morphism of $\mathscr{A}$ into $P(\omega)/\mathrm{Fin}$. For $A \in \mathscr{A}$ let $\chi_A \in P(\omega)$ be any element of the equivalent class of $\varphi(A)$. Since $\omega_1 \subseteq \mathscr{A}$, we have obtained a function $\chi\colon \omega_1 \to P(\omega)$.

For any $F \in \mathscr{F}$ consider the set $K_F = \{x \in P(\omega)\colon \operatorname{card}(\chi_F \setminus x) < \omega\}$. It is easy to see, K_F is σ-compact in $P(\omega)$ (with Cantor set topology). Hence K_F is a Borel set. Also, it is easy to see that $\chi^{-1}(K_F) = F$.

So $\mathscr{F} \subseteq \{\chi^{-1}(B) \mid B \subset P(\omega),\ B \text{ is a Borel set}\}$.

The above family is a σ-field which is a countable generating family. The proof of Proposition 1 is complete.

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On the homotopy classification of pairs of linked maps of manifolds into a linear space

by

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Abstract. It is shown that the linking number of maps $f\colon M \to E$ and $g\colon N \to E$ of connected closed manifolds into a linear space with disjoint images gives the homotopy classification of such pairs of maps if $\dim M + \dim N + 1 = \dim E$.

Let M and N be two closed connected oriented smooth manifolds of dimensions m and n respectively. We shall suppose that $0 \leqslant m \leqslant n$. Let E be a real oriented k-dimensional linear space with $k = m+n+1 \geqslant 2$. Denote $E_0 = E \setminus \{0\}$. A pair (f, g) of smooth maps $f\colon M \to E$ and $g\colon N \to E$ with disjoint images $f(M)$ and $g(N)$ will be called a *pair of linked maps*.

Two pairs of linked maps (f_0, g_0) and (f_1, g_1) are said to be *homotopic* if there are two smooth homotopies $f_t\colon M \to E$ and $g_t\colon N \to E$ with $f_t(M)$ and $g_t(N)$ disjoint for every $t \in I = [0, 1]$. We shall write $(f_0, g_0) \simeq (f_1, g_1)$ in this case. Denote by $\mathscr{H}$ the set of all homotopy classes of pairs of linked maps.

For a pair of linked maps (f, g) their linking number $l(f, g)$ is defined to be the winding number around 0 (comp. [2], p. 144) $W(\Phi)$ of the map $\Phi\colon M \times N \to E_0$ of oriented manifolds defined by $\Phi(u, v) = g(v) - f(u)$ (or the degree of the map $\Phi/|\Phi|\colon M \times N \to S^{m+n}$ if E is Euclidean). It is known ([8], p. 104) that homotopic pairs of linked maps have the same linking number.

The main result of this paper is the following

THEOREM 1. *The function $\mathscr{H} \to Z$ assigning to a homotopy class of a pair of linked maps (f, g) their linking number $l(f, g)$ is bijective.*

If $m = 0$ then M is a point and the theorem is really the Hopf classification theorem (comp. [7], § 7). If $m = n = 1$ then M and N are diffeomorphic to circles; this result was obtained by J. Milnor in [6], p. 190 by means of tools developed there. We shall give also another, more direct proof of this case.

We shall need some lemmas.

LEMMA 2. *Let D be an open connected set in a linear space E, (f_0, g_0) a pair of linked maps with $f_0(M), g_0(N) \subset D$ and $1 \leqslant m \leqslant n$. Then there are two pairs of linked maps, (f, g) and $(f, \bar{g})$, with $f(M), g(N), \bar{g}(N) \subset D$ such that $l(f, g) = l(f_0, g_0)+1$ and $l(f, \bar{g}) = l(f_0, g_0)-1$. As maps into D, f_0 and f belong to one homotopy class as well as g_0, g and $\bar{g}$.*

Proof. Let $v_0 \in N$. We can slightly deform g_0 in a small neighbourhood of v_0 to a smooth mapping $g_1: N \to D \setminus f_0(M)$ embedding some neighbourhood $V \subset N$ of v_0 into an n-dimensional affine subspace P of E. Similarly for $u_0 \in M$ we deform f_0 to $f: M \to D \setminus g_1(N)$ embedding some neighbourhood $U \subset M$ of u_0 in an m-dimensional affine subspace L of E. Since $k = m+n+1 \geqslant 2m+1$, f may be chosen as the embedding of the whole M in $D \setminus g_1(N)$ ([1], (7.9) Satz).

The set $f(M) \cup g_1(V)$ does not separate D because $m \leqslant n \leqslant k-2$. Therefore there exists an embedding $\gamma: R \to D$ such that $\gamma(R) \cap (f(M) \cup g_1(V))$ consists of exactly two points, $f(u_0) = \gamma(0)$ and $g_1(v_0) = \gamma(1)$. We can also suppose that the derivatives $\gamma'(0)$ and $\gamma'(1)$ are orthogonal to L and P respectively (after choosing some inner product in E). The normal bundle to $\gamma(R)$ in $D \subset E$ is trivial because $\gamma(R)$ is contractible ([4], 2.2.5). There is a diffeomorphism $\tilde{h}: R \times R^{k-1} \to T$ onto a tubular neighbourhood T of $\gamma(R)$ in D. We can suppose that for some linear subspaces L_0 and P_0 of R^{k-1} $f(U) = \tilde{h}(0, L_0) = f(M) \cap T$ and $g(V) = \tilde{h}(1, P_0)$. Let P_1 be the orthogonal complement of L_0 in R^{k-1}. There exists an orthogonal transformation $A \in SO(k-1)$ mapping P_1 onto P_0. Since $SO(k-1)$ is connected, there exists a smooth map $\lambda: R \to SO(k-1)$ such that $\lambda(0) = \mathrm{id}_{R^{k-1}}$ and $\lambda(1) = A$. Define $h: R \times R^{k-1} \to T$ by $h(t, w) = \tilde{h}(t, \lambda(t)w)$. Then h maps diffeomorphically $R \times R^{k-1}$ onto T, $\{0\} \times L_0$ onto open subset $f(U)$ of L and $\{1\} \times P_1$ onto open subset $g_1(V)$ of P.

Let $\Psi: R^{k-1} \to [-2, 0]$ be a smooth function with compact support equal to -2 on a neighbourhood of 0. Define the maps $G_s: R \times R^{k-1} \to R \times R^{k-1}$ by $G_s(t, w) = (t+s\psi(w), w)$ for $s \in I$. Define also $g_2: N \to D$ and $\Phi: I \times M \times N \to E$ by $g_2(v) = h \circ G_1 \circ h^{-1} \circ g_1(v)$ for $v \in V$, $g_2(v) = g_1(v)$ for $v \in N \setminus V$, $\Phi(s, u, v) = h \circ G_s \circ h^{-1} \circ g_1(v) - f(u)$ for $s \in I$, $u \in M$, $v \in V$ and $\Phi(s, u, v) = g_1(v) - f(u)$ for $s \in I$, $u \in M$, $v \in N \setminus V$. g_0, g_1 and g_2 are in one homotopy class of maps $N \to D$.

(f, g_2) is a pair of linked maps and Φ is the homotopy between the mappings Φ_0, Φ_1: $M \times N \to E$ defined by $\Phi_0(u, v) = g_1(v) - f(u)$, $\Phi_1(u, v) = g_2(v) - f(u)$. Φ has only one zero $z = (\frac{1}{2}, u_0, v_0)$. The differential Φ_* of Φ at z maps the tangent spaces $T_{1/2}I$, $T_{u_0}M$ and $T_{v_0}N$ onto the mutually orthogonal subspaces of $T_0E = E$ and so Φ_* is an isomorphism. From the well-known property of winding numbers (comp. [2], p. 144) $l(f, g_2) - l(f_0, g_0) = l(f, g_2) - l(f, g_1) = W(\Phi_1) - W(\Phi_2) = \pm 1$, the sign + occuring if Φ_* preserves the orientation $T_z(I \times M \times N) = R \oplus T_{u_0}M \oplus T_{v_0}N$ and $T_0E = E$, and the sign $-$ otherwise. We put $g = g_2$ in the first and $\bar{g} = g_2$ in the second case.

To get $\bar{g}$ in the first case and g in the second it is sufficient in the above construction to take, instead of A, an $\bar{A} \in SO(k-1)$ mapping P_1 onto P_0 with a change of orientation. This is possible because $1 \leqslant n = k-1-m \leqslant k-1$.

LEMMA 3. *If $f: M \to E$ is an embedding, $g_0, g_1: N \to E \setminus f(M)$ are smooth maps such that $l(f, g_0) = l(f, g_1)$ and $2 \leqslant m \leqslant n$, then g_0 and g_1 are smoothly homotopic.*

Proof. We may suppose that M is a submanifold of E and f is an inclusion. Let $G: I \times N \times E \to E$ be defined by $G(t, v, w) = g_0(v) + t(g_1(v) - g_0(v)) + w$. The map G is a submersion and by the parametric transversality theorem ([3], Th. 3.2.7) there is a small vector w_0 such that the maps $\bar{g}_0, \bar{g}_1: N \to E \setminus M$ defined by $\bar{g}_i(v) = g_i(v) + w_0$ are homotopic to g_0 and g_1 respectively and the homotopy $G_0: I \times N \to E$ defined by $G_0(t, v) = \bar{g}_0(v) + t(g_1(v) - g_0(v)) = G(t, v, w_0)$ is transversal to M. Define Φ_0, Φ_1: $M \times N \to E$ and Φ: $I \times M \times N \to E$ by $\Phi_i(u, v) = \bar{g}_i(v) - f(u)$ and $\Phi(t, u, v) = G_0(t, v) - f(u)$. The map Φ is a homotopy from Φ_0 to Φ_1 and has 0 as a regular value. There is a bijection between the finite sets $\Phi^{-1}(0)$ and $G_0^{-1}(M)$ such that the local degree of Φ at a point $(t, u, v) \in \Phi^{-1}(0)$ is equal to the intersection index of G_0 with M at the corresponding point $(t, v) \in G_0^{-1}(M)$ (comp. [4], § 5,2). By the well-known property of the winding number, from the equalities $0 = l(f, g_1) - l(f, g_0) = l(f, \bar{g}_1) - l(f, \bar{g}_0) = W(\Phi_1) - W(\Phi_0)$ it follows that the sum of intersection indices of G_0 with M for all points of $G_0^{-1}(M)$ is equal to 0.

Let z_0 and z_1 be two points of $G_0^{-1}(M)$ with the opposite intersection numbers 1 and -1 respectively. We may suppose that $G_0(z_0) \neq G_0(z_1)$ by a small modification of G_0 in a neighbourhood of z_0. By transversality G_0 is an embedding of some neighbourhood W of $\{z_0, z_1\}$ containing no other point of $G_0^{-1}(M)$. Let L be a smooth closed arc in the interior of $I \times N$ joining z_0 with z_1 and not passing through the other points of $G_0^{-1}(M)$. Let L_1 be an open arc such that $\bar{L}_1 \subset L \setminus \{z_0, z_1\}$ and $L \setminus L_1 \subset W$. The restriction $G_0|L$ may be approximated by an embedding $\gamma: L \to E$ extending $G_0|L \setminus L_1$. Then we choose a small tubular neighbourhood T of L_1 in the interior of $I \times N$ diffeomorphic to $R \times R^n$ and a small tubular neighbourhood V of $\gamma(L_1)$ in E diffeomorphic to $R \times R^{k-1}$ (the normal bundles of L_1 and $\gamma(L_1)$ are trivial). We can approximate $G_0|T$ by a monomorphism of trivial bundles G_1: $T \approx R \times R^n \to R \times R^{k-1} \approx V$ such that $G_1|L_1 = \gamma|L_1$ and the differential of G_1 on L_1 beyond some compact subset agrees with the differential of $G_0|T$. By means of G_1 we can construct a mapping G_2: $I \times N \to E$ which is equal to G_0 beyond T and is an embedding of some neighbourhood U of L diffeomorphic to a closed disc. Moreover, $G_2^{-1}(M) = G_0^{-1}(M)$ with the same intersection numbers and $G_2^{-1}(M) \cap U = \{z_0, z_1\}$.

The points $G_2(z_0)$ and $G_2(z_1)$ can be joined in M by an arc A because M is connected. By using Whitney's method of eliminating a pair of intersection points to the manifolds M and $G_2(U)$ ([5], Th. 6.6) we get a homotopy G_3: $I \times N \to E$ from $\bar{g}_0$ to $\bar{g}_1$ extending $G_2|I \times N \setminus U$ such that the number of points in $G_3^{-1}(M)$ is less by 2 than that for $G_0^{-1}(M)$. If $m \geqslant 3$ then $\dim G_2(U) = n+1 \geqslant m+1 \geqslant 4 > 3$, and those assumptions are sufficient for using Whitney's method. In the case $m = 2$ we use the fact that $E \setminus G_2(U)$ is simply connected, which follows from the isotopy of discs ([4], Th. 8.3.1) and the extension isotopy theorem ([4], Th. 8.1.3).

Proceeding similarly, we eliminate the remaining intersections and get a homotopy $\bar{g}_t$: $N \to E \setminus M$ from $\bar{g}_0$ to $\bar{g}_1$. Therefore g_0, g_1: $N \to E \setminus M$ are homotopic.

The next lemma is well known (comp. H. Whitney, *Differentiable manifolds*, Ann. of Math. 37 (1936), pp. 645–680, Th. 6, p. 657 or R. Thom, *La classification des immersions*, Séminaire Bourbaki 1957/58, Exp. 157).

LEMMA 4. *If f_0, f_1: $M \to E$ are embeddings of a closed m-dimensional manifold into a k-dimensional linear space and $k \geqslant 2m+2$, then there exists a diffeotopy h_t: $E \to E$ such that $h_0 = \mathrm{id}_E$ and $f_1 = h_1 \circ f_0$.*

Proof. By composing f_0 with a suitable translation of E we may suppose that $f_0(M)$ and $f_1(M)$ are disjoint. Since $k \geqslant 2(m+1)$, there exists an immersion H: $I \times M \to E$ which is a homotopy from f_0 to f_1 having no selfintersections when $k > 2(m+1)$ ([1], (7.8) Satz) or such that it selfintersections are transversal and no three points are mapped onto the same point ([8], § 4 B)). The number of the double points of H is finite. By the homogeneity of $I \times M$ relatively to the boundary ([7], § 4) there is a diffeomorphism G: $I \times M \to I \times M$ extending identity on $\partial(I \times M)$ such that the counter image of the set of double points by $H \circ G$ has in every set $\{t\} \times M$ at most one point. The map $H \circ G$ is an isotopy from f_0 to f_1 and the existence of the diffeotopy h_t follows by the isotopy extension theorem.

LEMMA 5. *Let $\bar{E} = E \cup \{\infty\}$ be a one-point compactification of a k-dimensional Euclidean vector space E diffeomorphic to the k-dimensional sphere and C an m-dimensional sphere lying in some $(m+1)$-dimensional linear subspace P of E with $1 \leqslant m \leqslant k-2$. Then there exists a submanifold S of $E \setminus C$ diffeomorphic to the $(n = k-m-1)$-dimensional sphere and a smooth strong deformation retraction r: $\bar{E} \setminus C \to S$. If C and S are suitably oriented then the linking number of the pair of inclusions $C \to E$ and $S \to E$ is equal to 1.*

Proof. We may suppose that C is the unit sphere in P. Identify the manifold $\bar{E}$ with the unit sphere S^k in the Euclidean space $E \times R$ by the stereographic projection from the point $p = (0, 1) \in E \times R$. Let L be the orthogonal complement of P in $E \times R$ and $S^n = S^k \cap L$ the $(n = k-m-1)$-dimensional sphere. We have the strong deformation retraction r_n: $S^k \setminus C \to S^n$ defined by $r_n(x, y) = y/|y|$ for $(x, y) \in S^k \setminus C \subset P \times L$. By the homogeneity of the sphere S^k there exists a diffeomorphism h: $S^k \to S^k$ which is an identity behind a small neighbourhood of the pole p disjoint with C such that $p \notin h(S^n)$. The map $r = h \circ r_n \circ h^{-1}$ is a strong deformation retraction of $\bar{E} \setminus C \approx S^k \setminus C$ onto the manifold $S \approx h(S^n)$. The sphere C is the boundary of the unit disc D in the subspace P and S intersects D transversally at exactly one point 0. Therefore if C and S are suitably oriented then the linking number of their inclusions is 1.

LEMMA 6. *Let C and S be connected compact oriented disjoint smooth manifolds without boundaries of dimensions $m \geqslant 1$ and $n \geqslant 1$ respectively in the oriented vector space E of dimension $k = m+n+1$ such that the linking number of the pair of inclusions $C \to E$ and $S \to E$ is equal to l. Let f: $M \to C \subset E$ and g: $N \to S \subset E$ be a pair of linked maps. Then $l(f, g) = l \cdot \deg f \cdot \deg g$.*

Proof. The map Φ: $M \times N \to E_0$ defined by $\Phi(u, v) = g(v) - f(u)$ can be represented as the composition $M \times N \xrightarrow{f \times g} C \times S \xrightarrow{\Psi} E_0$, where $\Psi(x, y) = y - x$. By the well-known properties of degrees concerning composition of maps and the Cartesian product of maps we have

$$l(f, g) = W(\Phi) = W(\Psi) \cdot \deg(f \times g) = l \cdot \deg f \cdot \deg g.$$

LEMMA 7. *Let M and N be two smooth disjoint simple closed curves in a 3-dimensional Euclidean space E. Then there is a smooth homotopy f_t: $M \to E \setminus N$ such that f_t are immersions for $t \in I$, f_0 is the inclusion $M \to E \setminus N$ and f_1 is an embedding onto an unknotted curve*, i.e., *there exists a diffeomorphism h: $E \to E$ diffeotopic to the identity on E such that $h \circ f_1(M)$ is a circle contained in some plane* (comp. [9], p. 159, Ex. 2).

Proof. For the 1-dimensional manifold $M \cup N$ there exists a plane P such that if π: $E \to P$ is an orthogonal projection then $\pi | M \cup N$ is an immersion with transversal selfintersections such that no three points of $M \cup N$ have the same image ([8], § 4 B)).

Choose a point x_0 in P and let x_1 be a point of M for which the function φ: $\pi(M) \to R$ defined by $\varphi(x) = |x - x_0|$ attains its maximum. There exists a line L contained in P orthogonal to the vector $x_1 - x_0$, intersecting $\pi(M)$ transversally at exactly two points, $\pi(a)$ and $\pi(b)$ near x_1, such that if the open arcs M_0, M_1 are two components of $M \setminus \{a, b\}$, then $\pi(M_0)$ and $\pi(M_1)$ are contained in different halfplanes determined by L on P, $\pi(M_0)$ contains all double points of $\pi(M)$ and $x_1 \in \pi(M_1)$.

Let $u_1, u_2, \ldots, u_k$ be all points of M mapped on the double points by $\pi | M$, arranged according to a chosen orientation of M. A line orthogonal to P may be identified with the oriented line R of reals. If $i < j$, $\pi(u_i) = \pi(u_j)$ and u_i is overcrossing, then we deform smoothly the inclusion f_0: $M \to E$ by immersions f_t: $M \to E$ in a small neighbourhood of u_i in such a way that $\pi \circ f_t = \pi \circ f_0$ and for the embedding f_1 u_i is undercrossing. According to the decomposition $E = P \times R$ we write $f_1 = (f_1', f_1'')$, where $f_1' = \pi \circ f_1$ and f_1'' is a real function. Let f_2'': $M \to R$ be a smooth function mapping the open arcs M_0 and M_1 diffeomorphically, preserving orientation on M_0 and reversing it on M_1. Put $f_2 = (f_1', f_2'')$: $M \to E$. The mappings $(1-t)f_1 + tf_2 = (f_1', (1-t)f_1'' + tf_2'')$ for $t \in I$ are immersions because f_1' is. They are embeddings because if $f_1'(u_i) = f_1'(u_j)$ and $i < j$ then $f_1''(u_i) < f_1''(u_j)$, $f_2''(u_i) < f_2''(u_j)$ and consequently

$$(1-t)f_1''(u_i) + tf_2''(u_i) < (1-t)f_1''(u_j) + tf_2''(u_j).$$

Let C be a circle in P with centre x_0 passing through x_1. Choose a diffeomorphism f_3' of M onto C mapping M_0 and M_1 into the halfplanes determined by L containing x_0 and x_1 respectively. Put $f_3 = (f_3', f_2'')$: $M \to E$. The mappings $(1-t)f_2 + tf_3 = ((1-t)f_1' + tf_3', f_2'')$ for $t \in I$ are embeddings of M because $f_2'' | M_0$

and $f_2''|M_1$ are embeddings, $f_1'(M_0)$ and $f_3'(M_0)$ lie in one halfplane and $f_1'(M_1)$ and $f_3'(M_1)$ in the other, f_1' and C are transversal to L and $f_2''(a) \neq f_2''(b)$.

We see that f_1 and f_3 are isotopic and by the isotopy extension theorem $f_1(M)$ is unknotted.

Proof of Theorem 1. If $m = 0$ then M is a point. For a pair of linked maps (f, g) the set $E \setminus f(M)$ has the homotopy type of an $(n = k-1)$-dimensional sphere S^n and the theorem follows from the Hopf homotopy classification theorem of maps of a manifold N into S^n, $n \geqslant 1$ ([7], § 7). In the sequel we assume $m \geqslant 1$.

The map $\mathscr{H} \to Z$ is surjective by Lemma 2. To prove its injectivity suppose that (f_0, g_0) and (f_1, g_1) are two pairs of linked maps and $l(f_0, g_0) = l(f_1, g_1)$. Since $k = m+n+1 \geqslant 2m+1$, we can assume that f_0 and f_1 are embeddings ([4], 2.2.13). We shall consider several cases.

a) $2 \leqslant m = n$. We may suppose that g_0 and g_1 are embeddings as well as f_0 and f_1. We can assume that $f_0(M)$, $g_0(N)$, $f_1(M)$, $g_1(N)$ are disjoint by composing f_0 and g_0 with a suitable translation of E. The set $D = E \setminus (f_1(M) \cup g_0(N))$ is open and connected because the dimensions of the manifolds $f_1(M)$ and $g_0(N)$ are less than $k-2$. By Lemma 2 applied to D, f_0 and g_1 there exist maps $f_2\colon M \to D$ and $g_2\colon N \to D$, homotopic to f_0 and g_1 respectively as maps into D, such that $l(f_2, g_2) = l(f_0, g_0)$. By Lemma 3 $(f_0, g_0) \simeq (f_2, g_0) \simeq (f_2, g_2) \simeq (f_1, g_2) \simeq (f_1, g_1)$.

b) $2 \leqslant m < n$. Since $k = m+n+1 \geqslant 2m+2$, by Lemma 4 there exists a diffeomorphism $h\colon E \to E$ isotopic to an identity such that $f_1 = h \circ f_0$. Then $(f_0, g_0) \simeq (f_1, h \circ g_0) \simeq (f_1, g_1)$ from Lemma 3.

c) $m = 1$, $n \geqslant 2$. M is diffeomorphic to a circle ([7], appendix). Since $k \geqslant 2m+2$, by Lemma 4 there exist diffeomorphisms $h_0, h_1\colon E \to E$ isotopic to the identity such that $h_0 \circ f_0$ and $h_1 \circ f_1$ map M onto an oriented circle C lying in some plane in E with degree 1 (if necessary we may compose h_0 or h_1 with the rotation of E preserving C and reversing the orientation of C). By Lemma 5 we choose an oriented manifold S diffeomorphic to the n-sphere, and a strong deformation retraction $r\colon \overline{E} \setminus C \to S$ such that the linking number of the inclusions $C \to E$ and $S \to E$ is 1. The maps $h_0 \circ g_0$ and $r \circ h_0 \circ g_0$ are homotopic as maps into $\overline{E} \setminus C$ and also as maps into $E \setminus C$, because $\dim I \times N = n+1 < k$ and the homotopy may be slightly deformed to avoid the point ∞ by the transversality extension theorem ([2], p. 72). Therefore $(f_0, g_0) \simeq (h_0 \circ f_0, h_0 \circ g_0) \simeq (h_0 \circ f_0, r \circ h_0 \circ g_0)$. Similarly

$$(f_1, g_1) \simeq (h_1 \circ f_1, h_1 \circ g_1) \simeq (h_1 \circ f_1, r \circ h_1 \circ g_1).$$

By Lemma 6 $\deg r \circ h_0 \circ g_0 = \deg r \circ h_1 \circ g_1$, and so by the Hopf classification theorem ([7], § 7) $(h_0 \circ f_0, r \circ h_0 \circ g_0) \simeq (h_1 \circ f_1, r \circ h_1 \circ g_1)$ and consequently $(f_0, g_0) \simeq (f_1, g_1)$.

d) $m = n = 1$. We may suppose that g_0 and g_1 are embeddings. The manifolds M and N are diffeomorphic to a circle. By Lemma 7 there are embeddings $\bar{f}_0\colon M \to E \setminus g_0(N)$ and $\bar{f}_1\colon M \to E \setminus g_1(N)$ such that $(f_0, g_0) \simeq (\bar{f}_0, g_0)$, $(f_1, g_1) \simeq (\bar{f}_1, g_1)$ and $\bar{f}_0(M)$ and $\bar{f}_1(M)$ are unknotted curves in the 3-dimensional space E. Choosing h_0, h_1 and r as in case c), we have

$$\begin{gathered}(f_0, g_0) \simeq (\bar{f}_0, g_0) \simeq (h_0 \circ \bar{f}_0, h_0 \circ g_0) \simeq (h_0 \circ \bar{f}_0, r \circ h_0 \circ g_0), \\ \wr\wr \\ (f_1, g_1) \simeq (\bar{f}_1, g_1) \simeq (h_1 \circ \bar{f}_1, h_1 \circ g_1) \simeq (h_1 \circ \bar{f}_1, r \circ h_1 \circ g_1).\end{gathered}$$

Thus the proof of Theorem 1 is complete.

In the case where M or N is nonorientable and all the remaining assumptions are the same, for a pair of linked maps $f\colon M \to E$ and $g\colon N \to E$ the linking number modulo 2 denoted by $l_2(f, g) \in Z_2$ can be defined as the winding number modulo 2 of the map $\Phi\colon M \times N \to E_0$ given by $\Phi(u, v) = g(v) - f(u)$.

THEOREM 8. *If M or N is nonorientable then the function $\mathscr{H} \to Z_2$ assigning to a homotopy class of a pair of linked maps (f, g) their linking number modulo 2 $l_2(f, g)$ is bijective.*

The proof, analogous to that of Theorem 1, can be based on the modified Lemma 3, in which M or N is nonorientable and the linking numbers $l(f, g_0) = l(f, g_1)$ are replaced by the linking numbers modulo 2 $l_2(f, g_0) = l_2(f, g_1)$. If N is nonorientable then the arc L in $I \times N$ should be chosen in such a way that, after setting an arc A in M joining the two points of intersection $G_0(z_0)$ and $G_0(z_1)$, the intersection numbers of oriented neighbourhoods of L and A in $I \times N$ and M respectively have opposite sings. If M is nonorientable then A should be chosen in such a way that after setting L the above condition is satisfied.

Remarks. Theorems 1 and 8 are valid also for continuous maps and continuous homotopies. This follows by the approximation of continuous maps by smooth ones.

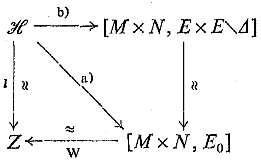
In the case $m = n$ it is possible to consider linked pairs of embeddings $f\colon M \to E$ and $g\colon N \to E$ instead of smooth maps, and isotopies instead of homotopies. From the results of Haefliger ([3], Th. 1′) it may be deduced that the analogues of Theorems 1 and 8 are valid in this case provided $2 \leqslant m = n$. This is not true if $m = n = 1$ because of the possibility of a different knotting of M and N.

COROLLARY 9. *Let M, N be closed connected smooth manifolds (orientable or not) of dimensions m and n respectively and E a linear space of dimension $k = m + n + 1 \geqslant 2$. Let Δ be the diagonal in $E \times E$.*

a) *The function $\mathscr{H} \to [M \times N, E_0]$ assigning to a homotopy class of a pair of linked maps (f, g) the homotopy class of the map Φ defined by $\Phi(u, v) = g(v) - f(u)$ is bijective.*

b) *The function $\mathscr{H} \to [M \times N, E \times E \setminus \Delta]$ assigning to a homotopy class of a pair of linked maps (f, g) the homotopy class of their Cartesian product $f \times g$ treated as a map into $E \times E \setminus \Delta$ is bijective.*

Proof. For oriented M and N this follows from the commutative diagram



in which the right vertical bijection is induced by the homotopy equivalence $h\colon E\times E\setminus\Delta \to E_0$ defined by $h(x,y) = y-x$, l is the bijection of Theorem 1 and W the bijection of the Hopf classification theorem. In the unoriented case we have a similar diagram.

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Products of normal spaces with Lašnev spaces

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Abstract. In this paper the equivalence of normality and countable paracompactness will be established for the product of a countably paracompact normal space with a Lašnev space. This extends Morita, Rudin and Starbird's theorem.

1. Introduction. All spaces considered in this paper are assumed to be Hausdorff and all maps continuous and onto. Closed images of metric spaces were characterized by Lašnev [7], and are called *Lašnev spaces*. Leĭbo [8, 9] applied Lašnev spaces to extend the well-known Katětov-Morita coincidence theorem and other properties of metric spaces in dimension theory (see [4]).

Let X be a countably paracompact normal space. It follows from the results of Morita [13] (for the proof see [5]) and Rudin and Starbird [18] that for a metric space Y the product space $X\times Y$ is normal if and only if $X\times Y$ is countably paracompact. However, no condition on Y other than metrizability seems to be known, under which the above equivalence is true. Indeed, in case Y is a paracompact M-space Rudin and Starbird [18] shows that the normality of $X\times Y$ implies the countable paracompactness of $X\times Y$, but the converse does not hold in general even if Y is compact. The aim of this paper is to show that the above is true in case of Y being Lašnev. We prove the following theorems:

THEOREM 1. *Let X be a normal space and Y a Lašnev space. If $X\times Y$ is countably paracompact, then $X\times Y$ is normal.*

THEOREM 2. *Let X be a space and Y a non-discrete Lašnev space. If $X\times Y$ is normal, then $X\times Y$ is countably paracompact.*

THEOREM 3. *Let X be a normal and countably paracompact space and Y a Lašnev space. Then $X\times Y$ is normal iff $X\times Y$ is countably paracompact.*

We note that in case Y is metrizable Theorems 1 and 2 are proved by Morita [13] and Rudin and Starbird [18] respectively. Also, our results will be applied to prove that if the product $X\times Y$ of a paracompact (resp. collectionwise normal) space X with a Lašnev space Y is normal then $X\times Y$ is paracompact (resp. collectionwise normal). This extends an analogous result for a metrizable space Y, implied by the results of Morita [12], Okuyama [17] and Rudin and Starbird [18].