Proof. For oriented M and N this follows from the commutative diagram

$$\begin{array}{ccc}
\mathcal{H} & \xrightarrow{\text{b)}} & [M\times N,\, E\times E\setminus\Delta] \\
{\scriptstyle l}\downarrow\wr & \searrow^{\text{a)}} & \downarrow\wr \\
Z & \xleftarrow[\;W\;]{\approx} & [M\times N,\, E_0]
\end{array}$$

in which the right vertical bijection is induced by the homotopy equivalence $h\colon E\times E\setminus\Delta\to E_0$ defined by $h(x,y)=y-x$, l is the bijection of Theorem 1 and W the bijection of the Hopf classification theorem. In the unoriented case we have a similar diagram.

References

[1] T. Bröcker and K. Jänich, *Einführung in die Differential-topologie*, Berlin 1973.
[2] V. Guillemin and A. Pollack, *Differential Topology*, Prentice Hall 1974.
[3] A. Haefliger, *Plongements différentiables dans le domaine stable*, Comm. Math. Helv. 37 (1962–63), pp. 155–176.
[4] M. Hirsch, *Differential Topology*, Springer Verlag, 1976.
[5] J. Milnor, *Lectures on the h-cobordism theorem*, Princeton Mathematical Notes, 1965.
[6] — *Link groups*, Ann. of Math. 59 (1954), pp. 177–195.
[7] — *Topology from the Differentiable Viewpoint*, The University Press of Virginia, 1965.
[8] Л. С. Понтрягин, *Гладкие многообразия и их применения в теории гомотопий*, Москва 1976.
[9] D. Rolfsen, *Knots and Links*, Berkeley 1976.
[10] H. Whitney, *The selfintersections of a smooth n-manifold in 2n-space*, Ann of Math. 45 (1944), pp. 247–293.

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Products of normal spaces with Lašnev spaces

by

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Abstract. In this paper the equivalence of normality and countable paracompactness will be established for the product of a countably paracompact normal space with a Lašnev space. This extends Morita, Rudin and Starbird's theorem.

1. Introduction. All spaces considered in this paper are assumed to be Hausdorff and all maps continuous and onto. Closed images of metric spaces were characterized by Lašnev [7], and are called *Lašnev spaces*. Leĭbo [8, 9] applied Lašnev spaces to extend the well-known Katětov-Morita coincidence theorem and other properties of metric spaces in dimension theory (see [4]).

Let X be a countably paracompact normal space. It follows from the results of Morita [13] (for the proof see [5]) and Rudin and Starbird [18] that for a metric space Y the product space $X\times Y$ is normal if and only if $X\times Y$ is countably paracompact. However, no condition on Y other than metrizability seems to be known, under which the above equivalence is true. Indeed, in case Y is a paracompact M-space Rudin and Starbird [18] shows that the normality of $X\times Y$ implies the countable paracompactness of $X\times Y$, but the converse does not hold in general even if Y is compact. The aim of this paper is to show that the above is true in case of Y being Lašnev. We prove the following theorems:

THEOREM 1. *Let X be a normal space and Y a Lašnev space. If $X\times Y$ is countably paracompact, then $X\times Y$ is normal.*

THEOREM 2. *Let X be a space and Y a non-discrete Lašnev space. If $X\times Y$ is normal, then $X\times Y$ is countably paracompact.*

THEOREM 3. *Let X be a normal and countably paracompact space and Y a Lašnev space. Then $X\times Y$ is normal iff $X\times Y$ is countably paracompact.*

We note that in case Y is metrizable Theorems 1 and 2 are proved by Morita [13] and Rudin and Starbird [18] respectively. Also, our results will be applied to prove that if the product $X\times Y$ of a paracompact (resp. collectionwise normal) space X with a Lašnev space Y is normal then $X\times Y$ is paracompact (resp. collectionwise normal). This extends an analogous result for a metrizable space Y, implied by the results of Morita [12], Okuyama [17] and Rudin and Starbird [18].

In § 3, the above equivalence from Theorem 3 will be proved also for a normal P-space (in the sense of Morita [11]) X and a paracompact σ-space Y. It is known that every Lašnev space is a paracompact σ-space.

2. Proofs of Theorems 1 and 2. Let N denote the set of positive integers. The following lemma is useful to prove Theorem 1, but the proof is easy and omitted:

LEMMA 2.1. *Let X be a countably paracompact space, and let E and F be a pair of disjoint subsets. Suppose that F is closed and there exist open sets U_n, $n \in N$ such that $E \subset \bigcap_n U_n$ and $\bigcap_n \mathrm{Cl}\, U_n \cap F = \emptyset$. Then E and F are separated by open sets.*

Let Y be a Lašnev space, and let Z be a metric space and $f\colon Z \to Y$ a closed map. By Lašnev [6] Y can be expressed as $Y = \bigcup_{i \geqslant 0} Y_i$, where for each $i \geqslant 1$ Y_i is a discrete closed subset and $Y_i \cap Y_0 = \emptyset$, and for each $y \in Y_0$ $f^{-1}(y)$ is compact.

Let $\mathscr{G}_n = \{G_{n\alpha} |\ \alpha \in \Omega_n\}$ be a locally finite open cover of Z such that $\mathscr{G}_{n+1}$ refines $\mathscr{G}_n$, each element of $\mathscr{G}_n$ has diameter $< 1/2^n$.

Let $\Phi = 1_X \times f\colon X \times Z \to X \times Y$ be the product map.

This notation will be used throughout this section.

Our proofs are based on the idea used by Rudin and Starbird [18].

Proof of Theorem 1. We shall use the following fact: if H and K are disjoint subsets of a space S and $\{U_i |\ i \in N\}$ and $\{V_i |\ i \in N\}$ are collections of open sets in S such that $H \subset \bigcup_i U_i$ and $K \subset \bigcup_i V_i$ and for each $i \in N$, $\mathrm{Cl}\, U_i \cap K = \emptyset$ and $H \cap \mathrm{Cl}\, V_i = \emptyset$, then H and K are separated by open sets.

To prove the theorem, suppose that X is normal and $X \times Y$ is countably paracompact. The proof is done by two claims.

CLAIM 1. *Let A and B be any closed sets in $X \times Y$ such that $A \subset X \times Y_0$ and $B \subset X \times (\bigcup_{i \geqslant 1} Y_i)$. Then A and B are separated by open sets.*

To see this, first we note that for each $i \geqslant 1$ $X \times Y_i$ is a zero-set of $X \times Y$ since Y is perfectly normal. Hence by Lemma 2.1 there exists an open set U_i of $X \times Y$ such that $A \subset U_i$ and $\mathrm{Cl}\, U_i \cap (X \times Y_i) = \emptyset$. Then we have $A \subset \bigcap_{i \geqslant 1} U_i$ and $\bigcap_{i \geqslant 1} \mathrm{Cl}\, U_i \cap B = \emptyset$, and consequently by Lemma 2.1 A and B are separated by open sets.

CLAIM 2. *Let C and D be any pair of disjoint closed sets of $X \times Y$, both contained in $X \times Y_0$. Then C and D are separated by open sets.*

For a moment let us assume Claim 2 and prove the normality of $X \times Y$.

Let E and F be a pair of disjoint closed sets of $X \times Y$. Let $i \geqslant 1$. Since $X \times Y_i$ is normal, there exist disjoint zero-sets Z_i and Z_i' of $X \times Y_i$ such that $E \cap (X \times Y_i) \subset Z_i$ and $F \cap (X \times Y_i) \subset Z_i'$. Since Y is paracompact and Y_i is closed and discrete, by [14, Theorem 4] $X \times Y_i$ is C-embedded in $X \times Y$ and is a zero-set of $X \times Y$. Hence, Z_i and Z_i' are zero-sets of $X \times Y$, and $Z_i \cap F = \emptyset$, $E \cap Z_i' = \emptyset$. By Lemma 2.1 there exist open sets U_i and V_i of $X \times Y$ such that $Z_i \subset U_i$, $\mathrm{Cl}\, U_i \cap F = \emptyset$ and $Z_i' \subset V_i$, $E \cap \mathrm{Cl}\, V_i = \emptyset$. Let $E_0 = E - \bigcup_{i \geqslant 1} U_i$ and $F_0 = F - \bigcup_{i \geqslant 1} V_i$. As in the proof in Claim 1 take an open set G_i for $i \geqslant 1$ so that $E_0 \subset G_i$ and $\mathrm{Cl}\, G_i \cap (X \times Y_i) = \emptyset$. Then we have $\bigcap_{i \geqslant 1} \mathrm{Cl}\, G_i \cap F \subset X \times Y_0$, and by Claim 2 there exists an open set G_0 such that $E_0 \subset G_0$ and $\mathrm{Cl}\, G_0 \cap (\bigcap_{i \geqslant 1} \mathrm{Cl}\, G_i \cap F) = \emptyset$. Hence by Lemma 2.1 there exists an open set U_0 such that $E_0 \subset U_0$ and $\mathrm{Cl}\, U_0 \cap F = \emptyset$. Similarly, we can choose an open set V_0 such that $F_0 \subset V_0$ and $E \cap \mathrm{Cl}\, V_0 = \emptyset$. We have now

$$E \subset \bigcup_{i \geqslant 0} U_i, \quad F \subset \bigcup_{i \geqslant 0} V_i$$

and

$$\mathrm{Cl}\, U_i \cap F = \emptyset, \quad E \cap \mathrm{Cl}\, V_i = \emptyset$$

for each $i \geqslant 0$. Therefore E and F can be separated by open sets, and consequently $X \times Y$ is normal.

Thus, to complete the proof it remains:

Proof of Claim 2. Note that for a subset E of $X \times Z$ we have $\Phi(\mathrm{Cl}\, E) \cap (X \times Y_0) = \mathrm{Cl}\, \Phi(E) \cap (X \times Y_0)$ since f is closed and $\{x\} \times f^{-1}(y)$ is compact for each $(x, y) \in X \times Y_0$, and which will be frequently used.

Let E_1 and E_2 be disjoint closed sets of $X \times Y$, both contained in $X \times Y_0$. Let us put for $\alpha \in \Omega_n$

$$C_{n\alpha}(1) = \bigcup \{P |\ P \text{ is an open set of } X \text{ such that } (P \times f(\mathrm{Cl}\, G_{n\alpha})) \cap E_2 = \emptyset\},$$

$$C_{n\alpha}(2) = \bigcup \{P |\ P \text{ is an open set of } X \text{ such that } (P \times f(\mathrm{Cl}\, G_{n\alpha})) \cap E_1 = \emptyset\},$$

$$C_{n\alpha} = C_{n\alpha}(1) \cup C_{n\alpha}(2),$$

and for $n \in N$

$$T_n = \bigcup \{(X - C_{n\alpha}) \times \mathrm{Cl}\, G_{n\alpha} |\ \alpha \in \Omega_n\}.$$

Since $\mathscr{G}_n$ is locally finite and $\mathscr{G}_{n+1}$ refines $\mathscr{G}_n$, T_n is closed and $T_{n+1} \subset T_n$. Moreover, $\bigcap_n T_n = \emptyset$, because if $\bigcap_n \mathrm{Cl}\, G_{n\alpha_n} \neq \emptyset$ then $X = \bigcup_n C_{n\alpha_n}$. Hence we have $\bigcap_n \Phi(T_n) \cap (X \times Y_0) = \emptyset$ since $\{x\} \times f^{-1}(y)$ is compact for each $(x, y) \in X \times Y_0$. From the fact above it follows that

$$\bigcap_n \mathrm{Cl}\, \Phi(T_n) \cap (X \times Y_0) = \bigcap_n \Phi(T_n) \cap (X \times Y_0) = \emptyset.$$

Thus, $\bigcap_n \mathrm{Cl}\, \Phi(T_n) \subset X \times (\bigcup_{i \geqslant 1} Y_i)$, and since $E_1 \cup E_2 \subset X \times Y_0$, by Claim 1 there exists an open set K of $X \times Y$ such that

$$\bigcap_n \mathrm{Cl}\, \Phi(T_n) \subset K, \quad \mathrm{Cl}\, K \cap (E_1 \cup E_2) = \emptyset.$$

Then $\{\mathrm{Cl}\, \Phi(T_n) - K |\ n \in N\}$ is decreasing and has the empty intersection. Hence, by the countable paracompactness of $X \times Y$ there exists an increasing open cover

$\mathscr{W} = \{W_n |\ n \in N\}$ of $X \times Y$ such that $\mathrm{Cl}\, W_n \cap (\mathrm{Cl}\, \Phi(T_n) - K) = \emptyset$ for each $n \in N$ (cf. [3]). Let us put for $\alpha \in \Omega_n$

$$D_{n\alpha} = \{x \in X|\ \{x\} \times f(\mathrm{Cl}\, G_{n\alpha}) \subset \mathrm{Cl}\, W_n - K\}.$$

Then $D_{n\alpha}$ is a closed set of X, and we have $D_{n\alpha} \subset C_{n\alpha}$ since $(D_{n\alpha} \times f(\mathrm{Cl}\, G_{n\alpha})) \cap \Phi(T_n) = \emptyset$. By the normality of X take open sets $U_{n\alpha}$ and $V_{n\alpha}$ of X so that

$$D_{n\alpha} - C_{n\alpha}(2) \subset U_{n\alpha} \subset \mathrm{Cl}\, U_{n\alpha} \subset C_{n\alpha}(1),$$

$$D_{n\alpha} - C_{n\alpha}(1) \subset V_{n\alpha} \subset \mathrm{Cl}\, V_{n\alpha} \subset C_{n\alpha}(2).$$

Let us set

$$M_n = \bigcup \{U_{n\alpha} \times G_{n\alpha}|\ \alpha \in \Omega_n\},$$

$$N_n = \bigcup \{V_{n\alpha} \times G_{n\alpha}|\ \alpha \in \Omega_n\},$$

Then it can be easily checked that

$$\Phi^{-1}(E_1) \subset \bigcup_n M_n, \quad \Phi^{-1}(E_2) \subset \bigcup_n N_n,$$

$$\mathrm{Cl}\, M_n \cap \Phi^{-1}(E_2) = \emptyset, \quad \Phi^{-1}(E_1) \cap \mathrm{Cl}\, N_n = \emptyset$$

for each $n \in N$. Therefore we can find open sets H_1' and H_2' of $X \times Z$ such that $\Phi^{-1}(E_1) \subset H_1'$, $\Phi^{-1}(E_2) \subset H_2'$ and $H_1' \cap H_2' = \emptyset$. Now the sets

$$H_1 = X \times Y - \mathrm{Cl}\, \Phi(X \times Z - H_1'),$$

$$H_2 = X \times Y - \mathrm{Cl}\, \Phi(X \times Z - H_2')$$

are open and disjoint, and we have $E_1 \subset H_1$, $E_2 \subset H_2$.

This proves Claim 2 and the proof of Theorem 1 is completed.

For the proof of Theorem 2 we need another lemma. Let

$$\Omega_n' = \{\alpha \in \Omega_n|\ G_{n\alpha} \text{ is not locally compact}\},$$

$$\mathscr{G}_n' = \{G_{n\alpha}|\ \alpha \in \Omega_n'\}.$$

LEMMA 2.2. *For each $n \in N$ and $\alpha \in \Omega_n'$ we can select two distinct points $p_{n\alpha}$ and $q_{n\alpha}$ of $G_{n\alpha}$ so that the following conditions are satisfied*:

(a) *If $\{f(p_{n\alpha}), f(q_{n\alpha})\} \cap Y_0 \neq \emptyset$, then $f(p_{n\alpha}) \neq f(q_{n\alpha})$,*

(b) *If $n \neq m$ or $n = m$ and $\alpha \neq \beta$, then $\{p_{n\alpha}, q_{n\alpha}\} \cap \{p_{m\beta}, q_{m\beta}\} = \emptyset$ and $\{f(p_{n\alpha}), f(q_{n\alpha})\} \cap \{f(p_{m\beta}), f(q_{m\beta})\} \cap Y_0 = \emptyset$.*

Proof. Let Ω_n' be the well-ordered set $(\Omega_n', <)$, and assume $\Omega_n' \cap \Omega_m' = \emptyset$ if $n \neq m$. We select $p_{n\alpha}$ and $q_{n\alpha}$ by induction on the lexicographic ordering of $\bigcup_n \Omega_n'$: for $\alpha \in \Omega_n'$ and $\beta \in \Omega_m'$ $\alpha < \beta$ if and only if $n < m$ or $n = m$ and $\alpha < \beta$. Let $\beta \in \Omega_m'$ and assume that $p_{n\alpha}$ and $q_{n\alpha}$ have been selected from $G_{n\alpha}$ for each $\alpha < \beta$ so that (a) and (b) are true for $p_{k\gamma}$ and $q_{k\gamma}$, $\gamma < \beta$. Let us put

$$C_\beta = \{p_{n\alpha}, q_{n\alpha}|\ \alpha < \beta\},$$

$$D_\beta = \{z \in C_\beta|\ f(z) \in Y_0\}.$$

Since C_β is a discrete closed set and f is closed, the set $C_\beta \cup f^{-1}f(D_\beta)$ is locally compact. Hence, $G_{m\beta} - C_\beta \cup f^{-1}f(D_\beta)$ is infinite. Select distinct points $p_{m\beta}$ and $q_{m\beta}$ from $G_{m\beta} - C_\beta \cup f^{-1}f(D_\beta)$. If $f(p_{m\beta}) \in Y_0$ may arise, select a point q from $G_{m\beta} - C_\beta \cup f^{-1}f(D_\beta) \cup f^{-1}f(p_{m\beta})$ and newly define $q_{m\beta}$ by q. Then we see that conditions (a), (b) are satisfied for all $p_{n\alpha}$, $q_{n\alpha}$ with $\alpha \leqslant \beta$. Hence, Lemma 2.2 is proved.

Proof of Theorem 2. Suppose that X is normal, Y is non-discrete and $X \times Y$ is normal. Since Y is a Fréchet space and non-discrete, Y contains an infinite sequence $\{y_n\}$ having y_0 as its limit. Then $X \times (\{y_0, y_n|\ n \in N\})$ is normal, and hence X is countably paracompact [2].

Let $\mathscr{F} = \{F_n|\ n \in N\}$ be a decreasing sequence of closed sets of $X \times Y$ with the empty intersection. To prove the countable paracompactness of $X \times Y$, since $X \times Y$ is normal, it is sufficient to obtain a countable closed cover of $X \times Y$, each member of which is disjoint from some member of $\mathscr{F}$.

Let $i \geqslant 1$. Since Y_i is discrete and closed in Y, $X \times Y_i$ is countably paracompact. Hence, there exists a sequence $\{C_{in}|\ n \in N\}$ of closed sets in $X \times Y_i$ (so also in $X \times Y$) such that $\bigcup_n C_{in} = X \times Y_i$ and $C_{in} \cap F_n = \emptyset$ for each $n \in N$. Since $X \times Y$ is normal, there exists an open set U_{in} such that $C_{in} \subset U_{in}$ and $\mathrm{Cl}\, U_{in} \cap F_n = \emptyset$. Les us put

$$E_n = F_n - \bigcup \{U_{ik}|\ i, k \in N\}.$$

Then E_n is a closed set of $X \times Y$ and contained in $X \times Y_0$ since

$$X \times (\bigcup_{i \geqslant 1} Y_i) \subset \bigcup \{U_{ik}|\ i, k \in N\}.$$

Let $\mathscr{E} = \{E_n|\ n \in N\}$.

CLAIM. *There exists a countable closed cover $\mathscr{D}$ of $X \times Y$ such that each member of $\mathscr{D}$ is disjoint from some member of $\mathscr{E}$.*

First suppose that the claim is valid. Let $\mathscr{D} = \{D_k|\ k \in N\}$, and set

$$E_{0n} = \bigcup \{D_k|\ D_k \cap E_n = \emptyset, k \leqslant n\} - \bigcup \{U_{ij}|\ i, j \in N\},$$

$$E_{in} = \mathrm{Cl}\, U_{in} \quad (i \geqslant 1)$$

for each $n \in N$. Since $\mathscr{E}$ is decreasing, $\{E_{in}|\ i \geqslant 0, n \in N\}$ is a closed cover of $X \times Y$, and we have $E_{in} \cap F_n = \emptyset$. Thus, $X \times Y$ is countably paracompact, and it suffices to prove the claim.

Proof of the claim. Let $p_{n\alpha}$ and $q_{n\alpha}$ be points of $G_{n\alpha}$, $\alpha \in \Omega_n'$ defined in Lemma 2.2. Let us put

$$C_{n\alpha} = \bigcup \{P|\ P \text{ is an open set of } X \text{ such that } (P \times f(\mathrm{Cl}\, G_{n\alpha})) \cap E_n = \emptyset\}$$

and

$$A_n = \bigcup \{(X - C_{n\alpha}) \times \{p_{n\alpha}\}|\ \alpha \in \Omega_n'\},$$

$$B_n = \bigcup \{(X - C_{n\alpha}) \times \{q_{n\alpha}\}|\ \alpha \in \Omega_n'\}.$$

Since $\mathscr{G}'_n$ is locally finite, A_n and B_n are closed sets of $X\times Z$ and disjoint. Let

$$A = \bigcup_n A_n, \quad B = \bigcup_n B_n.$$

Then by the first equality of (b) in Lemma 2.2 the same proof as in [18, Theorem 1*] yields that A and B are disjoint and closed in $X\times Z$. Furthermore, from (a) and the second equality of (b) in Lemma 2.2 it follows that $\Phi(A)\cap\Phi(B)\cap(X\times Y_0) = \emptyset$. Hence, by the fact mentioned in the proof of Claim 2 in Theorem 1, we have

$$\mathrm{Cl}\,\Phi(A)\cap\mathrm{Cl}\,\Phi(B)\cap(X\times Y_0) = \emptyset.$$

Since $E_1\subset X\times Y_0$, by the normality of $X\times Y$, there exist open sets U, V and W of $X\times Y$ such that

(1) $$\mathrm{Cl}\,\Phi(A)\cap\mathrm{Cl}\,\Phi(B)\subset W \quad\text{and}\quad \mathrm{Cl}\,W\cap E_1 = \emptyset,$$

$$\mathrm{Cl}\,\Phi(A)-W\subset U,\quad \mathrm{Cl}\,\Phi(B)-W\subset V\quad\text{and}\quad \mathrm{Cl}\,U\cap\mathrm{Cl}\,V = \emptyset.$$

Let us put

$$D_{n\alpha} = \{x\in X|\ \{x\}\times f(\mathrm{Cl}\,G_{n\alpha})\subset X\times Y-(U\cup W)\},$$
$$E_{n\alpha} = \{x\in X|\ \{x\}\times f(\mathrm{Cl}\,G_{n\alpha})\subset X\times Y-(V\cup W)\}$$

for $\alpha\in\Omega'_n$, and

$$H_n = \bigcup\{(D_{n\alpha}\cup E_{n\alpha})\times\mathrm{Cl}\,G_{n\alpha}|\ \alpha\in\Omega'_n\}.$$

Since $D_{n\alpha}$ and $E_{n\alpha}$ are closed and $\mathscr{G}'_n$ is locally finite, H_n is also closed. Moreover we have $\Phi(H_n)\cap E_n = \emptyset$. To see this, let $(x,z)\in H_n$ and suppose that $(x,z)\in D_{n\alpha}\times\mathrm{Cl}\,G_{n\alpha}$ for some $\alpha\in\Omega'_n$. Then $\{x\}\times f(\mathrm{Cl}\,G_{n\alpha})\subset X\times Y-(U\cup W)$. Hence, $(\{x\}\times\mathrm{Cl}\,G_{n\alpha})\cap A = \emptyset$, and consequently $x\in C_{n\alpha}$. Thus, we have $(x,f(z))\notin E_n$.

Since $E_n\subset X\times Y_0$, as before we have

(2) $$\mathrm{Cl}\,\Phi(H_n)\cap E_n = \emptyset\quad\text{for each } n\in N.$$

Let $\alpha\in\Omega_k-\Omega'_k$. Then $X\times G_{k\alpha}$ is countably paracompact since $G_{k\alpha}$ is locally compact metrizable and X is countably paracompact. Hence, there exists a closed set $K_{k\alpha n}$ of $X\times G_{k\alpha}$ such that

$$X\times G_{k\alpha} = \bigcup\{K_{k\alpha n}|\ n\in N\},$$
$$K_{k\alpha n}\cap\Phi^{-1}(E_n) = \emptyset\quad\text{for each } n\in N.$$

Since $G_{k\alpha}$ is an F_σ subset of Z, $K_{k\alpha n}$ can be written as $K_{k\alpha n} = \bigcup\{K_{k\alpha nj}|\ j\in N\}$ for some closed set $K_{k\alpha nj}$ of $X\times Z$. Let

$$L_{knj} = \bigcup\{K_{k\alpha nj}|\ \alpha\in\Omega_k-\Omega'_k\}.$$

Since $\mathscr{G}_k$ is locally finite, L_{knj} is closed in $X\times Z$, and $L_{knj}\cap\Phi^{-1}(E_n) = \emptyset$. Therefore,

(3) $$\mathrm{Cl}\,\Phi(L_{knj})\cap E_n = \emptyset\quad\text{for each } k,n,j\in N.$$

Finally we show that the collection

$$\mathscr{D} = \{\mathrm{Cl}\,W,\ \mathrm{Cl}\,\Phi(H_n),\ \mathrm{Cl}\,\Phi(L_{knj})|\ k,n,j\in N\}$$

covers $X\times Y$. To see this, let (x,y) be any point of $X\times Y$ and suppose that $(x,y)\notin\mathrm{Cl}\,W$. We may assume that $(x,y)\notin\mathrm{Cl}\,U$ since $\mathrm{Cl}\,U\cap\mathrm{Cl}\,V = \emptyset$. Let z be a point of Z with $y = f(z)$. Then there exists $G_{k\alpha}\in\mathscr{G}_k$ for some $k\in N$ such that $z\in G_{k\alpha}$ and $\{x\}\times f(\mathrm{Cl}\,G_{k\alpha})\subset X\times Y-(\mathrm{Cl}\,U\cup\mathrm{Cl}\,W)$. If $\alpha\in\Omega'_k$, then $x\in D_{k\alpha}$. Hence $(x,z)\in H_k$. That is, $(x,y)\in\mathrm{Cl}\,\Phi(H_k)$. If $\alpha\in\Omega_k-\Omega'_k$, then $(x,z)\in K'_{k\alpha nj}$ for some n and j. Hence, $(x,z)\in L_{knj}$ and $(x,y)\in\mathrm{Cl}\,\Phi(L_{knj})$. Thus, $\mathscr{D}$ covers $X\times Y$.

Therefore, in view of (1), (2) and (3) $\mathscr{D}$ is now the desired closed cover of $X\times Y$, which proves the claim and the proof of Theorem 2 is completed.

Proof of Theorem 3. The "if" part follows from Theorem 1. The "only if" part follows from Theorem 2 in case Y is non-discrete. If Y is discrete, $X\times Y$ is obviously countably paracompact.

Let us denote by $\mathfrak{m}$ an infinite cardinal number. A space is said to be $\mathfrak{m}$-*paracompact* if its every open cover with cardinality $\leqslant\mathfrak{m}$ admits a locally finite open refinement, and a space is $\mathfrak{m}$-*collectionwise normal* if for every discrete family $\{F_\alpha|\ \alpha\in\Omega\}$ with $\mathrm{Card}\,\Omega\leqslant\mathfrak{m}$ of its closed subsets there is a family $\{U_\alpha|\ \alpha\in\Omega\}$ of mutually disjoint open subsets such that $F_\alpha\subset U_\alpha$ for each $\alpha\in\Omega$.

Let $I^{\mathfrak{m}}$ be the product space of $\mathfrak{m}$ copies of $I = [0,1]$, and $A(\mathfrak{m})$ the one-point compactification of the discrete space of cardinality $\mathfrak{m}$. Then it is known by [10] and [1] respectively that a space X is $\mathfrak{m}$-paracompact and normal iff $X\times I^{\mathfrak{m}}$ is normal, and a space X is $\mathfrak{m}$-collectionwise normal and countably paracompact iff $X\times A(\mathfrak{m})$ is normal.

Using these results and Theorem 2.3 below and replacing C by $I^{\mathfrak{m}}$ or $A(\mathfrak{m})$, we immediately obtain Theorems 2.4 and 2.6 and their corollaries.

THEOREM 2.3. *Suppose that Y is Lašnev and C is compact and both $X\times Y$ and $X\times C$ are normal. Then $X\times Y\times C$ is normal.*

Proof. We may assume that Y is non-discrete. Then by Theorem 2 $X\times Y$ is countably paracompact. Since C is compact, $(X\times Y)\times C = (X\times C)\times Y$ is countably paracompact. Hence by Theorem 1 $X\times Y\times C$ is normal, which proves the theorem.

THEOREM 2.4. *Let X be an $\mathfrak{m}$-paracompact and normal space and Y a Lašnev space. If $X\times Y$ is normal, then $X\times Y$ is $\mathfrak{m}$-paracompact.*

COROLLARY 2.5. *If the product space $X\times Y$ of a paracompact space X with a Lašnev space Y is normal, then $X\times Y$ is paracompact.*

THEOREM 2.6. *Let X be an $\mathfrak{m}$-collectionwise normal space and Y a Lašnev space. If $X\times Y$ is normal, then $X\times Y$ is $\mathfrak{m}$-collectionwise normal.*

COROLLARY 2.7. *If the product space $X\times Y$ of a collectionwise normal space X with a Lašnev space Y is normal, then $X\times Y$ is collectionwise normal.*

Corollary 2.7 answers affirmatively to a question of K. Chiba asked in a letter to the author.

3. Product of a normal P-space with a paracompact σ-space. A space X is said to be a *P-space* if for any set Ω and for any family $\{G(\alpha_1, \ldots, \alpha_n) \mid \alpha_1, \ldots, \alpha_n \in \Omega;\ n \in N\}$ of open subsets of X such that

$$G(\alpha_1, \ldots, \alpha_n) \subset G(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}) \quad \text{for} \quad \alpha_1, \ldots, \alpha_n, \alpha_{n+1} \in \Omega;\ n \in N,$$

there exists a family $\{D(\alpha_1, \ldots, \alpha_n) \mid \alpha_1, \ldots, \alpha_n \in \Omega;\ n \in N\}$ of closed subsets of X such that the two conditions (a), (b) below are satisfied:

(a) $D(\alpha_1, \ldots, \alpha_n) \subset G(\alpha_1, \ldots, \alpha_n)$ for $\alpha_1, \ldots, \alpha_n \in \Omega$,

(b) $X = \bigcup_{n=1}^{\infty} D(\alpha_1, \ldots, \alpha_n)$ for any sequence (α_n) such that $X = \bigcup_{n=1}^{\infty} G(\alpha_1, \ldots, \alpha_n)$.

The notion of P-space is due to Morita [11], and by his theorem normal P-spaces are known to be those spaces whose product with any metric space is normal. σ-spaces are defined to be spaces with a σ-locally finite net (Okuyama [17]).

We shall now prove the following mentioned in the introduction.

THEOREM 3.1. *Let X be a normal P-space and Y a paracompact σ-space. Then $X \times Y$ is normal iff $X \times Y$ is countably paracompact.*

Proof. Since every σ-space is a strong Σ-space in the sense of Nagami [15], the "only if" part follows from [15, Theorem 4.10]. To show the "if" part, assume that $X \times Y$ is countably paracompact, and let A and B be disjoint closed subsets of $X \times Y$.

It is known that Y has a net $\mathscr{E} = \bigcup_{i=1}^{\infty} \mathscr{E}_i$, where each $\mathscr{E}_i$ is a discrete collection of closed sets. Let $\mathscr{F}_i = \mathscr{E}_i \cup \{Y\}$, and $\mathscr{F}_i = \{F_{i\alpha} \mid \alpha \in \Omega_i\}$. Here it may be assumed that all Ω_i's are equal to an index set Ω. Then we can write $\mathscr{F}_i = \{F_{i\alpha} \mid \alpha \in \Omega\}$.

Let us put for $i \in N$ and $\alpha \in \Omega$

$$U_{i\alpha}(1) = \bigcup \{P \mid P \text{ is an open set of } X \text{ such that } P \times F_{i\alpha} \subset X \times Y - B\},$$

$$U_{i\alpha}(2) = \bigcup \{P \mid P \text{ is an open set of } X \text{ such that } P \times F_{i\alpha} \subset X \times Y - A\}.$$

Further let us put for $\alpha_1, \ldots, \alpha_n \in \Omega$

$$G(\alpha_1, \ldots, \alpha_n; j) = U_{1\alpha_1}(j) \cup \ldots \cup U_{n\alpha_n}(j), \quad j = 1, 2,$$

$$G(\alpha_1, \ldots, \alpha_n) = G(\alpha_1, \ldots, \alpha_n;\ 1) \cup G(\alpha_1, \ldots, \alpha_n;\ 2).$$

Then we have $G(\alpha_1, \ldots, \alpha_n) \subset G(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$ for every sequence $\alpha_1, \ldots, \alpha_n, \alpha_{n+1} \in \Omega$. Since X is a P-space, there exists a family $\{D(\alpha_1, \ldots, \alpha_n) \mid \alpha_1, \ldots, \alpha_n \in \Omega;\ n \in N\}$ of closed subsets of X satisfying the conditions (a), (b) above. By (a) and the normality of X there exist zero-sets $K(\alpha_1, \ldots, \alpha_n; j)$ $(j = 1, 2)$ of X such that

$$D(\alpha_1, \ldots, \alpha_n) \subset K(\alpha_1, \ldots, \alpha_n;\ 1) \cup K(\alpha_1, \ldots, \alpha_n;\ 2),$$

$$K(\alpha_1, \ldots, \alpha_n; j) \subset G(\alpha_1, \ldots, \alpha_n; j), \quad j = 1, 2.$$

On the other hand, since Y is paracompact, there exists a locally finite family $\{H_{i\alpha} \mid \alpha \in \Omega\}$ of open sets of Y such that $F_{i\alpha} \subset H_{i\alpha}$ for each $\alpha \in \Omega$. We have then

$$K(\alpha_1, \ldots, \alpha_n;\ 1) \times (\bigcap_{i=1}^{n} F_{i\alpha_i}) \subset G(\alpha_1, \ldots, \alpha_n;\ 1) \times (\bigcap_{i=1}^{n} H_{i\alpha_i}) - B.$$

Since Y is perfectly normal, $\bigcap_{i=1}^{n} F_{i\alpha_i}$ is a zero-set of Y. Hence $K(\alpha_1, \ldots, \alpha_n;\ 1) \times (\bigcap_{i=1}^{n} F_{i\alpha_i})$ is a zero-set of $X \times Y$, and by the assumption and Lemma 2.1 there exists an open set $W(\alpha_1, \ldots, \alpha_n;\ 1)$ of $X \times Y$ such that

$$K(\alpha_1, \ldots, \alpha_n;\ 1) \times (\bigcap_{i=1}^{n} F_{i\alpha_i}) \subset W(\alpha_1, \ldots, \alpha_n;\ 1),$$

$$\operatorname{Cl} W(\alpha_1, \ldots, \alpha_n;\ 1) \subset G(\alpha_1, \ldots, \alpha_n;\ 1) \times (\bigcap_{i=1}^{n} H_{i\alpha_i}) - B.$$

Similarly we can choose an open set $W(\alpha_1, \ldots, \alpha_n;\ 2)$ of $X \times Y$ so that

$$K(\alpha_1, \ldots, \alpha_n;\ 2) \times (\bigcap_{i=1}^{n} F_{i\alpha_i}) \subset W(\alpha_1, \ldots, \alpha_n;\ 2),$$

$$\operatorname{Cl} W(\alpha_1, \ldots, \alpha_n;\ 2) \subset G(\alpha_1, \ldots, \alpha_n;\ 2) \times (\bigcap_{i=1}^{n} H_{i\alpha_i}) - A.$$

Let us put

$$V_n(j) = \bigcup \{W(\alpha_1, \ldots, \alpha_n; j) \mid \alpha_1, \ldots, \alpha_n \in \Omega\}, \quad j = 1, 2.$$

Since $\{\bigcap_{i=1}^{n} H_{i\alpha_i} \mid \alpha_1, \ldots, \alpha_n \in \Omega\}$ is locally finite and $\operatorname{Cl} W(\alpha_1, \ldots, \alpha_n;\ 1) \cap B = \emptyset$, we have

$$\operatorname{Cl} V_n(1) \cap B = \emptyset \quad \text{for each } n \in N.$$

Similarly

$$A \cap \operatorname{Cl} V_n(2) = \emptyset \quad \text{for each } n \in N.$$

Finally we show that $A \subset \bigcup_n V_n(1)$ and $B \subset \bigcup_n V_n(2)$. Let $(x, y) \in A$. Note that the set $\bigcap \{F \in \mathscr{F}_n \mid y \in F\}$ is equal to some $F_{n\alpha_n} \in \mathscr{F}_n$, and $\{F_{n\alpha_n} \mid n \in N\}$ has the property that any open set U with $y \in U$ contains some $F_{n\alpha_n}$. Hence it can be easily checked that $X = \bigcup_n G(\alpha_1, \ldots, \alpha_n)$. Therefore by (b) we have $X = \bigcup_n D(\alpha_1, \ldots, \alpha_n)$, and consequently $x \in D(\alpha_1, \ldots, \alpha_n)$ for some n. Then, since $(x, y) \in A$, we have $x \in K(\alpha_1, \ldots, \alpha_n;\ 1)$. Thus,

$$(x, y) \in K(\alpha_1, \ldots, \alpha_n;\ 1) \times (\bigcap_{i=1}^{n} F_{i\alpha_i}) \subset W(\alpha_1, \ldots, \alpha_n;\ 1) \subset V_n(1),$$

which shows $A \subset \bigcup_n V_n(1)$. By the same argument we have $B \subset \bigcup_n V_n(2)$. Hence, A and B are separated by open sets, which completes the proof of the theorem.

Remark. It should be noted that for the "if" part of Theorem 3.1 "a paracompact σ-space" cannot be weakened to "a strong Σ-space". Indeed, let $W(\omega_1) = \{\alpha|\ 0 \leqslant \alpha < \omega_1\}$ and $W(\omega_1+1) = \{\alpha|\ 0 \leqslant \alpha < \omega_1+1\}$ with the usual order topology, where ω_1 is the first uncountable ordinal. Then $W(\omega_1)$ is a normal P-space, and $W(\omega_1+1)$ is a strong Σ-space since it is compact. On the other hand, it is well known that $W(\omega_1) \times W(\omega_1+1)$ is countably compact, but is not normal.

In view of Theorem 3, it is unknown to the author whether the assumption "P-space" in Theorem 3.1 can be replaced by "countably paracompact space".

It should be pointed out that the product of a paracompact P-space and a paracompact σ-space is paracompact.

The following theorem of Nagami [16] can be proved using Theorem 3.1.

THEOREM 3.2. *Suppose that X is a collectionwise normal P-space and Y a paracompact σ-space and $X \times Y$ normal. Then $X \times Y$ is collectionwise normal.*

Proof. Note that $X \times A(\mathfrak{m})$ is normal and is a P-space. Therefore, with the aid of Theorem 3.1, the same argument as in the proof of Theorem 2.3 implies that $X \times Y \times A(\mathfrak{m})$ is normal. Hence $X \times Y$ is $\mathfrak{m}$-collectionwise normal for every $\mathfrak{m}$, and consequently it is collectionwise normal.

Added in proof. The author proved Theorem 3 also for the case Y is paracompact F_σ-metrizable; this fact and contents of this paper are announced in [19].

References

[1] O. T. Alas, *On a characterization of collectionwise normality*, Canad. Math. Bull. 14 (1971), pp. 13–15.
[2] C. H. Dowker, *On countably paracompact spaces*, Canad. J. Math. 3 (1951), pp. 219–224.
[3] R. Engelking, *General Topology*, Polish Scientific Publishers, Warszawa 1977.
[4] — *Dimension Theory*, North-Holland, Amsterdam 1978.
[5] T. Ishii, *On product spaces and product mappings*, J. Math. Soc. Japan, 18 (1966), pp. 166–181.
[6] N. S. Lašnev, *Continuous decompositions and closed mappings of metric spaces*, Soviet Math. Dokl. 6 (1965), pp. 1504–1506.
[7] — *Closed images of metric spaces*, Soviet Math. Dokl. 7 (1966), pp. 1219–1221.
[8] I. M. Leĭbo, *On the equality of dimensions for closed images of metric spaces*, Soviet Math. Dokl. 15 (1974), pp. 835–839.
[9] — *On closed images of metric spaces*, Soviet Math. Dokl. 16 (1975), pp. 1292–1295.
[10] K. Morita, *Paracompactness and product spaces*, Fund. Math. 50 (1962), pp. 223–236.
[11] — *Products of normal spaces with metric spaces*, Math. Ann. 154 (1964), pp. 365–382.
[12] — *Products of normal spaces with metric spaces*, II, Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A, 8 (1963), pp. 87–92.
[13] — *Note on products of normal spaces with metric spaces*, unpublished.
[14] — *On the dimension of the product of topological spaces*, Tsukuba J. Math. 1 (1977), pp. 1–6.

[15] K. Nagami, *Σ-spaces*, Fund. Math. 65 (1969), pp. 160–192.
[16] — *σ-spaces and product spaces*, Math. Ann. 181 (1969), pp. 109–118.
[17] A. Okuyama, *Some generalizations of metric spaces, their metrization theorems and product spaces*, Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A, 8 (1967), pp. 236–254.
[18] M. E. Rudin and M. Starbird, *Products with a metric factor*, Gen. Top. Appl. 5 (1975), pp. 235–248.
[19] T. Hoshina, *Normality and countable paracompactness of product spaces*, Question and Answers in Gen. Top. 4 (1983), pp. 54–56.

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