On minimal generators of $\sigma$-fields

by

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Abstract. In this note it is shown that
(i) there is a $\sigma$-field without minimal generators;
(ii) the $\sigma$-field $\Sigma_{01}$ has a minimal generator.

Let $\mathcal{A}$ be a $\sigma$-field of subsets of the set $X$. A family $\mathcal{F}$ is called a generating family (or generator) for $\mathcal{A}$ if $\mathcal{A}$ is the least $\sigma$-field containing $\mathcal{F}$, and a minimal generator if no proper subfamily of $\mathcal{F}$ is a generator for $\mathcal{A}$. This concept was introduced by D. Basu and studied by B. V. Rao and K. P. S. Bhaskara Rao, who raised the question whether every $\sigma$-field of subsets has a minimal generating family ([6, p. 685], [1, p. 2]). The aim of the first part of the paper is to give a negative answer to this question. The second part answers the question of M. Talagrand ([7, p. 59]) and shows that the family of all ultrafilters on $\omega$ is a minimal generator for the $\sigma$-field generated by them on $\{0, 1\}^\mathbb{N}$.

Let $P(X)$ denote the family of all subsets of $X$ and, for a family $\mathcal{F} \subseteq P(X)$, let $\sigma(\mathcal{F})$ denote a $\sigma$-field generated by $\mathcal{F}$.

1. Question of B. V. Rao and K. P. S. Bhaskara Rao. $\omega_1$ stands for the set of all countable ordinals and also the least uncountable cardinal.

Proposition 1 is the clue to our further investigations; for the proof see [1, remark after Proposition 2]. Another proof, based on a theorem of Parovičenko–Rudin, is given at the end of the paper.

PROPOSITION 1. Any family of subsets of $\omega_1$ of cardinality $\omega_1$ is contained in a $\sigma$-field generated by a countable family.

LEMMA ([2, Lemma 7.4]). For each family of cardinality $\omega_1$ of cubes on $\omega_1$ ("cub" means "an unbounded subset of $\omega_1$ closed in order topology"), there is a cub almost contained in any cub of that family ("almost" mean "all but countably many points").

PROPOSITION 2. The $\sigma$-field $\mathcal{A} \subseteq P(\omega_1)$ of sets containing a cub or disjoint with one does not have a minimal generator.
Proof. Let $\mathcal{F}$ be a generator for $\mathcal{B}$, card $\mathcal{F}$ > $\omega_1$. We may assume that any $G \in \mathcal{F}$ contains a cub. Let $\mathcal{G}_0 \subseteq \mathcal{F}$ be such that card $\mathcal{G}_0$ = $\omega_1$ and all countable sets are in $\sigma(\mathcal{G}_0)$. Take $\mathcal{F} = \mathcal{G}_0 \cup \mathcal{G}_0$, card $\mathcal{F}$ = $\omega_1$. By the lemma there is a cub $C$ almost contained in any element of $\mathcal{F}$. By Proposition 1 there is a countable family $\mathcal{G}_1 \subseteq \mathcal{F}$ such that $\mathcal{F} = \mathcal{G}_0 \cup \mathcal{G}_1$. Thus we have $\mathcal{F} \subseteq \sigma(\mathcal{G}_0 \cup \mathcal{G}_1)$. This shows that $\sigma(\mathcal{F}) \subseteq \sigma(\mathcal{G}_0 \cup \mathcal{G}_1)$ and that $\mathcal{F}$ is not a minimal generator.

**Proposition 3. (CH) $P(\omega_1)$ does not have a minimal generator.**

Proof. Assume that $\mathcal{F}$ is a generator for $P(\omega_1)$. Let $\mathcal{G} \subseteq \mathcal{F}$ be such that card $\mathcal{G}$ = $\omega_1$. By Proposition 1 there is a countable family $\mathcal{A}$ such that $\sigma(\mathcal{A}) = \mathcal{F}$. Any member of $\mathcal{A}$ is generated by a countable subfamily of $\mathcal{G}$; hence there is a countable family $\mathcal{G}_0 \subseteq \mathcal{G}$ such that $\sigma(\mathcal{G}_0) \subseteq \sigma(\mathcal{A})$. Therefore $\mathcal{F}$ is contained in $\sigma(\mathcal{G}_0)$. This shows that $\sigma(\mathcal{G}_0 \cup \mathcal{A}) = \sigma(\mathcal{A})$ and finally $\mathcal{F}$ is not a minimal generator. Because of the arbitrariness of $\mathcal{G}$ the proof is complete.

Remark 1. In view of Silver's result ([1], p. 162) Martin's Axiom + CH implies that $P(\omega_1)$ has a countable generator and has a minimal one [6]. Therefore, in Proposition 2 some additional assumption, in place of CH are needed.

**Proposition 4. (CH) Let $\mathcal{B}$ denote the $\sigma$-field of Lebesgue measurable sets on the real line does not have a minimal generator.**

Proof. By CH there is a family $\{B_i: n < \omega_1\}$ of measure zero Borel sets such that any null measure set is contained in some $B_n$ (see, for example, [5, Theorem 19.6 and the last remark in Chap. 19]). Let $\mathcal{F}$ be a generator for $\mathcal{B}$. Clearly $\mathcal{F} > \omega_1$. For any element $G \in \mathcal{G}$ we fix a Borel set $B(G)$ satisfying $B(G) \subseteq G$, $G \setminus B(G)$ having measure zero. Then there is a Borel set $B$ such that

$$
\text{card}(B: G(B) = B) > \omega_1;
$$

then, for some $\alpha < \omega_1$, $\mathcal{G}_\alpha = \{B: G(B) = B, \mathcal{G}_\alpha \subseteq B\}$ is of cardinality at least $\omega_1$. Choose a subfamily $\mathcal{G} \subseteq \mathcal{G}_\alpha$ of cardinality $\omega_1$. By Proposition 1 there is a countable family of subsets of $\mathcal{B}$, hence in $\mathcal{G}$-generating on $\mathcal{B}$, a $\sigma$-field which contains $\mathcal{G}$. Let $\mathcal{F} = \mathcal{G} \cup \{B: G(B) \subseteq \mathcal{F} \}$ where $\mathcal{G} \subseteq \mathcal{F}$. Finally $\sigma(\mathcal{F}) = \sigma(\mathcal{G} \cup \mathcal{A})$ and $\mathcal{F}$ is not a minimal generator.

Remark 2. The same proof works for the $\sigma$-field of subsets of real line with the Baire property (consult [5]).

2. Question of M. Talagrand. Let $\omega$ denote the set of all finite ordinals and $\mathcal{F}$ the set of all ultrafilters on $\omega$. We identify $[0, 1]^\omega$ with $P(\omega)$. The ultrafilter $u$ in $\mathcal{F}$ is principal if there is a $B \in \mathcal{F}$ such that $A \subseteq B \in \mathcal{F}$ and $u \in \mathcal{F}$ is free if it is not a principal. All principal ultrafilters generate on $P(\omega)$ the $\sigma$-field of usual Borel subsets of $[0, 1]^\omega$. It is well known from Sierpiński [8] that free ultrafilters on $\omega$ are non-Borel. M. Talagrand posed the following question at a conference on analytic sets in 1979: For a set $\mathcal{A}$ of ultrafilters on $\omega$, let $\sigma(\mathcal{A})$ be a $\sigma$-field of subsets of $P(\omega)$ generated by the elements of $\mathcal{A}$. Does there exist a $u \in \mathcal{F}$ such that $u \in \sigma(\mathcal{F} \setminus \{u\})$?

We prove the following.

**Theorem.** $\mathcal{F}$ is a minimal generator for $\sigma(\mathcal{F} \setminus \{u\})$.

Proof. Assume, to the contrary, that there is a $u \in \mathcal{F}$, $u \in \sigma(\mathcal{F} \setminus \{u\})$. Then obviously there are $\{u; i \in \omega\} \subseteq \sigma(\mathcal{F} \setminus \{u\})$ such that $u \not\in \sigma(\mathcal{F} \setminus \{u; i \in \omega\})$. The ultrafilter $u$ is not principal (if it were, for two points, one with each coordinate equal to zero and the other with a coordinate equal to one on only one coordinate which gives $u$, $u$ would separate them while $\sigma(\mathcal{F} \setminus \{u; i \in \omega\})$ would not, which is impossible if $u \in \sigma(\mathcal{F} \setminus \{u; i \in \omega\})$.

Put $\gamma = \{i: u_i \in \text{principal}\}; a = a_\gamma$.

Claim. $u \not\in \sigma(\mathcal{F} \setminus \{u; i \in \omega\})$ (closure in $\mathcal{F}$ of $\{u; i \in \omega\}$). Otherwise there is an $A \in [a]^{\omega_1}$ (finite subset of $a$), $\mathcal{A} \not\in A$ and $u \not\in \sigma(\mathcal{F} \setminus \{u; i \in \omega\})$ for $i \in \omega$.

$$
P(A) \not\in \sigma(\mathcal{F} \setminus \{u; i \in \omega\})
$$

is a subset of $P(\omega)$, is in one atom of $\sigma(\mathcal{F} \setminus \{u; i \in \omega\})$, namely

$$
\bigwedge \{P(\omega) \setminus u_i; i \in \omega\}
$$

$P(A) \cap \sigma(\mathcal{F} \setminus \{u; i \in \omega\})$ is a sub-$\sigma$-field of Borel sets on $P(A)$ (it is principal for $i \in \gamma$, while $P(A) \cap u_i$ is a non-Borel. Finally $u \not\in \sigma(\mathcal{F} \setminus \{u; i \in \omega\})$.

This contradiction proves the claim.

Since $u \not\in \sigma(\mathcal{F} \setminus \{u; i \in \omega\})$, there is a partition of $\omega$, $\{P_n: i \in \omega\}$ such that

1. $\forall n P_n \not\in u$

2. $\forall n \exists P_n \in u$

(it is easy to obtain such a partition by induction).

This partition defines a function $f: \omega \rightarrow \omega$ by $f(i) = n$ if $i \in P_n$. For $\mathcal{A} \subseteq P(\omega)$ put

$$
f_\mathcal{A}(\mathcal{A}) = \{A \in P(\omega): f^{-1}(A) \in \mathcal{A}\}.
$$

$f_\mathcal{A}$ is a $\sigma$-homomorphism on $P(\omega)$ (i.e., preserves complementation and countable unions in $P(\omega)$). Hence

$$
f_\mathcal{A}(\mathcal{F} \setminus \{u; i \in \omega\}) = \sigma(\mathcal{F} \setminus \{u; i \in \omega\})
$$

By (i), $f_\mathcal{A}(u)$ is a principal ultrafilter on $\omega$; thus $\sigma(\mathcal{F} \setminus \{u; i \in \omega\})$ consists only of Borel subsets of $P(\omega)$. On the other hand, by (i), $f_\mathcal{A}(u)$ is free, hence non-Borel. This contradiction of (iii) proves the theorem.

3. Another proof of Proposition 1. Our proof of Proposition 1 is based on the following theorem of Parovicenko and W. Rudin:

**Theorem.** Let $B$ be a Boolean algebra of cardinality not greater than $\omega_1$. There is a one-to-one homomorphism from $B$ into $P(\omega)/\text{Fin}$. The proof can be found in [2] (Theorem 14.12).

**Proof of Proposition 1.** Let $\mathcal{F} \subseteq P(\omega)$, card $\mathcal{F} = \omega_1$. Let $\mathcal{A}$ be an algebra of cardinality $\omega_1$ containing $\mathcal{F}$ and all one-point subsets of $\omega_1$. Let $\varphi$ be a homo-


On the homotopy classification of pairs of linked maps of manifolds into a linear space

by

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Abstract. It is shown that the linking number of maps $f: M \to E$ and $g: N \to E$ of connected closed manifolds into a linear space with disjoint images gives the homotopy classification of such pairs of maps if $\dim M + \dim N + 1 = \dim E$.

Let $M$ and $N$ be two closed connected oriented smooth manifolds of dimensions $m$ and $n$ respectively. We shall suppose that $0 \leq m \leq n$. Let $E$ be a real oriented $k$-dimensional linear space with $k = m + n + 1 \geq 2$. Denote $E_0 = E \setminus \{0\}$. A pair $(f, g)$ of smooth maps $f: M \to E$ and $g: N \to E$ with disjoint images $f(M)$ and $g(N)$ will be called a pair of linked maps.

Two pairs of linked maps $(f_0, g_0)$ and $(f_1, g_1)$ are said to be homotopic if there are two smooth homotopies $f_i: M \to E$ and $g_i: N \to E$ with $f_i(M)$ and $g_i(N)$ disjoint for every $t \in [0, 1]$. We shall write $(f_0, g_0) \simeq (f_1, g_1)$ in this case. Denote by $\mathcal{H}$ the set of all homotopy classes of pairs of linked maps.

For a pair of linked maps $(f, g)$ their linking number $l(f, g)$ is defined to be the winding number around 0 (comp. [2], p. 144) $W(\phi)$ of the map $\phi: M \times N \to E_0$ of oriented manifolds defined by $\phi(u, v) = g(v) - f(u)$ (or the degree of the map $\phi|\phi: M \times N \to S^{m+n}$ if $E$ is Euclidean). It is known ([8], p. 104) that homotopic pairs of linked maps have the same linking number.

The main result of this paper is the following theorem.

**Theorem 1.** The function $\mathcal{H} \to \mathbb{Z}$ assigning to a homotopy class of a pair of linked maps $(f, g)$ their linking number $l(f, g)$ is bijective.

If $m = 0$ then $M$ is a point and the theorem is really the Hopf classification theorem (comp. [7], § 7). If $m = n = 1$ then $M$ and $N$ are diffeomorphic to circles; this result was obtained by J. Milnor in [6], p. 190 by means of tools developed there. We shall give also another, more direct proof of this case.

We shall need some lemmas.