$, $B = E_{n-1}$, $\varphi = \chi_n$, $h_1 = f_0 s_0$, $h_2 = g_n$ and $h_3 = f_{n+1} s_{n+1}$, we get a homotopy $\chi_{n+1}\colon X_1 \times [0, 1] \to P$ and a map $g_{n+1}\colon X_1 \to P$ such that (2.5) is satisfied for $k \leqslant n+1$.

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$$\varphi(x, t) = \chi_{n+2}(x, t) \quad \text{if} \quad (x, t) \in E_n \times [0, 1],$$

we get a homotopy $\varphi\colon X_1 \times [0, 1] \to P$ such that $\varphi(x, t) = f_0 s_0(x) = f_1(x)$ for $(x, t) \in X_1 \times \{0\} \cup E_{-1} \times [0, 1]$ and $\varphi_1(x) = \varphi(x, 1) \in Q$ for every $x \in X$.

Assume that (X, A) satisfies the condition (a). Then $E_{-1} \supset A_1$ and $f \simeq f_0 p_0 \simeq f_0 p_0^1 p_1 = f_1 p_1 \simeq \varphi_1 p_1$ as maps from (X, A) to (P, Q). Since $\varphi_1 p_1(X) \subset Q$, the proof is finished.

Assume that (X, A) satisfies (b). Then $E_{-1} = X_1^{(m-1)}$ and $f_1 \simeq \varphi_1$ rel. $X_1^{(m-1)}$ as maps from X_1 to P. Since (P, Q) is a $(\underline{C}, m-1)$-pair and $\varphi_1(X_1) \subset Q$, there exist $f_1'\colon (K, L) \to (P, Q)$ and $\alpha\colon (X_1, A_1) \to (K, L) \in \underline{C}$ such that $f_1' \alpha \simeq f_1$ as maps from (X_1, A_1) to (P, Q) and $f_1' \simeq g'$ as maps from $(K, L^{(m-1)})$ to (P, Q), where $g'(K) \subset Q$. The deformation dimension of A in (X, A) with respect to $\underline{C}$ is less than m; thus there is $p\colon (X, A) \to (K, L)$ such that $p(A) \subset L^{(m-1)}$ and $p \simeq \alpha p_1$ as maps from (X, A) to (K, L). Observe that $f \simeq f_1 p_1 \simeq f_1' \alpha p_1 \simeq f_1' p \simeq g' p$ as maps from (X, A) to (P, Q) and $g' p(X) \subset Q$. The proof is finished.

3. Some applications. By $\underline{C}^n$ we denote the subclass of Pol^2 consisting of all pairs $(P, Q) \in \mathrm{Pol}^2$ such that Q is n-connected. Let us observe that if $(P, Q) \in \underline{C}^n$, then (P, Q) is a $(\underline{C}^n, m)$-pair for every $m \geqslant n$ and thus (P, Q) is a $\underline{C}^n$-pair. From Theorem (2.1) we obtain the following.

(3.1) COROLLARY. *Suppose that* $(P, Q) \in \underline{C}^n$ $(n \geqslant 1)$ *and that* $f\colon (X, A) \to (P, Q)$ *in a map* m*-deformable into* Q*, where* $m \geqslant 1$. *If* $H^n(X, A; \underline{\Pi_n(f)}) = 0$ *for every* $n > m$, *X is movable and the deformation dimension of A in (X, A) with respect to* $\underline{C}^n$ *is finite, then f is deformable into Q.*

The deformation dimension of (X, A) (respectively X) with respect to Pol^2 (respectively Pol) will be denoted by $\operatorname{def-dim}(X, A)$ (respectively $\operatorname{def-dim} X$).

From Proposition (1.5) and the Theorem (2.1) one can obtain a cohomological characterization of the deformation dimension of pairs of spaces (compare [N] and [N-S]).

(3.2) COROLLARY. *If (X, A) is a pair of spaces such that* $\operatorname{def-dim}(X, A)$ *is finite then* $c[(X, A)] \leqslant \operatorname{def-dim}(X, A) \leqslant \max(2, c(X, A))$. *If* $\operatorname{def-dim}(X, A)$ *is infinite and (X, A) satisfies one of the conditions* (a) *and* (b) *from Theorem* (2.1), *then* $c[(X, A)]$ *is infinite.*

Suppose that $F\colon (X, *) \to (Y, *)$ is a shape morphism of pointed spaces which induces isomorphism of all homotopy pro-groups. By Theorem (4.3) and Corollary (4.4) of [A-M], pp. 36–37, it follows that F induces isomorphism of the Čech cohomology groups (in each dimension) for every generalized Čech local system. Hence $c[X] = c[Y]$. Thus, if $\operatorname{def-dim} X$ is finite and Y is movable, then $\operatorname{def-dim} Y$ is finite; also if $\operatorname{def-dim} Y$ is finite and X is movable then $\operatorname{def-dim} X$ is finite (see Theorem (2.3) [N-S] or Corollary (3.2)). Hence, by the Whitehead theorem in shape theory (see for example [M-S], p. 28) we obtain the following corollary due to J. Dydak ([D_2] p. 28).

(3.3) COROLLARY. *Suppose that $F\colon (X, *) \to (Y, *)$ is a pointed shape morphism between connected spaces such that* pro-$\Pi_n(F)$ *is an isomorphism for every n. If one of the following two conditions is satisfied:*

(a) def-dim *X is finite and Y is movable,*

(b) def-dim *Y is finite and X is movable,*

then F is a shape isomorphism.

The idea of the proof of this corollary is also due to J. Dydak.

4. Cohomotopy groups. Let n be a natural number, $s_0 \in S^n$; define a map $\Omega\colon B = S^n \times \{s_0\} \cup \{s_0\} \times S^n \to S^n$ by the formula $\Omega(s, s_0) = s = \Omega(s_0, s)$ for $s \in S^n$.

$\pi^n(X, A)$ denotes the pointed set of all homotopy classes from a pair (X, A) of topological spaces to $(S^n, \{s_0\})$ with the distinguished element $e^n_{(X,A)}$ which is determined by the constant map $f\colon (X, A) \to (S^n, s_0)$ with $f(X) = \{s_0\}$.

Suppose that (X, A) is a topological pair such that for every two maps $\alpha, \beta\colon (X, A) \to (S^n, \{s_0\})$ there exists a map $\gamma\colon (X, A) \to (S^n \times S^n, \{(s_0, s_0)\})$ with $\gamma(X) \subset B$ and $\alpha \triangle \beta \simeq \gamma$ (as maps from (X, A) to $(S^n \times S^n, \{(s_0, s_0)\})$. We say that (X, A) is *cohomotopically n-admissible* ([B] p. 43) or (X, A) admits the *existence of the nth cohomotopy group* iff the homotopy class of $\Omega\gamma\colon (X, A) \to (S^n, \{s_0\})$ depends only on the homotopy classes of α and β and the addition defined by $[\alpha] + [\beta] = [\Omega\gamma]$ makes the set $\pi^n(X, A)$ into an Abelian group. The element $e^n_{(X,A)}$ is the trivial element for this addition. We say that a space X is *cohomotopically n-admissible* iff the pair $(X, \emptyset)$ admits the existence of the nth cohomotopy group $\pi^n(X) = \pi^n(X, \emptyset)$.

If $f: X, A \to (Y, B)$ is a shaping, then for every pair $(P, Q) \in \mathrm{Pol}^2$ the shape morphism f induces the function $f^{(P,Q)}: [(Y, B); (P, Q)] \to [(X, A); (P, Q)]$. In the case where $(P, Q) = (S^n, \{s_0\})$ this function is denoted by $\pi^n(f)$. It is clear that $\pi^n(f)(e^n_{(Y,B)}) = e^n_{(X,A)}$.

Denote by $\underline{SC}^n$ the full subcategory of the shape category whose objects are cohomotopically n-admissible pairs.

If (X, A) and (Y, B) are objects of $\underline{SC}^n$, then $\pi^n(f): \pi^n(Y, B) \to \pi^n(X, A)$ is a homomorphism.

(4.1) THEOREM. *π^n is a contravariant functor from SC^n to the category of abelian groups*

Theorem (4.1) in the case of $A = \emptyset = B$ has been proved by S. Godlewski ([Go]). His proof is valid in the relative case too.

K. Borsuk has proved that if A is a closed subset of a metric space X with $\dim X < 2n-1$, then (X, A) admits the existence of the nth cohomotopy group.

J. Dydak has showed ([D$_1$] p. 85) that every topological space X with shape dimension $< 2n-1$ admits the existence of the nth cohomotopy group.

It is also known ([Hu$_1$]) that every compact pair $(X, A) \in \mathrm{Pol}^2$ with $c[(X, A)] < 2n-1$ is cohomotopically n-admissible.

Examining the proofs of the above-mentioned theorems one can observe that they use only the fact that in all of these cases the deformation dimension of (X, A) (or X) with respect to a class $\underline{\mathscr{S}}_0$ of all pairs $(P, \{s_0\})$, where P is connected and simply connected and $s_0 \in P$, is less than $2n-1$.

(4.2) THEOREM. *Suppose that (X, A) is a pair of compacta and the deformation dimension of (X, A) with respect to $\underline{\mathscr{S}}_0$ is less than $2n-1$. Then (X, A) is cohomotopically n-admissible.*

Using (2.1) and (3.1) we get the following corollaries.

(4.3) COROLLARY. *Suppose that (X, A) is a movable pair of topological spaces such that $H^m(X, A; G) = 0$ for every $m \geqslant 2n-1$ and every abelian group G. Then (X, A) is cohomotopically n-admissible.*

(4.4) COROLLARY. *Suppose that X is a movable metrizable space and A is a closed movable subset of X. Assume also that the pair (X, A) satisfies the following conditions:*

(a) $\mathrm{Max}(n$: *there exists an abelian group G with $H^n(A; G) \neq 0) < \infty$,*

(b) *$H^m(X, A; G) = 0$ for every $m \geqslant 2n-1$ and every abelian group G.*

Then (X, A) admits the existence of the n-th cohomotopy group.

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INSTYTUT MATEMATYKI, UNIWERSYTET WARSZAWSKI
INSTITUTE OF MATHEMATICS, UNIVERSITY OF WARSAW

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
INSTITUTE OF MATHEMATICS, POLISH, POLISH ACADEMY OF SCIENCES

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