On minimal generators of σ-fields

by

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Abstract. In this note it is shown that
(i) there is a σ-field without minimal generators;
(ii) the σ-field $\Sigma_{\beta\omega}$ has a minimal generator.

Let $\mathscr{B}$ be a σ-field of subsets of the set X. A family $\mathscr{G}$ is called a *generating family* (or *generator*) for $\mathscr{B}$ if $\mathscr{B}$ is the least σ-field containing $\mathscr{G}$, and a minimal generator if no proper subfamily of $\mathscr{G}$ is a generator for $\mathscr{B}$. This concept was introduced by D. Basu and studied by B. V. Rao and K. P. S. Bhaskara-Rao, who raised the question whether every σ-field of subsets has a minimal generating family ([6, p. 685], [1, p. 2]). The aim of the first part of the paper is to give a negative answer to this question. The second part answers the question of M. Talagrand ([7, p. 59]) and shows that the family of all ultrafilters on ω is a minimal generator for the σ-field generated by them on $\{0, 1\}^{\omega}$.

Let $P(X)$ denote the family of all subsets of X and, for a family $\mathscr{F} \subseteq P(X)$, let $\sigma(\mathscr{F})$ denote a σ-field generated by $\mathscr{F}$.

1. Question of B. V. Rao and K. P. S. Bhaskara-Rao. ω_1 stands for the set of all countable ordinals and also the least uncountable cardinal.

Proposition 1 is the clue to our further investigations; for the proof see [1, remark after Proposition 2]. Another proof, based on a theorem of Parovicenko–Rudin, is given at the end of the paper.

PROPOSITION 1. *Any family of subsets of ω_1 of cardinality ω_1 is contained in a σ-field generated by a countable family.*

LEMMA ([2, Lemma 7.4]. *For each family of cardinality ω_1 of cubs on ω_1 ("cub" means "an unbounded subset of ω_1 closed in order topology"), there is a cub almost contained in any cub of that family ("almost" means "all but countably many points").*

PROPOSITION 2. *The σ-field $\mathscr{B} \subseteq P(\omega_1)$ of sets containing a cub or disjoint with one does not have a minimal generator.*

Proof. Let $\mathcal{G}$ be a generator for $\mathcal{B}$. card$\mathcal{G}>\omega_1$. We may assume that any $G\in\mathcal{G}$ contains a cub. Let $\mathcal{G}_0\subseteq\mathcal{G}$ be such that card$\mathcal{G}_0=\omega_1$ and all countable sets are in $\sigma(\mathcal{G}_0)$. Take $\mathcal{F}\subseteq\mathcal{G}\setminus\mathcal{G}_0$, card$\mathcal{F}=\omega_1$. By the lemma there is a cub C almost contained in any element of $\mathcal{F}$. By Proposition 1 there is a countable family $\mathcal{G}_1\subseteq\mathcal{G}$ such that $\{F\setminus C\colon F\in\mathcal{F}\}\cup\{C\}\subseteq\sigma(\mathcal{G}_1)$. Thus we have $\mathcal{F}\subseteq\sigma(\mathcal{G}_0\cup\mathcal{G}_1)$. This shows that $\sigma(\mathcal{G})=\sigma(\mathcal{G}\setminus(\mathcal{F}\setminus\mathcal{G}_1))$ and that $\mathcal{G}$ is not a minimal generator.

PROPOSITION 3. (CH) *$P(\omega_1)$ does not have a minimal generator.*

Proof. Assume that $\mathcal{G}$ is a generator for $P(\omega_1)$. Let $\mathcal{F}\subseteq\mathcal{G}$ be such that card$(\mathcal{F})=\omega_1$. By Proposition 1 there is a countable family $\mathcal{A}$ such that $\sigma(\mathcal{A})\supseteq\mathcal{F}$. Any member of $\mathcal{A}$ is generated by a countable subfamily of $\mathcal{G}$; hence there is a countable family $\mathcal{G}_0\subseteq\mathcal{G}$ such that $\mathcal{A}$ and therefore $\mathcal{F}$ is contained in $\sigma(\mathcal{G}_0)$. This shows that $\sigma(\mathcal{G}\setminus(\mathcal{F}\setminus\mathcal{G}_0))=\sigma(\mathcal{G})$ and finally $\mathcal{G}$ is not a minimal generator. Because of the arbitrariness of $\mathcal{G}$ the proof is complete.

Remark. 1 In view of Silver's result ([4, p. 162]) Martin's Axiom + CH implies that $P(\omega_1)$ has a countable generator and has a minimal one [6]. Therefore, in Proposition 2 some additional assumption, in place of CH are needed.

PROPOSITION 4 (CH). *Let $\mathcal{L}$ denote the σ-field of Lebesgue measurable sets on the real line does not have a minimal generator.*

Proof. By CH there is a family $\{B_\alpha\colon \alpha<\omega_1\}$ of increasing zero measure Borel sets such that any zero measure set is contained in some B_α (use, for example, [5, Theorem 19.6 and the last remark in Ch. 19]). Let $\mathcal{G}$ be a generator for $\mathcal{L}$. Obviously card$\mathcal{G}>\omega_1$. For any element $G\in\mathcal{G}$ fix a Borel set $B(G)$ satisfying $B(G)\subseteq G$, $G\setminus B(G)$ having measure zero. There is a Borel set B such that

$$\operatorname{card}\{G\colon B(G)=B\}>\omega_1;$$

then, for some $\alpha<\omega_1$, $\mathcal{G}_\alpha=\{G\colon B(G)=B,\ G\setminus B\subseteq B_\alpha\}$ is of cardinality at least ω_1. Choose a subfamily $\mathcal{F}$ of G_α of cardinality ω_1. By Proposition 1 there is a countable family of subsets of B_α-hence in $\mathcal{L}$-generating on B_α a σ-field which contains $\{G\setminus B\colon G\in\mathcal{F}\}$. So there is a countable $\mathcal{G}_0\subseteq\mathcal{G}$ such that $\{G\setminus B\colon G\in\mathcal{F}\}\cup\cup\{B,B_\alpha\}\subseteq\sigma(\mathcal{G}_0)$. Finally $\sigma(\mathcal{G})=\sigma(\mathcal{G}\setminus(\mathcal{F}\setminus\mathcal{G}_0))$ and $\mathcal{G}$ is not a minimal generator.

Remark 2. The same proof works for the σ-field of subsets of real line with the Baire property (consult [5]).

2. Question of M. Talagrand. Let ω denote the set of all finite ordinals and $\beta\omega$ the set of all ultrafilters on ω. We identify $\{0,1\}^\omega$ with $P(\omega)$. The ultrafilter $u\in\beta\omega$ is principal if there is a $j\in\omega$ such that $A\in u$ iff $j\in A$ and $u\in\beta\omega$ is free if it is not a principal. All principal ultrafilters generate on $P(\omega)$ the σ-field of usual Borel subsets of $\{0,1\}^\omega$. It is well known from Sierpiński [8] that free ultrafilters on ω are non-Borel. M. Talagrand posed the following question at a conference on analytic sets in 1979: For a set $\mathcal{A}$ of ultrafilters on ω, let $\sigma(\mathcal{A})$ be a σ-field of subsets of $P(\omega)$ generated by the elements of $\mathcal{A}$. Does there exist a $u\in\beta\omega$ such that $u\in\sigma(\beta\omega\setminus\{u\})$?

We prove the following

THEOREM. *$\beta\omega$ is a minimal generator for $\sigma(\beta\omega)$.*

Proof. Assume, a contrario, that there is a $u\in\beta\omega$, $u\in\sigma(\beta\omega\setminus\{u\})$. Then obviously there are $\{u_i\colon i\in\omega\}\subseteq\beta\omega\setminus\{u\}$ such that $u\in\sigma(\{u_i\colon i\in\omega\})$. The ultrafilter u is not principal (if it were, for two points, one with each coordinate equal to zero and the other with a coordinate equal to one only on the coordinate which gives u, u would separate them while $\sigma(\{u_i\colon i\in\omega\})$ would not, which is impossible if $u\in\sigma(\{u_i\colon i\in\omega\})$.

Put $\gamma=\{i\colon u_i$ is principal$\}$; $\alpha=\omega\setminus\gamma$.

CLAIM. *$u\in\mathrm{cl}_{\beta\omega}\{u_i\colon i\in\alpha\}$ (closure in $\beta\omega$ of $\{u_i\colon i\in\alpha\}$). Otherwise there is an $A\in[\omega]^\omega$ (infinite subset of ω), $A\in u$ and $A\notin u_i$ for $i\in\alpha$.*

$P(A)$, as a subset of $P(\omega)$, is in one atom of $\sigma(\{u_i\colon i\in\alpha\})$, namely

$$\bigcap\{P(\omega)\setminus u_i\colon i\in\alpha\}.$$

$P(A)\cap\sigma(\{u_i\colon i\in\gamma\})$ is a sub-σ-field of Borel sets on $P(A)$ (u_i is principal for $i\in\gamma$), while

$P(A)\cap u$, as a free ultrafilter on A, is non-Borel. Finally $u\notin\sigma(\{u_i\colon i\in\omega\})$. This contradiction proves the claim.

Since $u\in\mathrm{cl}_{\beta\omega}\{u_i\colon i\in\omega\}$, there is a partition of ω, $\{P_n\colon n\in\omega\}\subseteq\{\omega\}^\omega$ such that

(i) $$\forall n\,P_n\notin u,$$

(ii) $$\forall j\,\exists n\,P_n\in u_j$$

(it is easy to obtain such a partition by induction).

This partition defines a function $f\colon\omega\to\omega$ by $f(i)=n$ if $i\in P_n$. For $\mathcal{A}\subseteq P(\omega)$ put

$$f_*(\mathcal{A})=\{A\in P(\omega)\colon f^{-1}(A)\in\mathcal{A}\}.$$

f_* is a σ-homomorphism on $P(\omega)$ (i.e., preserves complementation and countable unions in $P(P(\omega))$). Hence

(iii) $$f_*(u)\subseteq f_*(\sigma(\{u_i\colon i\in\omega\}))=\sigma(\{f(u_i)\colon i\in\omega\}).$$

By (ii), $f_*(u_i)$ is a principal ultrafilter on ω; thus $\sigma(\{f_*(u_i)\colon i\in\omega\})$ consists only of Borel subsets of $P(\omega)$. On the other hand, by (i), $f_*(u)$ is free, hence non-Borel. This contradiction of (iii) proves the theorem.

3. Another proof of Proposition 1. Our proof of Proposition 1 is based on the following theorem of Parovicenko and W. Rudin:

THEOREM. *Let B be a Boolean algebra of cardinality not greater than ω_1. There is a one-to-one homomorphism from B into $P(\omega)/\mathrm{Fin}$.*

The proof can be found in [2] (Theorem 14.12).

Proof of Proposition 1. Let $\mathcal{F}\subseteq P(\omega_1)$ card$(\mathcal{F})\leqslant\omega_1$. Let $\mathcal{A}$ be an algebra of cardinality ω_1 containing $\mathcal{F}$ and all one-point subsets of ω_1. Let φ be a homomorphism of $\mathscr{A}$ into $P(\omega)/\mathrm{Fin}$. For $A \in \mathscr{A}$ let $\chi_A \in P(\omega)$ be any element of the equivalent class of $\varphi(A)$. Since $\omega_1 \subseteq \mathscr{A}$, we have obtained a function $\chi\colon \omega_1 \to P(\omega)$.

For any $F \in \mathscr{F}$ consider the set $K_F = \{x \in P(\omega)\colon \operatorname{card}(\chi_F \setminus x) < \omega\}$. It is easy to see, K_F is σ-compact in $P(\omega)$ (with Cantor set topology). Hence K_F is a Borel set. Also, it is easy to see that $\chi^{-1}(K_F) = F$.

So $\mathscr{F} \subseteq \{\chi^{-1}(B) \mid B \subset P(\omega),\ B$ is a Borel set$\}$.

The above family is a σ-field which is a countable generating family. The proof of Proposition 1 is complete.

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On the homotopy classification of pairs of linked maps of manifolds into a linear space

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Abstract. It is shown that the linking number of maps $f\colon M \to E$ and $g\colon N \to E$ of connected closed manifolds into a linear space with disjoint images gives the homotopy classification of such pairs of maps if $\dim M + \dim N + 1 = \dim E$.

Let M and N be two closed connected oriented smooth manifolds of dimensions m and n respectively. We shall suppose that $0 \leqslant m \leqslant n$. Let E be a real oriented k-dimensional linear space with $k = m+n+1 \geqslant 2$. Denote $E_0 = E \setminus \{0\}$. A pair (f, g) of smooth maps $f\colon M \to E$ and $g\colon N \to E$ with disjoint images $f(M)$ and $g(N)$ will be called a *pair of linked maps*.

Two pairs of linked maps (f_0, g_0) and (f_1, g_1) are said to be *homotopic* if there are two smooth homotopies $f_t\colon M \to E$ and $g_t\colon N \to E$ with $f_t(M)$ and $g_t(N)$ disjoint for every $t \in I = [0, 1]$. We shall write $(f_0, g_0) \simeq (f_1, g_1)$ in this case. Denote by $\mathscr{H}$ the set of all homotopy classes of pairs of linked maps.

For a pair of linked maps (f, g) their linking number $l(f, g)$ is defined to be the winding number around 0 (comp. [2], p. 144) $W(\Phi)$ of the map $\Phi\colon M \times N \to E_0$ of oriented manifolds defined by $\Phi(u, v) = g(v) - f(u)$ (or the degree of the map $\Phi/|\Phi|\colon M \times N \to S^{m+n}$ if E is Euclidean). It is known ([8], p. 104) that homotopic pairs of linked maps have the same linking number.

The main result of this paper is the following

THEOREM 1. *The function $\mathscr{H} \to Z$ assigning to a homotopy class of a pair of linked maps (f, g) their linking number $l(f, g)$ is bijective.*

If $m = 0$ then M is a point and the theorem is really the Hopf classification theorem (comp. [7], § 7). If $m = n = 1$ then M and N are diffeomorphic to circles; this result was obtained by J. Milnor in [6], p. 190 by means of tools developed there. We shall give also another, more direct proof of this case.

We shall need some lemmas.