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Embeddings, amalgamation and elementary equivalence: the representation of compact logics

by

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Abstract. Any logic L generates the L-embedding relation $\underset{L}{\to}$ just as first-order logic $L_{\omega\omega}$ generates the elementary embeddability relation. By abstracting from L, given a transitive relation $\to$ between structures we may ask whether there is a (perhaps unique) logic L such that $\to = \underset{L}{\to}$. We prove that if L is compact and $\to = \underset{L}{\to}$, then L is uniquely determined by $\to$: thus in particular $L_{\omega\omega}$ is uniquely determined by the elementary embeddability relation. We give necessary and sufficient conditions for the existence and uniqueness of a logic L such that $\to = \underset{L}{\to}$, in case $\to$ has a strong form of amalgamation property, called AP^+. Upon restriction to countable structures of finite type there are exactly two nontrivial embedding relations with AP^+.

0. Introduction. Given a logic L, say in the sense of [12], one defines the L-elementary embedding relation $\underset{L}{\to}$ just as for first-order logic $L_{\omega\omega}$ one defines the elementary embeddability relation (1.2(b), (c)). By abstracting from L, we may consider an arbitrary transitive relation $\to$ between structures which is preserved under isomorphism, reduct and renaming (1.1). Any such relation $\to$ generates an equivalence relation $\sim \, = \, \to^*$ between structures, by saying that $\mathfrak{A}\sim\mathfrak{B}$ iff $\mathfrak{A}$ and $\mathfrak{B}$ are connected by a finite path of arrows between structures of the same type (1.4). Conversely, any equivalence relation $\sim$ as defined in (1.3) generates an embedding relation $\to \, = \, \sim^*$, by saying that $\mathfrak{A} \to \mathfrak{B}$ iff the type $\tau(\mathfrak{A})$ of $\mathfrak{A}$ is contained in $\tau(\mathfrak{B})$ and some expansion of $\mathfrak{B} \restriction \tau(\mathfrak{A})$ is $\sim$- equivalent to the diagram expansion $\mathfrak{A}_A$ of $\mathfrak{A}$ (1.4).

Given an abstract embedding relation $\to$ we consider the problem of *existence* and *uniqueness* of a logic L such that $\underset{L}{\to} = \to$. In Section 2 we give criteria for the *uniqueness* of L: Theorem 2.3 states that if $\underset{L}{\to} = \to$ and L is compact, then L is uniquely determined by $\to$: thus in particular $L_{\omega\omega}$ is uniquely determined by its own embeddability relation $\precsim$; the same holds for the logic with the cofinality ω quantifier (2.4). Theorem 2.6 establishes the following: if $\sim$ is an abstract equivalence relation, then $\sim \, = \, \equiv_L$ for at most one logic L, provided the pair $(\sim, \sim^*)$ has JEP: the latter means that whenever $\mathfrak{A}\sim\mathfrak{B}$ there is $\mathfrak{N}$ with $\mathfrak{A} \to \mathfrak{N} \leftarrow \mathfrak{B}$, where $\rightarrow = \sim^*$ is the embedding generated by $\sim$ (1.4). For the proof we use the main theorem of [14]. From Theorem 2.1 we then obtain as a byproduct the following criterion for the union of two compact logics L' and L'' to be compact (see [10, Problem 2, p. 218]): $L'' \cup L'$ is compact if $\equiv_{L'}$ and $\equiv_{L''}$ are comparable.

In Section 3 we give criteria for the *existence* of a logic L such that $\underset{L}{\rightarrow} = \rightarrow$; we mainly deal with embeddings satisfying a strong form of amalgamation, denoted by AP^+ (3.1): this turns out to be the counterpart of the Robinson property (3.2) of $\rightarrow^*$, as proved in (3.9). Using Theorems 3.9 and 2.3 together with the main result in [17], we give in Theorem 3.10 a necessary and sufficient condition for an embedding relation $\rightarrow$ to determine exactly one compact logic L with interpolation such that $\underset{L}{\rightarrow} = \rightarrow$. Any such $L \neq L_{\omega\omega}$ would provide a positive solution to the fourth problem in [7]. Whatever the answer to this problem, the results of Section 3 can be translated, with no essential modification, into the realm of enriched structures, e.g., topological, uniform, proximity, modal structures (see [6], [1] and [19], the latter for the general framework). In addition, the fact that equivalence (resp., embedding) relations in this paper are *not* required to imply elementary equivalence (resp., elementary embeddability) makes Theorem 3.9 also applicable to the study of *weaker* languages than $L_{\omega\omega}$. Work along these lines is in progress.

In Section 4 we relativize all our definitions to the class K of countable structures of finite type, and prove that there are exactly two nontrivial embedding relations with AP^+ on K; for the proof we use the main result of [13].

This paper could be also viewed as progress toward a problem raised in [20, § 2], namely determining necessary and sufficient conditions for the existence of a strongest logic with prescribed L-elementary equivalence (or, L-embedding) relation.

Abstract embeddings, AP^+ and the function $*$ were originally introduced by the author in the Summer of 1979, when 1.5, 3.3, 3.4, 4.1-4.4 were also proved. Theorems 3.9 and 3.10 were first proved in 1980. The present paper is based on a set of lectures delivered at the Mathematical Institute of the University of Florence during the academic year 1981/82. The author thanks the referee for his suggestions.

1. Embedding and equivalence relations between structures. See [12, § 1] for the necessary background in abstract model theory. As usual $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{D}$, $\mathfrak{M}$, $\mathfrak{N}$, and $\mathfrak{S}$ denote structures whose universes are respectively denoted by A, B, D, M, N and S. For any structure $\mathfrak{A}$, $\tau(\mathfrak{A})$ is the (usually many-sorted) type of $\mathfrak{A}$, and τ_A is the diagram type (also called, diagram language) of $\mathfrak{A}$, i.e., the type of the diagram expansion $\mathfrak{A}_A$ of $\mathfrak{A}$, which is obtained by adding a constant c_a for each element $a \in A$, and by interpreting each c_a by the element a, as in [2, p. 68]. We let $\mathrm{Str}(\tau)$ denote the class of all structures of type τ. A name-changer, or renaming, $\varrho: \tau \rightarrow \varrho(\tau)$, is an isomorphism between types. If $\mathfrak{A} \in \mathrm{Str}(\tau')$ and $\tau' \subseteq \tau$, then ϱ naturally transforms $\mathfrak{A}$ into the structure $\mathfrak{A}_\varrho \in \mathrm{Str}(\varrho(\tau'))$ obtained by stipulating that each symbol $R \in \tau'$ is interpreted in $\mathfrak{A}$ exactly as $\varrho(R)$ is interpreted in $\mathfrak{A}_\varrho$; compare with [4, p. 155].

1.1. Definition. A binary relation $\rightarrow$ on the class of all structures is called an (*abstract, model-theoretical*) *embedding relation* iff $\rightarrow$ satisfies the following conditions:

(1) *isomorphism*: $\mathfrak{A} \cong \mathfrak{B}$ implies $\mathfrak{A} \rightarrow \mathfrak{B}$;

(2) *renaming*: $\mathfrak{A} \rightarrow \mathfrak{B}$ implies $\mathfrak{A}^\varrho \rightarrow \mathfrak{B}^\varrho$;

(3) *reduct*: $\mathfrak{A} \rightarrow \mathfrak{B}$ implies $\mathfrak{A} \upharpoonright \tau \rightarrow \mathfrak{B} \upharpoonright \tau$;

(4) *type*: $\mathfrak{A} \rightarrow \mathfrak{B}$ implies $\tau(\mathfrak{A}) \subseteq \tau(\mathfrak{B})$; $\mathfrak{A} \rightarrow \mathfrak{B}$ iff $\mathfrak{A} \rightarrow \mathfrak{B} \upharpoonright \tau(\mathfrak{A})$;

(5) *transitivity*: $\mathfrak{A} \rightarrow \mathfrak{B}$ and $\mathfrak{B} \rightarrow \mathfrak{N}$ implies $\mathfrak{A} \rightarrow \mathfrak{N}$,

where $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{N}$ are arbitrary structures, τ is any type $\subseteq \tau(\mathfrak{A})$, and ϱ any name-changer whose domain contains $\tau(\mathfrak{B})$.

1.2. Examples. (a) Let $\mathfrak{A} \underset{\exp}{\rightarrow} \mathfrak{B}$ mean that $\tau(\mathfrak{A}) \subseteq \tau(\mathfrak{B})$ and $\mathfrak{A} \cong \mathfrak{B} \upharpoonright \tau(\mathfrak{A})$; then $\underset{\exp}{\rightarrow}$ is an embedding relation.

(b) Let $\mathfrak{A} \underset{L_{\omega\omega}}{\rightarrow} \mathfrak{B}$ mean that $\tau(\mathfrak{A}) \subseteq \tau(\mathfrak{B})$ and $\mathfrak{A} \precsim \mathfrak{B} \upharpoonright \tau(\mathfrak{A})$, i.e., $\mathfrak{A}$ is elementarily embeddable in $\mathfrak{B} \upharpoonright \tau(\mathfrak{A})$; equivalently [2, 3.1.3], $\mathfrak{A} \underset{L_{\omega\omega}}{\rightarrow} \mathfrak{B}$ iff $\mathfrak{B}^+ \equiv \mathfrak{A}_A$ for some expansion $\mathfrak{B}^+$ of $\mathfrak{B} \upharpoonright \tau(\mathfrak{A})$, where $\equiv$ is the elementary equivalence relation; then $\underset{L_{\omega\omega}}{\rightarrow}$ is an embedding relation.

(c) Generalizing example (b), let L be any logic, say in the sense of [12, §1]; let $\mathfrak{A} \underset{L}{\rightarrow} \mathfrak{B}$ mean that $\tau(\mathfrak{A}) \subseteq \tau(\mathfrak{B})$ and $\mathfrak{B}^+ \equiv_L \mathfrak{A}_A$ for some expansion $\mathfrak{B}^+$ of $\mathfrak{B} \upharpoonright \tau(\mathfrak{A})$, where $\equiv_L$ is the L-elementary equivalence relation (two structures are $\equiv_L$-equivalent iff they satisfy the same sentences of L); then $\underset{L}{\rightarrow}$ is an embedding relation, and is called the *L-embedding relation*. Compare with [14], [15].

1.3. Definition. An equivalence relation $\sim$ on the class of all structures is an (*abstract, model-theoretical*) *equivalence relation* iff $\sim$ satisfies the following conditions:

(1) *isomorphism*: $\mathfrak{M} \cong \mathfrak{N}$ implies $\mathfrak{M} \sim \mathfrak{N}$;

(2) *renaming*: $\mathfrak{M} \sim \mathfrak{N}$ implies $\mathfrak{M}^\varrho \sim \mathfrak{N}^\varrho$;

(3) *reduct*: $\mathfrak{M} \sim \mathfrak{N}$ implies $\mathfrak{M} \upharpoonright \tau \sim \mathfrak{N} \upharpoonright \tau$;

(4) *type*: $\mathfrak{M} \sim \mathfrak{N}$ implies $\tau(\mathfrak{M}) = \tau(\mathfrak{N})$,

where $\mathfrak{M}$ and $\mathfrak{N}$ are arbitrary structures, τ is any type $\subseteq \tau(\mathfrak{M})$, and ϱ is any name-changer whose domain contains $\tau(\mathfrak{M})$. Compare with [20].

1.4. Definition. Given any embedding $\rightarrow$ and equivalence relation $\sim$ we say that the pair $(\sim, \rightarrow)$ has the *Joint Embedding Property* (JEP) iff for every $\mathfrak{M}$ and $\mathfrak{N}$ with $\mathfrak{M} \sim \mathfrak{N}$ there is $\mathfrak{B}$ of the same type as $\mathfrak{M}$ and $\mathfrak{N}$ such that $\mathfrak{M} \rightarrow \mathfrak{B} \leftarrow \mathfrak{N}$. A logic L is said to have JEP iff the pair $(\equiv_L, \underset{L}{\rightarrow})$ has JEP; compare with [14], [15]. Every abstract equivalence relation $\sim$ *generates* an embedding relation $\rightarrow$ also denoted $\sim^*$, by saying that $\mathfrak{A} \rightarrow \mathfrak{B}$ iff $\tau(\mathfrak{A}) \subseteq \tau(\mathfrak{B})$ and $\mathfrak{A}_A \sim \mathfrak{B}^+$ for some expansion $\mathfrak{B}^+$ of $\mathfrak{B} \upharpoonright \tau(\mathfrak{A})$.

Conversely, every abstract embedding relation $\to$ *generates* an equivalence relation $\sim$, also denoted $\to^*$, by saying that $\mathfrak{M} \sim \mathfrak{N}$ iff $\mathfrak{M}$ and $\mathfrak{N}$ have the same type τ and are connected by a finite path of arrows over $\mathrm{Str}(\tau)$, i.e., there are $\mathfrak{B}_1, \ldots, \mathfrak{B}_n \in \mathrm{Str}(\tau)$ such that:

$$(+) \qquad \mathfrak{M} \underset{1}{\text{---}} \mathfrak{B}_1 \underset{2}{\text{---}} \mathfrak{B}_2 \underset{3}{\text{---}} \cdots \underset{n}{\text{---}} \mathfrak{B}_n = \mathfrak{N}, \text{ where } \underset{i}{\text{---}} \in \{\to, \leftarrow\}$$

for each $i = 1, \ldots, n$.

1.5. Proposition. *Let $\sim$ be an abstract equivalence relation; let $\to\, = \,\sim^*$ be the embedding relation generated by $\sim$, and let $\approx\, = \,\to^* = \,\sim^{**}$ be the equivalence relation generated by $\to$. Then $\approx$ is finer than $\sim$, i.e., for all $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{M} \approx \mathfrak{N}$ implies $\mathfrak{M} \sim \mathfrak{N}$.*

Proof. Assume $\mathfrak{M} \approx \mathfrak{N}$; then we can write

$$\mathfrak{M} \underset{1}{\text{---}} \mathfrak{B}_1 \underset{2}{\text{---}} \mathfrak{B}_2 \underset{3}{\text{---}} \cdots \underset{n}{\text{---}} \mathfrak{B}_n = \mathfrak{N},$$

as in (+) above. The fact that $\mathfrak{B}_i \underset{i+1}{\text{---}} \mathfrak{B}_{i+1}$ (say, $\underset{i+1}{\text{---}} = \to$) is to the effect that some expansion of the structure $\mathfrak{B}_{i+1}$ is $\sim$-equivalent to the diagram expansion of $\mathfrak{B}_i$ (or vice versa, if $\underset{i+1}{\text{---}} = \leftarrow$), hence, by 1.3(3), $\mathfrak{B}_i \sim \mathfrak{B}_{i+1}$. Similarly, $\mathfrak{M} \sim \mathfrak{B}_1$, hence we have $\mathfrak{M} \sim \mathfrak{B}_1 \sim \ldots \sim \mathfrak{B}_n = \mathfrak{N}$. ■

1.6. Proposition. *For any logic L we have:*

(i) $(\equiv_L)^* = \underset{L}{\to}$;

(ii) *if L has* JEP *then* $(\underset{L}{\to})^* = \equiv_L$.

Proof. (i) is immediate from the definition of $\underset{L}{\to}$ and of $\sim^*$; to prove (ii), note that for all $\mathfrak{M}, \mathfrak{N} \in \mathrm{Str}(\tau)$, if $\mathfrak{M} \equiv_L \mathfrak{N}$ then by JEP we have $\mathfrak{M} \underset{L}{\to} \mathfrak{B} \underset{L}{\leftarrow} \mathfrak{N}$ for some $\mathfrak{B} \in \mathrm{Str}(\tau)$; hence, by definition of $\to^*$, we also have that $\mathfrak{M}(\underset{L}{\to})^*\mathfrak{N}$. This shows that $\equiv_L$ is finer than $(\underset{L}{\to})^*$. By Proposition 1.5, $\equiv_L^{**}$, i.e., $(\underset{L}{\to})^*$, is finer than $\equiv_L$. Then we conclude that $(\underset{L}{\to})^* = \equiv_L$. ■

2. Unique representability, compact logics and JEP. Every logic L uniquely determines the L-embedding relation $\underset{L}{\to}$ as in 1.2(c). We now consider the problem whether an abstract embedding relation $\to$ can be identified with $\underset{L}{\to}$ for some (perhaps uniquely determined) logic L. We shall mainly consider logics $L = L(Q^i)_{i \in I}$ for I a set of relativizing Lindström quantifiers as defined in [12, § 1]. Following common usage, we let α and β denote ordinals, and $\varkappa$, λ, μ cardinals, so that we can equivalently write $L = L(Q^\alpha)_{\alpha<\varkappa}$.

Given logics L' and L'' we define $L = L' \cup L''$ to be the logic whose sentences ψ are precisely those of the form

$$(++) \qquad \psi = Q_1 k_1 \ldots Q_r k_r B(\varphi_1', \ldots, \varphi_n', \varphi_1'', \ldots, \varphi_m''),$$

where each k_t is a constant symbol, $Q_t \in \{\forall, \exists\}$ for every $t = 1, \ldots, r$, B is a boolean function, i.e., a composition of $\wedge$, $\vee$, and $\neg$'s, each φ_i' is a sentence of L', and each φ_j'' is a sentence of L''; compare with [17, 2.1]. Using high-school gymnastics one can verify that L is closed under negation, conjunction, existential quantification and relativization to boolean combinations of atomic sentences (see [17, § 5] for details). One now defines $\mathfrak{A} \models_L \psi$ in the natural sense; note that L need not (*prima facie*) have the form $L = L(Q^\alpha)_{\alpha<\varkappa}$. We shall use the following result of P. Lipparini (see [8] and references quoted therein):

2.1. Theorem. *Let $L' = L(Q^\alpha)_{\alpha<\varkappa}$, $L'' = L(Q^\beta)_{\beta<\lambda}$, $L = L' \cup L''$. If $\equiv_{L'}$ is finer than $\equiv_{L''}$* (i.e., $\forall \mathfrak{N}, \mathfrak{M}$, *$\mathfrak{N} \equiv_{L'} \mathfrak{M}$ implies $\mathfrak{N} \equiv_{L''} \mathfrak{M}$) then $\equiv_{L'} = \equiv_L$.*

Proof. Assume $\equiv_L \neq \equiv_{L'}$ so that $\equiv_L$ is strictly finer than $\equiv_{L'}$; then for some ψ in L and $\mathfrak{A}$, $\mathfrak{B}$ with $\mathfrak{A} \equiv_{L'} \mathfrak{B}$ we have $\mathfrak{B} \models_L \neg\psi$ and $\mathfrak{A} \models_L \psi$. As in (++) above, ψ has the form

$$\psi = Q_1 k_1 \ldots Q_r k_r B(\varphi_1', \ldots, \varphi_n', \varphi_1'', \ldots, \varphi_m'').$$

Let $\mathfrak{A}^+ = \langle \mathfrak{A}, R_1^{\mathfrak{A}^+}, \ldots, R_n^{\mathfrak{A}^+} \rangle$ and $\mathfrak{B}^+ = \langle \mathfrak{B}, R_1^{\mathfrak{B}^+}, \ldots, R_n^{\mathfrak{B}^+} \rangle$ be the (definitional) expansions of $\mathfrak{A}$ and $\mathfrak{B}$ respectively, defined by:

$$(1) \qquad \mathfrak{A}^+, \mathfrak{B}^+ \models_{L'} \bigwedge_{i=1}^{n} \forall k_1 \ldots \forall k_r (R_i k_1 \ldots k_r \leftrightarrow \varphi_i'),$$

where $R_1, \ldots, R_n$ are new r-ary relation symbols. Define now the sentence ε in L'' by $\varepsilon = Q_1 k_1 \ldots Q_r k_r B(R_1 \bar{k}, \ldots, R_n \bar{k}, \varphi_1'', \ldots, \varphi_m'')$, where $\bar{k}$ is short for $k_1, \ldots, k_r$. By (1) we have that $\mathfrak{A}^+ \models_{L''} \varepsilon$ and $\mathfrak{B}^+ \models_{L''} \neg\varepsilon$, hence $\mathfrak{A}^+ \not\equiv_{L''} \mathfrak{B}^+$. Since $\equiv_{L'}$ is finer than $\equiv_{L''}$ then $\mathfrak{A}^+ \not\equiv_{L'} \mathfrak{B}^+$, too. Then $\mathfrak{A}^+ \models_{L'} \chi$ and $\mathfrak{B}^+ \models_{L'} \neg\chi$ for some sentence χ in L'. Let ϑ be the sentence in L' defined by $\vartheta = \chi(\varphi_1'/R_1\bar{k}, \ldots, \varphi_n'/R_n\bar{k})$, i.e., ϑ is obtained from χ by replacing each (material) occurrence of $R_i\bar{k}$ in χ by φ_i': this is perfectly legitimate in a logic with quantifiers such as L', where sentences are finite strings of symbols obtained from the atomic sentences using the same formation rules as in $L_{\omega\omega}$ together with Q^α-quantification, for each Q^α in L'. By (1) we have $\mathfrak{A}^+ \models_{L'} \vartheta$ and $\mathfrak{B}^+ \models_{L'} \neg\vartheta$, hence $\mathfrak{A} \models_{L'} \vartheta$ and $\mathfrak{B} \models_{L'} \neg\vartheta$ (since the R_i's are no more present in ϑ), which contradicts the assumption that $\mathfrak{A} \equiv_{L'} \mathfrak{B}$. ■

2.2. Lemma. *Let $\to$ be an embedding relation such that $\to\, = \underset{L'}{\to}$ for some compact logic $L' = L(Q^\alpha)_{\alpha<\varkappa}$. Let $L'' = L(Q^\beta)_{\beta<\lambda}$ be such that $\mathfrak{A} \underset{L'}{\to} \mathfrak{B}$ implies $\mathfrak{A} \underset{L''}{\to} \mathfrak{B}$ (for all $\mathfrak{A}$, $\mathfrak{B}$). Then L'' is compact, too.*

Proof. Since L' is compact, then L' has JEP, by the same argument used for $L_{\omega\omega}$ in [2, 3.1.4]. This means that whenever $\mathfrak{A} \equiv_{L'} \mathfrak{B}$ there is $\mathfrak{S}$ such that $\mathfrak{A} \underset{L'}{\to} \mathfrak{S} \underset{L'}{\leftarrow} \mathfrak{B}$; by hypothesis, $\mathfrak{A} \underset{L''}{\to} \mathfrak{S} \underset{L''}{\leftarrow} \mathfrak{B}$, whence $\mathfrak{A} \equiv_{L''} \mathfrak{B}$; this shows that $\equiv_{L'}$ is finer than $\equiv_{L''}$. By Theorem 2.1 we have $\equiv_L = \equiv_{L'}$, where $L = L' \cup L''$ (see (++) above).

Claim. *L is compact.*

Proof of the claim. Deny (absurdum hypothesis): then some cardinal $\mu \geqslant \omega$ is cofinally (or, weakly) characterizable in L; in other words, there is an expansion $\mathfrak{M}$ of the structure $\langle \mu, <, c_\gamma \rangle_{\gamma<\mu}$ such that whenever $\mathfrak{N} \equiv_L \mathfrak{M}$ the set $\{c_\gamma^{\mathfrak{N}}\}_{\gamma<\mu}$ is unbounded in the order $<^{\mathfrak{N}}$, i.e., there is no element c in $\mathfrak{N}$ such that $\langle \mathfrak{N}, c\rangle \models c_\gamma < c$ for all $\gamma<\mu$: for a proof see [10, § 6], [11, § 2]; the fact that L is closed under relativizations to boolean combinations of atomic sentences is enough for μ to exist; for details see [9, 1.3, proof of the claim]. Since $\equiv_L = \equiv_{L'}$ then we can as well say that whenever $\mathfrak{M} \equiv_{L'} \mathfrak{N}$ the set $\{c_\gamma^{\mathfrak{N}}\}_{\gamma<\mu}$ is unbounded in $<^{\mathfrak{N}}$. But this implies that L' is not compact, a contradiction. Thus L is compact, which proves our claim. Now, L'' is a sublogic of L, so L'' is compact, too, which proves the lemma. ■

The following is a "unique representability" result: recall that two logics L' and L'' are equivalent iff they have the same sentences, up to logical equivalence; in this case L' and L'' are often identified.

2.3. THEOREM. *Let $\to$ be an embedding relation such that $\to = \underset{L'}{\to}$ for some compact logic $L' = L(Q^\alpha)_{\alpha<\varkappa}$. Assume $\to = \underset{L''}{\to}$ for some $L'' = L(Q^\beta)_{\beta<\lambda}$. Then L'' is equivalent to L'.*

Proof. Let $\approx = \to^*$. By Lemma 2.2, L'' is compact, hence L'' has JEP. By Proposition 1.6(ii) we have $(\underset{L''}{\to})^* = \equiv_{L''}$, i.e., $\approx = \equiv_{L''}$. Similarly L' has JEP, and $\approx = \equiv_{L'} = \equiv_{L''}$. Letting $L = L' \cup L''$, by Theorem 2.1 we have $\equiv_L = \equiv_{L'} = \equiv_{L''}$, and arguing as in the proof of the claim of Lemma 2.2 we have that L is compact. A familiar finite open-cover argument (e.g., an obvious generalization of [5, 2.4]) yields the equivalence of L and L', and the equivalence of L and L'', hence the equivalence of L' and L''. ■

We may apply Theorem 2.3 to all compact logics found in the literature (see, e.g., [22], [21], [11]) — including $L_{\omega\omega}$. To fix ideas we shall only deal with $L_{\omega\omega}$ and with $L(Q^{\mathrm{cf}\omega})$, the logic with the cofinality ω quantifier introduced in [22] and proved therein to be compact. Recall from 1.2(b) the definition of $\underset{L_{\omega\omega}}{\to}$ in terms of $\gtrsim$.

2.4. COROLLARY. (i) *First-order logic is uniquely determined by the elementary embeddability relation $\gtrsim$;*

(ii) *Let $L' = L(Q^{\mathrm{cf}\omega})$. Assume $L'' = L(Q^\beta)_{\beta<\lambda}$ is a logic with $\underset{L'}{\to} = \underset{L''}{\to}$. Then L'' is equivalent to L'.* ■

2.5. Remark. Corollary 2.4(i) can also be proved without using Theorems 2.1 and 2.3 but using instead the (JEP and) Robinson property of $\equiv$, together with Propositions 1.5 and 1.6 and [18,1.1].

We give a final criterion for an abstract equivalence relation $\sim$ to have *at most one* logic L such that $\equiv_L = \sim$. Following [14] we let ♮ (read: *natural*) denote the following set-theoretical hypothesis:

Every uniform ultrafilter on every regular $\mu \geqslant \omega$ is λ-descendingly incomplete whenever $\omega \leqslant \lambda \leqslant \mu$.

See [2, p. 210] for this terminology. In [3, p. 91] it is proved that ♮ is weaker than $\neg L^\mu$ (there is no inner model with an uncountable measurable cardinal), hence ♮ is weaker than $\neg O^\#$ and $V = L$.

2.6. THEOREM. (♮) *Let $\sim$ be an abstract equivalence relation such that $(\sim, \sim^*)$ has* JEP. *Then there is at most one logic $L' = L(Q^\alpha)_{\alpha<\varkappa}$ such that $\equiv_{L'} = \sim$. Further, if any such L' exists, then L' is compact.*

Proof. Assume both L' and $L'' = L(Q^\beta)_{\beta<\lambda}$ are such that $\equiv_{L'} = \equiv_{L''} = \sim$. Let $L = L' \cup L''$; using Theorem 2.1 we have $\equiv_L = \equiv_{L'} = \equiv_{L''} = \sim$. Now L has the following properties: (i) for every type τ the collection of sentences of L of type τ forms a set; (ii) L has the atomic sentences and is closed under conjunction, negation, existential quantification, and relativization to boolean combinations of atomic sentences, and (iii) L has JEP or, stated otherwise, the pair $(\equiv_L, \underset{L}{\to})$ has JEP: indeed $(\sim, \sim^*)$ has JEP by hypothesis, $\equiv_L = \sim$, and $\underset{L}{\to} = (\equiv_L)^* = \sim^*$ by Proposition 1.6(i). Then we can apply the main theorem of [14] to the effect that L is compact (here assumption ♮ is apparently used, but see also [11]). By a familiar finite open-cover argument (as in [5, 2.4]), from the compactness of L and the fact that $\equiv_L = \equiv_{L'} = \equiv_{L''}$ one infers that L is equivalent to each of its sublogics L' and L''. ■

3. Criteria for the existence of representations. Amalgamation. The results of Section 2 provide a number of criteria for an embedding relation to be representable as L-embedding for *at most one* logic L. Criteria for the existence of *exactly one* such L will be given in this section by strengthening the compactness (or JEP) conditions as follows:

3.1. DEFINITION. An embedding relation $\to$ on the class of all structures has the *Strong Amalgamation Property*, AP^+ (resp., the *Amalgamation Property*, AP) iff for every τ, τ', τ'', with $\tau' \cap \tau'' = \tau$ (resp., with $\tau' = \tau'' = \tau$) and structures $\mathfrak{A}, \mathfrak{A}', \mathfrak{A}''$ of type τ, τ', τ'' respectively, if $\mathfrak{A}' \leftarrow \mathfrak{A}$ and $\mathfrak{A} \to \mathfrak{A}''$ then $\mathfrak{A}' \to \mathfrak{D}$ and $\mathfrak{D} \leftarrow \mathfrak{A}''$ for some $\mathfrak{D}$ of type $\tau' \cup \tau''$. Compare with [15].

3.2. DEFINITION. An equivalence relation $\sim$ on the class of all structures has the *Robinson property* iff for every $\mathfrak{A}' \in \mathrm{Str}(\tau')$, $\mathfrak{A}'' \in \mathrm{Str}(\tau'')$, if $\mathfrak{A}' \restriction \tau' \cap \tau'' \sim \mathfrak{A}'' \restriction \tau' \cap \tau''$, then there is $\mathfrak{D} \in \mathrm{Str}(\tau' \cup \tau'')$ such that $\mathfrak{D} \restriction \tau' \sim \mathfrak{A}'$ and $\mathfrak{D} \restriction \tau'' \sim \mathfrak{A}''$.

Equivalence relations and logics with the Robinson property were extensively studied by the author in [13]–[19]. It is easy to see that when $\sim = \equiv_L$ then $\sim$ has the Robinson property iff L satisfies the Robinson consistency theorem. Referring to examples (a)–(c) in 1.2, observe that $\underset{\exp}{\to}$ has AP^+; to see that $\underset{L_{\omega\omega}}{\to}$ has AP^+ use the Robinson consistency theorem for first-order logic, together with the identity $\equiv^* = \underset{L_{\omega\omega}}{\to}$ (in the light of 1.6(i), since $L_{\omega\omega}$ has JEP [2, 3.1.4]); then argue as in Theorem 3.5(i) below. More generally, for any logic L generated by a set of quantifiers, we have that $\underset{L}{\to}$ has AP^+ if $\equiv_L$ has the Robinson property (again by

Theorem 3.5(i)); the latter holds iff L satisfies compactness + interpolation [14], [15], [11], [9, 1.4], iff L provides a positive solution of the fourth problem in [7].

We now investigate the relationship between AP^+ and Robinson property. Recall from 1.4 the definition of JEP and of the map $*$.

3.3. THEOREM. *Let $\to$ be an embedding relation with AP^+; let $\sim\ =\ \to^*$ be the equivalence relation generated by $\to$. Then $\sim$ has the Robinson property.*

For the proof we prepare the following lemma, dealing with AP:

3.4. LEMMA. *If $\to$ has AP and $\sim\ =\ \to^*$, then $(\sim, \to)$ has JEP.*

Proof of the lemma. Assume $\mathfrak{A}\sim\mathfrak{B}$, so that the following holds for some path of length n:

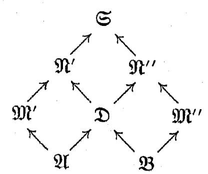
$$\mathfrak{A} \underset{1}{—} \mathfrak{M}_1 \underset{2}{—} \cdots \underset{n}{—} \mathfrak{M}_n = \mathfrak{B}\,; \qquad \mathfrak{A}, \mathfrak{B}, \mathfrak{M}_1, \ldots, \mathfrak{M}_n \in \mathrm{Str}(\tau)\,.$$

We now prove JEP arguing by induction on $n = 1, 2, \ldots$:

Basis. Trivial, because $\mathfrak{A}\to\mathfrak{B}$ implies $\mathfrak{A}\to\mathfrak{B}\leftarrow\mathfrak{B}$ by 1.1(1).

Induction Step. We can write $\mathfrak{A}\sim\mathfrak{M}_{n-1} \underset{n}{—} \mathfrak{B}$ by definition of $\sim\ =\ \to^*$; by induction hypothesis there exists $\mathfrak{S}\in\mathrm{Str}(\tau)$ such that $\mathfrak{A}\to\mathfrak{S}\leftarrow\mathfrak{M}_{n-1} \underset{n}{—} \mathfrak{B}$. Now, in case $\underset{n}{—}$ is $\leftarrow$, then $\mathfrak{S}$ jointly embeds $\mathfrak{A}$ and $\mathfrak{B}$ by transitivity, 1.1(5); in case $\underset{n}{—}$ is $\to$, then by AP there is $\mathfrak{D}\in\mathrm{Str}(\tau)$ such that $\mathfrak{S}\to\mathfrak{D}\leftarrow\mathfrak{B}$ and, again by transitivity, we see that $\mathfrak{D}$ jointly embeds $\mathfrak{A}$ and $\mathfrak{B}$. ■

Proof of Theorem 3.3. Let $\tau = \tau'\cap\tau''$, $\mathfrak{M}'$, $\mathfrak{M}''$ be structures of type τ' and τ'' respectively, with $\mathfrak{A} = \mathfrak{M}'\restriction\tau\sim\mathfrak{M}''\restriction\tau = \mathfrak{B}$. We have to find $\mathfrak{S}$ of type $\tau'\cup\tau''$ such that $\mathfrak{S}\restriction\tau'\sim\mathfrak{M}'$ and $\mathfrak{S}\restriction\tau''\sim\mathfrak{M}''$. By Lemma 3.4 we have $\mathfrak{A}\to\mathfrak{D}\leftarrow\mathfrak{B}$ for some $\mathfrak{D}\in\mathrm{Str}(\tau)$. Thus by 1.1(1), (4) we have $\mathfrak{M}'\leftarrow\mathfrak{A}\to\mathfrak{D}\leftarrow\mathfrak{B}\to\mathfrak{M}''$. By AP^+ (twice) there are $\mathfrak{N}'\in\mathrm{Str}(\tau')$ and $\mathfrak{N}''\in\mathrm{Str}(\tau'')$ such that $\mathfrak{M}'\to\mathfrak{N}'\leftarrow\mathfrak{D}\to\mathfrak{N}''\leftarrow\mathfrak{M}''$. Again by AP^+ there is a structure $\mathfrak{S}\in\mathrm{Str}(\tau'\cup\tau'')$ such that $\mathfrak{N}'\to\mathfrak{S}\leftarrow\mathfrak{N}''$. Graphically:



By 1.1(3), (5) we see that $\mathfrak{M}'\to\mathfrak{S}\restriction\tau'$ and $\mathfrak{M}''\to\mathfrak{S}\restriction\tau''$; hence, by definition of $\sim\ =\ \to^*$, $\mathfrak{M}'\sim\mathfrak{S}\restriction\tau'$ and $\mathfrak{M}''\sim\mathfrak{S}\restriction\tau''$. ■

3.5. THEOREM. *Let $\sim$ be an equivalence relation having the Robinson property; let $\to\ =\ \sim^*$ be the embedding relation generated by $\sim$. Then*

(i) *$\to$ has AP^+, and*

(ii) *the pair $(\sim, \to)$ has* JEP.

Proof. (i) Let $\mathfrak{A}\leftarrow\mathfrak{N}\to\mathfrak{B}$, with $\tau(\mathfrak{A})\cap\tau(\mathfrak{B}) = \tau(\mathfrak{N})$. By definition of $\to\ =\ \sim^*$ we have $\mathfrak{A}\restriction\tau(\mathfrak{N})\sim\mathfrak{N}\sim\mathfrak{B}\restriction\tau(\mathfrak{N})$, if one argues as in the proof of 1.5. Let $\mathfrak{A}_A$ and $\mathfrak{B}_B$ be the diagram expansions of $\mathfrak{A}$ and $\mathfrak{B}$ respectively, where different constant symbols are used in τ_A and τ_B. Again, $\mathfrak{A}_A\restriction\tau(\mathfrak{N})\sim\mathfrak{N}\sim\mathfrak{B}_B\restriction\tau(\mathfrak{N})$; by the Robinson property of $\sim$ there is $\mathfrak{M}$ such that $\mathfrak{M}\restriction\tau_A\sim\mathfrak{A}_A$ and $\mathfrak{M}\restriction\tau_B\sim\mathfrak{B}_B$. By definition of $\to$ we have $\mathfrak{A}\to\mathfrak{M}\leftarrow\mathfrak{B}$, hence $\to$ has AP^+.

(ii) If $\mathfrak{A}\sim\mathfrak{B}$, let $\mathfrak{A}_A$ and $\mathfrak{B}_B$ be the diagram expansions of $\mathfrak{A}$ and $\mathfrak{B}$ respectively, with $\tau_A\cap\tau_B = \tau(\mathfrak{A})$. By the Robinson property of $\sim$ there is $\mathfrak{M}$ such that $\mathfrak{M}\restriction\tau_A\sim\mathfrak{A}_A$ and $\mathfrak{M}\restriction\tau_B\sim\mathfrak{B}_B$; thus by definition of $\to$ we have $\mathfrak{A}\to\mathfrak{M}\restriction\tau(\mathfrak{A})\leftarrow\mathfrak{B}$, whence $(\sim, \to)$ has JEP as required. ■

3.6. PROPOSITION. *Let $\sim$, $\sim_1$, $\sim_2$ be equivalence relations with the Robinson property. Then we have:*

(i) $\sim^{**} = \sim$;

(ii) *if $\sim_1\neq\sim_2$ then $\sim_1^*\neq\sim_2^*$.*

Proof. (i) By 1.5, $\sim^{**}$ is finer than $\sim$; on the other hand, if $\mathfrak{A}\sim\mathfrak{B}$ then, letting $\to\ =\ \sim^*$, by Theorem 3.5(ii) there is $\mathfrak{M}$ such that $\mathfrak{A}\to\mathfrak{M}\leftarrow\mathfrak{B}$, hence $\mathfrak{A}\sim^{**}\mathfrak{B}$, which shows that $\sim$ is finer than $\sim^{**}$; therefore $\sim\ =\ \sim^{**}$, as required.

(ii) Without loss of generality assume $\mathfrak{A}\sim_1\mathfrak{B}$ and not-$\mathfrak{A}\sim_2\mathfrak{B}$. Let $\to_1\ =\ \sim_1^*$, $\to_2\ =\ \sim_2^*$. By Theorem 3.5(ii) there is $\mathfrak{M}$ with $\mathfrak{A}\to_1\mathfrak{M}_1\leftarrow\mathfrak{B}$. If $\to_1\ =\ \to_2$ (absurdum hypothesis), then $\mathfrak{A}\to_2\mathfrak{M}_2\leftarrow\mathfrak{B}$ holds, hence we have $\mathfrak{A}\sim_2\mathfrak{B}$ by arguing as in the proof of 1.5; this is a contradiction. ■

Remark. In the light of 3.6(ii) one might ask whether different embedding relations with AP^+ generate different equivalence relations: to see that this need not be the case, consider $\underset{L_{\omega\omega}}{\to}$ (1.2(b)) and the relation $\to^\#$ given by $\mathfrak{A}\to^\#\mathfrak{B}$ iff $\tau(\mathfrak{A})\subseteq\tau(\mathfrak{B})$ and $\mathfrak{A}$ is completely embeddable into $\mathfrak{B}\restriction\tau(\mathfrak{A})$, as in [2, 6.4]. Both embedding relations have AP^+, as a corollary of Robinson's consistency theorem. Further, both $\underset{L_{\omega\omega}}{\to}$ and $\to^\#$ generate elementary equivalence, $\equiv$ (argue as in 1.6 in the light of [2, 3.1.4 and 6.4.23]). But they are different.

We are thus led to say that two embedding relations $\to_1$ and $\to_2$ are *equivalent* iff they generate the same equivalence relation. If AP holds, hence a fortiori if AP^+ holds, then we have the following simple characterization of equivalence:

3.7. PROPOSITION. *Let $\to_1$ and $\to_2$ both have* AP. *Then the following two statements are equivalent:*

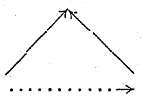
(i) *$\to_1$ and $\to_2$ are equivalent embedding relations;*

(ii) *for any $\mathfrak{A}, \mathfrak{B}\in\mathrm{Str}(\tau)$ with $\mathfrak{A}\to_1\mathfrak{B}$ there is $\mathfrak{M}\in\mathrm{Str}(\tau)$ such that $\mathfrak{A}\to_2\mathfrak{M}_2\leftarrow\mathfrak{B}$, and, symmetrically, for any $\mathfrak{D}, \mathfrak{N}\in\mathrm{Str}(\tau)$ with $\mathfrak{D}\to_2\mathfrak{N}$ there is $\mathfrak{S}\in\mathrm{Str}(\tau)$ such that $\mathfrak{D}\to_1\mathfrak{S}_1\leftarrow\mathfrak{N}$.*

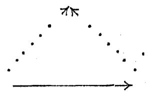
Proof. (i) ⇒ (ii) Assume $\to_1^* = \to_2^* = \sim$; if $\mathfrak{A}\to_1\mathfrak{B}$ then by definition of $\sim$, $\mathfrak{A}\sim\mathfrak{B}$; by Lemma 3.4 $(\sim, \to_2)$ has JEP, hence $\mathfrak{A}\to_2\mathfrak{M}_2\leftarrow\mathfrak{B}$ for some $\mathfrak{M}$. By symmetry we also have the other half of the proof.

(ii) ⇒ (i) Let $\sim_1 = \to_1^*$ and $\sim_2 = \to_2^*$. If $\mathfrak{A}\sim_1\mathfrak{B}$ then there is $\mathfrak{D}$ such that $\mathfrak{A}\to_1\mathfrak{D}_1\leftarrow\mathfrak{B}$, since $(\sim_1, \to_1)$ has JEP by Lemma 3.4. By assumption there are $\mathfrak{M}$ and $\mathfrak{N}$ such that $\mathfrak{A}\to_2\mathfrak{M}_2\leftarrow\mathfrak{D}\to_2\mathfrak{N}_2\leftarrow\mathfrak{B}$; this shows that $\mathfrak{A}\sim_2\mathfrak{B}$, by definition of $\sim_2$. Therefore $\sim_1$ is finer than $\sim_2$. By symmetry, $\sim_2$ is finer than $\sim_1$, hence $\to_1^* = \to_2^*$. ■

Remark. Thus two embedding relations $\cdots\to$ and $\longrightarrow$ with AP generate the same equivalence relation iff each arrow of the former can be replaced by two arrows of the latter, and vice versa, as follows:



and



One might ask if each equivalence class of embedding relations with AP$^+$ has one canonical representative which is well-behaved with respect to the map $*$; let us first agree to say that an embedding relation $\to$ is *involutive* if $(\to^*)^* = \to$. Then we have:

3.8. PROPOSITION. *Each equivalence class of embedding relations with* AP$^+$ *has exactly one involutive element.*

Proof. Let Z be any such class, and $\to$ an arbitrary element of Z. We shall prove that $\to^{**}$ is the unique involutive element of Z. To this purpose, let $\sim\, = \,\to^*$; since $\to$ has AP$^+$ then $\sim$ has the Robinson property by Theorem 3.3; thus $\sim\, = \,\sim^{**}$ by 3.6(i); hence $(\sim^*)^{**} = (\sim^{**})^* = \sim^*$, which shows that $\to^{**}$ is involutive. Since $\sim$ has the Robinson property, then $\sim^*$ has AP$^+$ by Theorem 3.5(i); the embeddings $\sim^*$ and $\to$ are equivalent since $(\sim^*)^* = \sim\, = \,\to^*$. By definition of Z we have that $\to^{**} = \sim^* \in Z$. To see that $\sim^*$ is the unique involutive embedding in Z, recall that every embedding $\dot\to$ in Z is equivalent to $\to$, hence $\dot\to^* = \sim$; therefore, if $\dot\to$ is involutive, then $\dot\to\, = \,\dot\to^{**} = \sim^* = \,\to^{**}$. ■

We now show that there is a natural correspondence between the family **R** of equivalence relations with the Robinson property and the family **A** of involutive embeddings with AP$^+$. We remark here that the isomorphism and renaming conditions for embeddings (1.1) and equivalence relations (1.3) could be considerably weakened without affecting the validity of this correspondence.

3.9. THEOREM. *The function $*$ maps* **R** *one-one onto* **A** *and vice versa. Also,* $\sim^{**} = \sim$ *and* $\to^{**} = \to$, *for any* $\sim$ *in* **R** *and* $\to$ *in* **A**.

Proof. By definition of **A**, $\to^{**} = \to$. By Proposition 3.6(i), $\sim^{**} = \sim$. For any $\sim$ in **R**, $\sim^*$ has AP$^+$ by Theorem 3.5(i), and $\sim^*$ is involutive, since $\sim^{***} = (\sim^{**})^* = \sim^*$; thus $*$ maps **R** into **A**. Furthermore, the function $*\colon \mathbf{R}\to\mathbf{A}$ is one-one by 3.6(ii); the function $*$ is onto **A** because for each $\to$ in **A** we have $\to\, = \,\to^{**}$, i.e., $\to$ is generated by $\to^*$, the latter being in **R** by Theorem 3.3. ■

From Theorem 3.9 together with 2.4 and the Duality Theorem in [17] we shall now obtain criteria for an embedding relation $\to$ to be representable as $\underset{L}{\to}$.

Recall [20, Theorem 7(1)], [17], that an abstract equivalence relation $\sim$ is *bounded* iff for every type τ there is a *set* $S_\tau \subseteq \mathrm{Str}(\tau)$ such that $\forall\mathfrak{A}\in\mathrm{Str}(\tau)\ \exists\mathfrak{B}\in S_\tau$ with $\mathfrak{B}\sim\mathfrak{A}$. Also, [17], $\sim$ is *separable* iff whenever $\tau(\mathfrak{A}) = \tau(\mathfrak{B})$ and not- $\mathfrak{A}\sim\mathfrak{B}$, there is a quantifier Q such that $\mathfrak{A}\not\equiv_{L(Q)}\mathfrak{B}$ and $\sim$ is finer than $\equiv_{L(Q)}$.

3.10. THEOREM. *For any embedding relation* $\to$ *the following are equivalent:*

(i) *there is exactly one compact logic with interpolation* $L' = L(Q^\alpha)_{\alpha<\varkappa}$ *such that* $\underset{L'}{\to} = \to$;

(ii) *there is exactly one logic* $L' = L(Q^\alpha)_{\alpha<\varkappa}$ *such that* $\underset{L'}{\to} = \to$; *in addition,* L' *is compact and satisfies interpolation;*

(iii) $\to$ *is involutive with* AP$^+$, *and* $\to^*$ *is bounded, separable and finer than elementary equivalence* $\equiv$.

Proof. (i) ⇒ (iii) The basic properties of logic L' ensure that $\equiv_{L'}$ satisfies the conditions of Definition 1.3; $\equiv_{L'}$ has the Robinson property as an immediate corollary of compactness and interpolation in L'. By Proposition 1.6, $\underset{L'}{\to} = (\equiv_{L'})^*$, whence by Theorem 3.9 $\underset{L'}{\to} = \to$ has AP$^+$ and is involutive. The identity $\equiv_{L'}^{****} = \to^{***} = \to^*$ now yields $\to^* = \equiv_{L'}$. One concludes that $\to^*$ is bounded, separable and finer than $\equiv$ recalling that L' is generated by a *set* of quantifiers, and that $L_{\omega\omega}$ is a sublogic of L.

(iii) ⇒ (i) Let $\sim\, = \,\to^*$. Then $\sim$ satisfies the conditions of Definition 1.3, and $\sim$ has the Robinson property by Theorem 3.3. This, together with our hypotheses about $\sim$ are sufficient to apply the Duality Theorem [17, 5.5] to the effect that $\sim\, = \,\equiv_{L'}$ for exactly one (up to equivalence) logic $L' = L(Q^\alpha)_{\alpha<\varkappa}$; further, L' is compact and obeys interpolation. We also have $\to\, = (\to^*)^* = (\equiv_{L'})^* = \underset{L'}{\to}$. Assume $L'' = L(Q^\beta)_{\beta<\lambda}$ is a compact logic with interpolation such that $\underset{L''}{\to} = \to$. Then $(\equiv_{L''})^* = \underset{L''}{\to} = \to$, hence $(\equiv_{L''})^{**} = \to^* = \equiv_{L'}$. Since $\equiv_{L''}$ has the Robinson property, then $\equiv_{L''} = (\equiv_{L''})^{**}$ by Theorem 3.9, hence $\equiv_{L''} = \equiv_{L'}$. Another application of [17, 5.5] yields that L'' is equivalent to L'.

(ii) ⇒ (i) Trivial.

(iii) ⇒ (ii) Use Theorem 2.3 and the implication (iii) ⇒ (i) established above. ■

Remark. The Duality Theorem [17, 5.5] can be applied to prove 3.10 without mentioning the special set-theoretical hypothesis used in the quantifier-free framework of [17], along the lines of [18, 1.1]: indeed for any $L = L(Q^\alpha)_{\alpha<\varkappa}$ the identity "Robinson consistency = compactness + interpolation" can be proved without any set-theoretical hypothesis (see, e.g., [9, 1.4] for a short proof due to Lindström). Once this is accomplished, the proof of the Duality Theorem for logics generated by quantifiers can be completed without mentioning any special set-theoretical axioms as well, exactly as the author did in [17] (see [17, 6.9], in particular).

4. Restricting to countable structures. Let C be a nonempty class of types closed under union, intersection, renaming, reduct, (i.e., $\tau' \subseteq \tau \in C$ implies $\tau' \in C$). For

$\varkappa \geqslant \omega$ a fixed but arbitrary cardinal, consider the class of structures $X = \bigcup_{\tau \in C} \mathrm{Str}_{\varkappa}(\tau)$, where $\mathrm{Str}_{\varkappa}(\tau)$ is the class of all structures of type τ and of cardinality $<\varkappa$. Then one can define the notion of an *abstract embedding relation* $\to$ *on* X, by relativizing to X Definition 1.1. One can similarly define the notion of an *abstract equivalence relation* $\sim$ *on* X. Any equivalence relation $\sim$ on X *generates* an embedding relation $\to\ = \ \sim^*$ on X by stipulating that (for all $\mathfrak{A}, \mathfrak{B} \in X$), $\mathfrak{A} \to \mathfrak{B}$ iff $\tau(\mathfrak{A}) \subseteq \tau(\mathfrak{B})$ and there exists some expansion $\mathfrak{B}^+$ of $\mathfrak{B} \restriction \tau(\mathfrak{A})$ having the following property: $\mathfrak{A}_A \restriction \tau \sim \mathfrak{B}^+ \restriction \tau$, for each $\tau \subseteq \tau_A$ with $\tau \in C$. Any embedding relation $\to$ on X *generates* an equivalence relation $\sim\ = \ \to^*$ on X by stipulating that two structures $\mathfrak{M}, \mathfrak{N} \in X$ are $\sim$-equivalent iff they have the same type τ and are connected by some finite path of arrows,

$$\mathfrak{M} \underset{1}{—} \mathfrak{B}_1 \underset{2}{—} \mathfrak{B}_2 \underset{3}{—} \dots \underset{n}{—} \mathfrak{B}_n = \mathfrak{N}; \quad \mathfrak{B}_i \in \mathrm{Str}(\tau) \cap X,$$

just as in 1.4.

An embedding relation $\to$ on X has AP^+ *on* X iff for every $\mathfrak{A}, \mathfrak{A}', \mathfrak{A}'' \in X$ of type τ, τ', τ'' respectively, if $\tau' \cap \tau'' = \tau$ and $\mathfrak{A}' \leftarrow \mathfrak{A} \to \mathfrak{A}''$ then there is $\mathfrak{B} \in X \cap \mathrm{Str}(\tau' \cup \tau'')$ such that $\mathfrak{A}' \to \mathfrak{B} \leftarrow \mathfrak{A}''$. One similarly defines the notion of an equivalence relation $\sim$ on X having the Robinson property *on* X, by relativizing Definition 3.2 to X (compare with [13]). *In this section K will denote the class of all countable structures of finite type*: we also let $\equiv|_K$ and $\cong|_K$ respectively denote the restriction to K of elementary equivalence $\equiv$ and isomorphism $\cong$. Refer to Examples (a), (b) in 1.2 for the definition of $\underset{L_{\omega\omega}}{\to}$ and $\underset{\exp}{\to}$.

4.1. Proposition. *Both* $\equiv|_K$ *and* $\cong|_K$ *have the Robinson property on* K. *Also, both* $\underset{\exp}{\to}$ *and* $\underset{L_{\omega\omega}}{\to}$, *upon restriction to* K, *have* AP^+ *on* K.

Proof. Clearly $\cong$ has the Robinson property, and $\underset{\exp}{\to}$ has AP^+; one similarly notes that these properties are preserved upon restriction to K, in the present case. The fact that $\equiv|_K$ has the Robinson property on K is a corollary of the Robinson consistency theorem for $L_{\omega\omega}$ together with the generalized downward Löwenheim–Skolem theorem, see [2]. One can similarly prove that $\underset{L_{\omega\omega}}{\to}$ restricted to K has AP^+ on K. ■

The following theorems show that AP^+ is quite rare on K:

4.2. Theorem. *Let $\to$ be an embedding relation on K, with AP^+ on K, and $\to\ \neq\ \underset{\exp}{\to}$ on K. Assume that the equivalence relation $\sim$ on K generated by $\to$ is finer than $\equiv|_K$. Then $\sim\ =\ \equiv|_K$.*

Proof. Since $\to\ \neq\ \underset{\exp}{\to}$ on K, then for at least two nonisomorphic structures $\mathfrak{A}, \mathfrak{B} \in K$ we have $\mathfrak{A} \to \mathfrak{B}$; hence $\sim\ \neq\ \cong$ on K. On the other hand, the relation $\sim$ has the Robinson property on K, as can be seen by relativizing to K the argument of Theorem 3.3. Therefore, by the main theorem in [13], $\sim\ =\ \equiv|_K$. ■

4.3. Theorem. *Let $\sim$ be an abstract equivalence relation on K. Assume that $\sim$ is finer than $\equiv|_K$ and that the embedding relation $\to$ on K generated by $\sim$ has AP^+ and is different from $\underset{\exp}{\to}$ on K. Then $\sim\ =\ \equiv|_K$.*

Proof. Let $\approx$ be the equivalence relation on K generated by $\to$. By relativizing to K the argument of 1.5 one has that $\approx$ is finer than $\sim$, hence finer than $\equiv|_K$. By hypothesis and by (relativizing to K) Theorem 3.3 we have that $\approx$ has the Robinson property on K; finally, by Theorem 4.2, $\approx\ =\ \equiv|_K$. Therefore $\sim\ =\ \equiv|_K$. ■

The above two theorems have a counterpart in abstract model theory:

4.4. Theorem. *Let $L = L(Q^{\alpha})_{\alpha<\varkappa}$ be a compact logic. Then the following are equivalent*:

(i) $\equiv_L\ =\ \equiv$ *on the class of countable structures*;

(ii) $\underset{L}{\to}$ *restricted to K has AP^+ on K.*

Proof. For the direction (i) $\Rightarrow$ (ii) see Proposition 4.1. To prove the other direction it suffices to prove that $\mathfrak{A} \equiv_L \mathfrak{B}$ iff $\mathfrak{A} \equiv \mathfrak{B}$ for all $\mathfrak{A}, \mathfrak{B} \in K$, since by hypothesis each sentence of L only has a finite number of symbols. For the same reason, letting $\sim$ be the restriction of $\equiv_L$ to K, one sees that $\sim$ generates the restriction of $\underset{L}{\to}$ to K. By Theorem 4.3 we have that $\sim\ =\ \equiv|_K$, unless $\sim\ =\ \cong|_K$. Assume this latter alternative actually holds (absurdum hypothesis). Let T be a first-order complete theory of finite type τ with one sort s; assume T ω_1-categorical and not ω-categorical. By the theorem of Baldwin and Lachlan [2, 7.1.27], T has exactly ω-many nonisomorphic countable models, say

$$\mathfrak{A}_0, \mathfrak{A}_1, \dots, \mathfrak{A}_n, \dots; \quad n<\omega,\ \mathfrak{A}_n \in K,\ \mathfrak{A}_i \not\cong \mathfrak{A}_j \text{ for all } i \neq j.$$

By the absurdum hypothesis for each $n<\omega$ the complete theory in L of $\mathfrak{A}_n$, $\mathrm{Th}_L \mathfrak{A}_n$, is ω-categorical. Let S_n be the class of cardinals λ such that $\mathrm{Th}_L \mathfrak{A}_n$ has a model of cardinality λ. Using Morley's theorem [2, 7.1.14] we have $S_j \cap S_i = \{\omega\}$ whenever $i \neq j$, since $L_{\omega\omega}$ is a sublogic of L. Let $\varrho\colon \tau \to \varrho(\tau)$ be a renaming such that $\{s\} = \tau \cap \varrho(\tau)$, and let $\mathfrak{A}_0^{\varrho}$ be the renamed structure corresponding to $\mathfrak{A}_0$ as in the introductory discussion of Section 1. Clearly $\mathrm{Th}_L \mathfrak{A}_0^{\varrho}$ has a model of cardinality λ iff $\lambda \in S_0$. Let T' be the theory given by $T' = \mathrm{Th}_L \mathfrak{A}_0^{\varrho} \cup \mathrm{Th}_L \mathfrak{A}_1$: then T' has only countable models, since $S_0 \cap S_1 = \{\omega\}$. We have thus exhibited a set T' of sentences which is a counterexample to the assumed compactness of L, a contradiction. ■

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Countable subsets of Suslinian continua

by

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Abstract. An example of a Suslinian continuum Y with no countable set intersecting all nondegenerate subcontinua of Y is given.

All spaces are considered to be metric. A *continuum* is a connected and compact space. A continuum is *Suslinian* if it does not contain uncountably many mutually exclusive nondegenerate subcontinua.

In 1971 A. Lelek posed the following question: If Y is a Suslinian continuum, does there exist a countable set A in Y such that A intersects every nondegenerate subcontinuum of Y? ([2], Problem 10, P 726). A partial positive answer was given by A. Lelek in the case where Y is hereditarily unicoherent ([3], Th. 2.2, p. 133). The aim of this note is to describe an example which gives a negative answer to this question.

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CONSTRUCTION OF THE EXAMPLE. Denote by I the unit interval $[0,1]$. Let $h\colon I\to I$ be a mapping defined by the following formula:

$$h(t)=\begin{cases}2t & \text{for } 0\leqslant t\leqslant \frac{1}{2},\\ 2-2t & \text{for } \frac{1}{2}\leqslant t\leqslant 1.\end{cases}$$

For an arbitrary finite collection A of subintervals of I, we will say that A has the property $(*)$ provided that for every $J_1, J_2\in A$ either $J_1=J_2$ or $J_1\subset \operatorname{int} J_2$ or $J_2\subset \operatorname{int} J_1$ or $J_1\cap J_2=\emptyset$.

For any collection A with the property $(*)$ let us adopt the following notation. $L(A)$ is the set of all left ends of intervals from A, and $R(A)$ is the set of all right ends of intervals from A. For a point $p\in L(A)\cup R(A)$ let $d(p)$ denote the length of the interval from A having p as an endpoint.

Set $r(A)=\frac{1}{4}\min\{|a-b|\colon a\neq b,\ a,b\in L(A)\cup R(A)\}$.

For $n=1,2,\ldots$ let us define a mapping $g_n[A]\colon I\to I$ by the following formula: