On dendroids and their end-points and ramification points in the classical sense

by

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Abstract. It is shown that a dendroid has uncountably many end-points (in the classical sense) if and only if it contains a homeomorphic copy of either the Cantor fan or the Cantor comb or the Gehman dendrite. Besides, it is shown that a set of all ramification points of a plane dendroid X which lie on some subarc of X is a $G_{\delta\sigma}$-set.

1. Introduction. In this paper we consider dendroids with uncountably many end-points. We show that such dendroids contain subcontinua of a special type. Moreover, we start an investigation of a Borel class of sets of all ramification points in plane dendroids.

This paper is a modification of J. Nikiel "On planable dendroids and their end-points and ramification points in the classical sense" (which was sent to Fundamenta Mathematicae in 1981) and it follows papers [7] and [8]. The next paper dealing with related topics is [9].

The referee of the earlier version of this paper sketched proofs of some more general results than those obtained there (first of all Theorem 2 of the present paper), stated Theorems 3 and 4, and re-proved some results of [7] and [8] which lead to Theorem 4. In the paper "On planable dendroids and their end-points and ramification points in the classical sense" it was shown that a plane dendroid fulfilling the assumptions of Theorem 2 (resp. Theorem 3) of the present paper contains a semi-smooth fan the set of all end-points of which is homeomorphic to the Cantor set (resp. a semi-smooth comb of a special type). The results of Theorems 2 and 3 were proved under the assumption that the dendroid in question is smooth and planable. The referee's methods used in the proof of Theorem 2 are quite different from those explored in the earlier version of the present paper. The importance of the referee's contribution to the final version of the paper made the author suggest co-authorship to the referee, but he (or she) preferred to remain anonymous.

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2. Preliminaries. A metrizable continuum is said to be a *dendroid* if it is arcwise connected and hereditarily unicoherent. Therefore each subcontinuum of a dendroid is again a dendroid.

If X is a dendroid and $x, y \in X$, then there is a unique arc I contained in X the end-points of which are x and y. We denote this arc by $[x, y]$ (if confusion is possible we write $[x, y]_X$ instead of $[x, y]$). The only exception from the above rule is $[0, 1]$ denoting the set of real numbers which are not less than 0 and not greater than 1. We also use the notation $(x, y] = [y, x) = [x, y]\setminus\{x\}$ and $(x, y) = [x, y]\setminus\{x, y\}$. If x is a point of a dendroid X, then we define its order (in the classical sense), $r(X, x)$, as a cardinality of the set of all arc-components of $X\setminus\{x\}$ (for equivalent definitions see [1], p. 229 and [6], p. 301). If $r(X, x) = 1$ (resp. $r(X, x) \geqslant 3$), then x is said to be an *end-point* (resp. a *ramification point*) *of* X. The set of all end-points (resp. ramification points) of X is denoted by $E(X)$ (resp. $R(X)$). We say that a dendroid X is a *fan with the top* p if $p \in X$ and $R(X) = \{p\}$. X is said to be a *comb* provided $R(X) \subset I$ for some arc I contained in X. If X is a comb, then I_X denotes the minimal arc of X (in the sense of inclusion) which contains $R(X)$. We say that a fan (resp. comb) X is *uncountable* if $r(X, p) > \aleph_0$ (resp. if $R(X)$ is uncountable).

There is exactly one dendrite G such that $E(G)$ is homeomorphic to the Cantor set and $r(G, x) \leqslant 3$ for each $x \in G$. We call G the *Gehman dendrite* (see [7] and [8]).

Let C denote the Cantor set constructed in $[0, 1]$ in the usual way. Now, we construct two dendroids in the plane E^2. Put

$$F_C = \{\langle x, y\rangle \in E^2\colon\ x + (\tfrac{1}{2} - c)y = \tfrac{1}{2} \text{ for some } c \in C\}$$

and

$$C_C = [0, 1] \times \{0\} \cup C \times [0, 1].$$

F_C is called the *Cantor fan* and C_C — the *Cantor comb.*

All the required definitions and facts on smooth dendroids can be found in [3].

In this paper d always denotes a metric on a given space (and its subspaces). This metric is fixed during the whole reasoning. If X is a metric space, Y is a subset of X, ε is a positive real number and d is a metric on X, then we denote

$$B(Y, \varepsilon) = \{x \in X\colon\ d(x, Y) < \varepsilon\}, \qquad S(Y, \varepsilon) = \{x \in X\colon\ d(x, Y) = \varepsilon\}$$

and

$$\overline{B}(Y, \varepsilon) = B(Y, \varepsilon) \cup S(Y, \varepsilon).$$

If A is a set, then $|A|$ denotes its cardinality.

3. Some lemmas. A dendroid X is said to be a *Gehman dendroid* if there is a one-to-one and continuous map from $G\setminus E(G)$ onto some dense subset of X.

LEMMA 1 ([7], Theorem 1). *A dendroid X has uncountably many end-points if and only if either* (i) *there is an arc I (maybe degenerate) in X such that the set $X\setminus I$ has uncountably many arc-components, or* (ii) *X contains some Gehman dendroid.*

LEMMA 2 ([8], Theorem 8). *Each Gehman dendroid contains topologically the Gehman dendrite.*

LEMMA 3. *Let X_1 and X_2 be smooth fans such that $E(X_1)$ is a compact set. Then X_1 is homeomorphic to X_2 if and only if $E(X_1)$ is homeomorphic to $E(X_2)$.*

Proof. Let us suppose that $E(X_1)$ is compact and $f\colon E(X_1) \to E(X_2)$ is a homeomorphism. For $i = 1, 2$ let p_i denote the top of X_i, and if $x \in X_i\setminus\{p_i\}$, then let $g_i(x)$ be the only point of $E(X_i)$ such that $x \in [p_i, g_i(x)]$. Since X_i is smooth and $E(X_i)$ is compact, we see that $g_i\colon X_i\setminus\{p_i\} \to E(X_i)$ is continuous for $i = 1, 2$. Let d_i be a radially convex metric on X_i with respect to the point p_i (such a metric does exist by [3], Theorem 10, p. 310). If $x \in X_1\setminus\{p_1\}$, then let $F(x)$ be the unique point of $(p_2, f(g_1(x))]$ such that

$$d_2(F(x), p_2) = d_1(x, p_1)\frac{d_2(f(g_1(x)), p_2)}{d_1(g_1(x), p_1)}$$

and put $F(p_1) = p_2$. It is clear that the map F is well-defined, one-to-one and onto; moreover, $F|_{E(X_1)} = f$. We show that F is continuous, and so F is a homeomorphism from X_1 onto X_2.

Since the maps f, g_1 and g_2 are continuous and $g_2(F(x)) = f(g_1(x))$ for $x \in X_1\setminus\{p_1\}$, we see that $F|_{X_1\setminus\{p_1\}}$ is continuous. It only remains to show that F is continuous at the point p_1. Let $\langle x_n\rangle$ be such a sequence in X_1 that $\lim x_n = p_1$. Therefore $\lim d_1(x_n, p_1) = 0$ and since $d_1(p_1, E(X_1)) > 0$ (by the compactness of $E(X_1)$) we see that $\lim d_2(F(x_n), p_2) = 0$, i.e., $\lim F(x_n) = p_2$.

LEMMA 4. *Let X be a dendroid. Then X is homeomorphic to the Cantor comb if and only if*

(i) *X is a comb,*
(ii) *$r(X, x) \leqslant 3$ for each $x \in X$,*
(iii) *$E(X)$ is homeomorphic to the Cantor set, and*
(iv) *If $I_X = [p, q]$, then X is smooth with respect to p and X is smooth with respect to q.*

Proof. Let us consider the mapping $i\colon X\setminus I_X \to I_X$, where $i(x)$ is the first point of $[p, x]$ meeting I_X. By (iv) i is continuous. By (ii) $i_{E(X)}\colon E(X) \to I_X$ is one-to-one, and so (by (iii)) $i(E(X)) = R(X) \cup \{p, q\}$ is homeomorphic to the Cantor set. Let $k\colon I_X \to [0, 1]$ be any homeomorphism such that $k(R(X) \cup \{p, q\}) = C$. By (iv) and [3], Corollary 10, p. 309 X/I_X is a smooth dendroid, and so by (i) X/I_X is a fan. By (iii) and Lemma 3 it is homeomorphic to the Cantor fan. Let $h\colon X \to F_C$ be ac omposition of the quotient map $X \to X/I_X$ and a homeomorphism $X/I_X \to F_C$, $h(x) = \langle h_1(x), h_2(x)\rangle \in F_C \subset E^2$ for $x \in X$. The mapping $H\colon X \to C_C$,

$$H(x) = \begin{cases}\langle k(x), 0\rangle & \text{for } x \in I_X,\\ \langle k(i(x)), h_2(x)\rangle & \text{for } x \in X\setminus I_X,\end{cases}$$

is easily seen to be a homeomorphism.

LEMMA 5. *Let I, J be arcs with their natural orderings "$\leqslant_I$" and "$\leqslant_J$". Let K be a closed subset of I and let $f\colon K \to J$ be any function either increasing or decreasing. Then $f(K)$ is a G_δ-set in J.*

Proof. Let us assume that f is increasing (the case where f decreases is similar). Let a_0 (resp. b_0) be the least (resp. the greatest) point of K. We may assume that a_0 and b_0 are end-points of I and that $f(a_0)$ and $f(b_0)$ are end-points of J. Since K is closed in I, we see that $K = I \setminus \bigcup_{n=1}^{\infty} (a_n, b_n)_I$ for some points $a_n, b_n \in I$, $a_n \leqslant_I b_n$, $n = 1, 2, \ldots$ Put

$D_1 = \{a \in I\colon\ \sup\{f(x)\colon\ x <_I a\} <_J f(a)\}$, $D_2 = \{a \in I\colon\ f(a) <_J \inf\{f(x)\colon\ a <_I x\}\}$

and $D = \{a \in I\colon f$ is not continuous at $a\}$. Then by [10], Theorem 4.29, p. 95 we have $D = D_1 \cup D_2$, and so by [10], Theorem 4.30, p. 96 D, D_1 and D_2 are countable sets. For each $a \in D_1$ (resp. $a \in D_2$) put $e_a = \sup\{f(x)\colon\ x <_I a\}$ (resp. $f_a = \inf\{f(x)\colon\ a <_I x\}$). Since the countable union of F_σ-sets is an F_σ-set, we see that the set

$$L = \bigcup_{n=1}^{\infty} (f(a_n), f(b_n))_J \cup \bigcup_{a \in D_1} [e_a, f(a))_J \cup \bigcup_{a \in D_2} (f(a), f_a]_J$$

(here $(x, x)_J = \emptyset$ for $x \in J$) is an F_σ-set in J. It is not difficult to see that $f(K) = J \setminus L$.

LEMMA 6. *If X is a dendroid, then for any points $a, b \in X$, $c \in [a, b]$, and any open set U containing c there is a neighbourhood V of b such that $[a, x] \cap U \neq \emptyset$ for each $x \in \operatorname{cl} V$.*

Proof. Suppose that for each $n = 1, 2, \ldots$ there is an $x_n \in B(b, 1/n)$ such that $[a, x_n] \cap U = \emptyset$. Put $Y = \operatorname{Ls}[a, x_n]$; so by [4], Theorem 2-101, p. 101 Y is a subcontinuum of X such that $a, b \in Y$ and $c \notin Y$. This means that X is not hereditarily unicoherent.

Let Z be a dendroid. For any positive number ε and any arc I contained in Z define a number $s(I, \varepsilon)$ as the greatest integer n such that there is a monotone sequence $x_0, x_1, \ldots, x_n$ of points of I satisfying $d(x_{i-1}, x_i) > \varepsilon$ for $i = 1, \ldots, n$. Obviously $s(I, \varepsilon)$ is finite.

LEMMA 7. *Let p and q be two points of a dendroid Z and let t be a point of $[p, q]$. Then for every positive number ε there are neighbourhoods U of q and V of t such that for each $x \in \operatorname{cl} U$*

$$s([x, p], \varepsilon) \geqslant s([p, q], \varepsilon).$$

Moreover, if $z \in [x, p] \cap \operatorname{cl} V$ and $[z, p] \setminus B([t, p], 4\varepsilon) \neq \emptyset$ then

$$s([x, p], \varepsilon) > s([p, q], \varepsilon).$$

Proof. Put $n = s([p, q], \varepsilon)$ and let $x_0, x_1, \ldots, x_n$ be a monotone sequence of points of $[p, q]$ such that $d(x_{i-1}, x_i) > \varepsilon$ for $i = 1, \ldots, n$. Let k be an integer such that $x_0, \ldots, x_k \in [p, t]$ and $x_{k+1}, \ldots, x_n \in [t, q]$. Using Lemma 6, we construct inductively a sequence $V_0, \ldots, V_k, V, V_{k+1}, \ldots, V_n, U$ of neighbourhoods of points $x_0, \ldots, x_k, t, x_{k+1}, \ldots, x_n, q$, respectively, such that

(a) $\operatorname{diam} V_i < \varepsilon$ for $i = 0, \ldots, n$,
(b) $d(V_{i-1}, V_i) > \varepsilon$ for $i = 1, \ldots, n$,
(c) $[y, p] \cap V_{i-1} \neq \emptyset$ for $i = 1, \ldots, k$ and $y \in \operatorname{cl} V_i$,
(d) $[z, p] \cap V_k \neq \emptyset$ for $z \in \operatorname{cl} V$,
(e) $[y, p] \cap V \neq \emptyset$ for $y \in \operatorname{cl} V_{k+1}$,
(f) $[y, p] \cap V_j \neq \emptyset$ for $j = k+1, \ldots, n-1$ and $y \in \operatorname{cl} V_{j+1}$,
(g) $[x, p] \cap V_n \neq \emptyset$ for $x \in \operatorname{cl} U$.

Each V_i is constructed as a subset of $B(x_i, \varepsilon_i)$, where

$$\varepsilon_i = \tfrac{1}{3}\min\{\varepsilon, \varepsilon - d(x_{i-1}, x_i), \varepsilon - d(x_i, x_{i+1})\}.$$

Thus (a) and (b) hold. Put $V_0 = B(x_0, \varepsilon_0)$. By Lemma 6 there is a neighbourhood V_1' of x_1 such that for each $y \in \operatorname{cl} V_1'$ the intersection $[y, p] \cap V_0$ is nonempty. Put $V_1 = V_1' \cap B(x_1, \varepsilon_1)$. In a similar way $V_2, \ldots, V_k, V, V_{k+1}, \ldots, V_n$ and U are constructed. By (c)–(g), if $x \in \operatorname{cl} U$, then $[x, p] \cap V \neq \emptyset$ and $[x, p] \cap V_i \neq \emptyset$ for $i = 1, \ldots, n$.

Now, take any $x \in \operatorname{cl} U$. Let y_i be the last point in the ordered arc $[x, p]$ meeting $\operatorname{cl} V_i$ for $i = 0, \ldots, n$. Note that the sequence $y_0, \ldots, y_n$ is monotone, and so by (b) we have $s([x, p], \varepsilon) \geqslant n$.

Take any $z \in [x, p] \cap \operatorname{cl} V$ and suppose that there is a point

$$v \in [z, p] \setminus B([t, p], 4\varepsilon).$$

If $v \in [y_k, p]$, then in order to prove that $s([x, p], \varepsilon) > n$ it suffices to arrange points $v, y_0, \ldots, y_n$ into a monotone sequence. So assume that $v \in [z, y_k]$. It suffices to show that $d(v, y_{k+1}) > \varepsilon$. Observe that by (a) $V_{k+1} \subset B([t, p], 3\varepsilon)$ (because if $V_{k+1} \setminus B([t, p], 3\varepsilon) \neq \emptyset$ then $d(x_k, x_{k+1}) > 2\varepsilon$; so $s([p, q], \varepsilon) > n$ — a contradiction). Therefore $d(v, y_{k+1}) > \varepsilon$.

For a point p of a dendroid X and for any positive number ε let $p(\varepsilon)$ be the set of all points $x \in X$ such that for every neighbourhood U of x there is a $y \in U$ with $s([y, p], \varepsilon) > s([x, p], \varepsilon)$.

LEMMA 8. *If X is a dendroid, $p \in X$, and ε is a positive number, then $p(\varepsilon)$ is a closed and nowhere dense subset of X.*

Proof. By Lemma 7, $p(\varepsilon)$ is closed in X. Suppose that $\operatorname{int} p(\varepsilon) \neq \emptyset$. We will construct inductively a decreasing sequence $V_0, V_1, \ldots$ of nonvoid open subsets of $\operatorname{int} p(\varepsilon)$ such that $s([y, p], \varepsilon) \geqslant n$ for any $y \in \operatorname{cl} V_n$. Obviously, such a construction leads to a contradiction — because for $y \in \bigcap_{n=0}^{\infty} \operatorname{cl} V_n$ we have $s([y, p], \varepsilon) \geqslant n$ for every n.

Put $V_0 = \operatorname{int} p(\varepsilon)$. Suppose that $V_0, \ldots, V_{n-1}$ are constructed. Take a point $x \in V_{n-1}$; so $s([x, p], \varepsilon) \geqslant n-1$. Since $x \in p(\varepsilon)$, there is a point $y \in V_{n-1}$ such that

$s([y,p],\varepsilon)\geqslant n$ (because V_{n-1} is an open set). By Lemma 7 there is a neighbourhood V of y such that $s([z,p],\varepsilon)\geqslant n$ for each $z\in \mathrm{cl}\,V$. Now, put $V_n=V\cap V_{n-1}$.

LEMMA 9. *If X is a dendroid, $p\in X$, and Y is an arc-component of $X\setminus\{p\}$, then Y is an F_σ-subset of X.*

Proof. Let q be any point of Y. For each integer k such that $d(p,q)>1/k$ let Y_k be the component of $X\setminus B(p,1/k)$ which contains q. Hence Y_k is closed in X and connected, i.e., Y_k is a subdendroid of X. Therefore each Y_k is contained in Y. It remains to show that Y is contained in $\bigcup Y_k$. If $x\in Y$, then $d(p,[q,x])>0$; so let m be such that $1/m<d(p,[q,x])$. Then $x\in[q,x]\subset Y_m$.

4. Main facts.

THEOREM 1. *Let X be a planable dendroid and let $I\subset X$ be an arc. Then the set $R(X)\cap I$ is a $G_{\delta\sigma}$-set.*

Proof. We consider X as a subset of the plane E^2. All balls and their boundaries are taken in E^2, "cl" and "bd" denote, respectively, the closure and the boundary of a set in E^2.

Denote $I=[a_1,a_2]$. Let I_1 and I_2 be arcs in E^2 (they are not subsets of X) such that $I_1\cap I_2=\emptyset$, $E(I_i)=\{a_i,a_i^1\}$, $I_i\cap I=\{a_i\}$ and $I_i\cap S(I,1)=\{a_i^1\}$ for $i=1,2$. For each positive integer n and $i=1,2$ let $a_i^n\in I_i$ be such that $[a_i,a_i^n]_{I_i}\cap S(I,1/n)=\{a_i^n\}$. By [11], Chapter VI, (1.6), p. 104 the set $\bar{B}(I,1/n)\setminus([a_1^n,a_1]_{I_1}\cup I\cup[a_2,a_2^n]_{I_2})$ has exactly two components D_1^n and D_2^n such that

$$\mathrm{bd}\,D_i^n=[a_1^n,a_1]_{I_1}\cup I\cup[a_2,a_2^n]_{I_2}\cup J_i^n,\quad \text{where}\quad J_1^n,J_2^n\subset S(I,1/n)$$

are two distinct arcs with end-points a_1^n and a_2^n, for each positive integer n and $i=1,2$. Put

$$K_i^n=[a_1,a_1^n]_{I_1}\cup J_i^n\cup[a_2^n,a_2]_{I_2}\quad\text{for}\quad i=1,2;$$

so both J_1^n and J_2^n are arcs with end-points a_1 and a_2. Below we consider the arcs I and K_i^n; $i=1,2$; $n=1,2,\ldots$; with their natural orderings "$\leqslant_I$", "$\leqslant_{K_i^n}$", respectively, from a_1 to a_2.

For each positive integer n and $i=1,2$ let X_i^n be the component of $X\cap\mathrm{cl}\,D_i^n$ containing I; so X_i^n is a dendroid. For each point $x\in R(X_i^n)\cap I$ let $P(x,i,n)$ be the union of all arc-components of $X_i^n\setminus\{x\}$ which are disjoint with I. By [11], Chapter VI, (1.7), p. 105 and the Jordan curve theorem we have the following:

if $x,x'\in R(X_i^n)\cap I$, $x\neq x'$, $y\in P(x,i,n)\cap K_i^n$, $y'\in P(x',i,n)\cap K_i^n$, then $x<_I x'$ if and only if $y<_{K_i^n}y'$.

For each $y\in X_i^n\cap K_i^n$ let $f_i^n(y)$ be the unique point of $R(X_i^n)\cap I$ such that $[y,f_i^n(y)]\cap I=\{f_i^n(y)\}$. Then for each positive integer n and $i=1,2$ the map $f_i^n\colon X_i^n\cap K_i^n\to I$ is increasing. Moreover, $X_i^n\cap K_i^n$ is compact; so by Lemma 5 $f_i^n(X_i^n\cap K_i^n)$ is a G_δ-set. Note that

$$f_i^n(X_i^n\cap K_i^n)=\big(R(X_i^n)\cap I\big)\cup\{a_1,a_2\}.$$

By the Jordan curve theorem it is clear that

$$R(X)\cap(a_1,a_2)=\bigcup\{f_i^n(X_i^n\cap K_i^n)\colon i=1,2;\ n=1,2,\ldots\}\setminus\{a_1,a_2\}.$$

Therefore $R(X)\cap I$ is a countable union of G_δ-sets.

Remarks. 1. It can easily be seen from the above proof that if $d(E(X)\setminus E(I),I)>0$ then $R(X)\cap I$ is a G_δ-set.

2. In [9] we show that the set of all ramification points of a plane dendroid can be covered by countably many arcs, and so it is a $G_{\delta\sigma}$-set.

COROLLARY 1. *If X is a planable comb, then $R(X)$ is a $G_{\delta\sigma}$-set.*

THEOREM 2. *Let X be a dendroid. If for some point $p\in X$ we have $r(X,p)>\aleph_0$, then X contains a fan Y homeomorphic to the Cantor fan. The top of Y is p.*

Proof. Let H denote the union of all open sets contained in $X\setminus\{p\}$ meeting a countable number of arc-components of $X\setminus\{p\}$. Since X is separable metric, H meets only a countable number of arc-components of $X\setminus\{p\}$. Put $Z=X\setminus H$. Observe that for every $z\in Z$ we have $[p,z]\subset Z$ (by Lemma 6). Hence Z is a dendroid such that $r(Z,p)>\aleph_0$. Moreover, each arc-component of $Z\setminus\{p\}$ is an F_σ-subset of Z (by Lemma 9) which is of the first category in Z.

For $n=1,2,\ldots$ consider the set F_n consisting of all functions f mapping $\{1,2,\ldots,n\}$ into $\{0,1\}$. For $f\in F_n$, $n=2,3,\ldots$ let $f'\in F_{n-1}$ denote f restricted to the set $\{1,2,\ldots,n-1\}$.

For each $n=1,2,\ldots$ and each $f\in F_n$ we will construct a nonvoid open set $V_f\subset\mathrm{cl}\,V_f\subset Z\ \{p\}$ such that

(i) $\mathrm{diam}\,V_f<1/n$,

(ii) $\mathrm{cl}\,V_f\cap p(1/n)=\emptyset$,

(iii) $\mathrm{cl}\,V_f\subset V_{f'}$ (if $n\geqslant 2$),

(iv) if $f,g\in F_n$, $f\neq g$, $x\in\mathrm{cl}\,V_f$ and $y\in\mathrm{cl}\,V_g$ then $[x,y]\cap B(p,1/n)\neq\emptyset$.

Suppose that for every $k<n$ and for every $f\in F_k$ a set V_f is constructed. Using Lemma 8 and the fact that arc-components of $Z\setminus\{p\}$ are of the first category in Z, one can choose for every $f\in F_n$ a point $y_f\in\big(V_{f'}\setminus\bigcup_{k=1}^{\infty}p(1/k)\big)$ such that $p\in(y_f,y_g)$ for $f,g\in F_n$, $f\neq g$. Since F_n is finite, there is a positive integer m such that $m>4n$ and

$$B([y_f,p],4/m)\cap B([y_g,p],4/m)\subset B(p,1/n)\quad\text{for}\quad f,g\in F_n,\ f\neq g.$$

Since $y_f\notin p(1/n)\cup p(1/m)$, there is a neighbourhood U_f of y_f such that

$$\mathrm{cl}\,U_f\subset V_{f'}\cap B(y_f,1/m)\setminus p(1/n)\quad\text{and}\quad s([z,p],1/m)=s([y_f,p],1/m)$$

for each $z\in\mathrm{cl}\,U_f$. By Lemma 7 there is a neighbourhood $V_f\subset U_f$ of y_f such that $[z,p]\subset B([y_f,p],4/m)$ for each $z\in\mathrm{cl}\,V_f$. It is clear that the conditions (i)–(iv) are fulfilled.

The set $C_0 = \bigcap_{n=1}^{\infty} \bigcup_{f\in F_n} V_f$ is homeomorphic to the Cantor set. Moreover, $p \notin C_0$, $C_0 \cap \bigcup_{n=1}^{\infty} p(1/n) = \emptyset$ (by (ii)) and any two different points of C_0 are contained in two different arc-components of $Z\setminus\{p\}$ (by (iv)).

Put $Y = \bigcup_{c\in C_0} [c, p]$. For a point $y \in Y\setminus\{p\}$ let $e(y)$ denote the unique point of C_0 such that $y \in [e(y), p]$. We will show that, for every sequence $\langle y_n\rangle$ of points of $Y\setminus\{p\}$, if $y = \lim y_n$, then $y \in Y$ and $[y, p] = \mathrm{Ls}[y_n, p]$.

Let $\langle y_n\rangle$ be fixed, $y = \lim y_n$, $y \neq p$. Consider the sequence $\langle e(y_n)\rangle$. Since C_0 is compact, there is a subsequence $\langle e(y_{n_k})\rangle$ converging to some point $c \in C_0$. Suppose that $y \notin [c, p]$. Take an integer m such that $4/m < d(y, [c, p])$. By Lemma 7 we have $s([e(y_{n_k}), p], 1/m) > s([c, p], 1/m)$ for sufficiently large k, and this contradicts the fact that $c \notin p(1/m)$. Thus $y \in [c, p] \subset Y$.

Now, suppose that $[y, p] \neq \mathrm{Ls}[y_n, p]$. Then there is an integer k such that $[y_n, p]\setminus B([y, p], 4/k) \neq \emptyset$ for infinitely many n's. By Lemma 7 it follows that $c \in p(1/k)$, which is impossible for $c \in C_0$.

Thus Y is a smooth fan and by Lemma 3 we see that it is homeomorphic to the Cantor fan.

COROLLARY 2. *Each uncountable fan contains topologically the Cantor fan.*

THEOREM 3. *Let X be a dendroid. If for some arc I contained in X we have $|R(X) \cap I| > \aleph_0$ and $r(X, x) \leqslant \aleph_0$ for each $x \in R(X) \cap I$, then X contains a comb Y homeomorphic to the Cantor comb. Moreover, $R(Y) \subset I$.*

Proof. Denote the end-points of I by p and q. Put $Z = X/I$ and let $h\colon X \to Z$ denote the quotient map. Put $p = h(I) \in Z$. By [2], Corollary 2, p. 219, Z is a dendroid. Moreover, $r(Z, p) > \aleph_0$; so, by Theorem 2, Z contains a fan Z_0 homeomorphic to the Cantor fan and such that $R(Z_0) = \{p\}$. Put $Y_1 = h^{-1}(Z_0)$; so

$$R(Y_1) \subset I \quad\text{and}\quad |R(Y_1)| > \aleph_0 .$$

Furthermore, $E(Y_1) \subset C_1 \cap \{p, q\}$, where $C_1 = h^{-1}(E(Z_0))$ is homeomorphic to the Cantor set.

For each $x \in Y_1\setminus I$ let $i(x)$ be the first point of $[x, p]$ which meets I and let $j(x)$ be a point of C_1 such that $x \in (i(x), j(x)]$. Let D be the set of all points of C_1 such that if $c \in D$ then for some $x \in (i(c), c]$ the function i is not continuous at x. We show that

(*) $C_1\setminus D$ contains an uncountable G_δ-subset of C_1.

First we prove that if $c \in D$ then

$$(i(c), c] \subset \bigcup_{n=1}^{\infty} (p(1/n \cup q(1/n)) .$$

Indeed, suppose that for some $x \in (i(c), c]$ there is a sequence $\langle x_n\rangle$ of points of $Y_1\setminus I$ such that $\lim x_n = x$ and $\lim i(x_n) = z \neq i(x)$. Using the map $h\colon Y_1 \to Z_0$, one can see that $\lim j(x_n) = j(x) = c$ and that $z \in I\setminus\{i(x)\}$. We may assume that $z \in [p, i(x))$. Then for each $y \in (i(c), c]$ there is a sequence $\langle y_n\rangle$ of points of $(i(x_n), j(x_n)]$ with $y = \lim y_n$ (to see this again use the map h). Moreover, there is an integer m such that $[y_n, q]\setminus B([z, q], 4/m) \neq \emptyset$. By Lemma 7 $y \in q(1/m)$.

By Lemma 8 the set $\bigcup_{c\in D} (i(c), c]$ is a union of countably many nowhere dense subsets of $Y_1\setminus I$. This proves (*).

By (*) and [5], Corollary (Theorem of Alexandrov and Hausdorff), p. 427 there is a subset C_2 of $C_1\setminus D$ which is homeomorphic to the Cantor set. Put $Y_2 = I \cup \bigcup_{c\in C_2} [i(c), c]$; so Y_2 is a dendroid. Moreover, $i|_{Y_2\setminus I}\colon Y_2\setminus I \to I$ is continuous. Therefore Y_2 is smooth with respect to p and q.

Let us use the same notation for F_n and f' as in the proof of Theorem 2.

For each n and each $f \in F_n$ one can easily construct a closed-open nonvoid set $V_f \subset C_2$ such that

$$\operatorname{diam} V_f < 1/n, \quad V_f \subset V_{f'} \quad (\text{for } n \geqslant 2),$$

$$i(V_f) \cap i(V_g) = \emptyset \quad \text{for} \quad f, g \in F_n, f \neq g .$$

It suffices to notice that $R(Y_2) = i(C_2)$ is an uncountable subset of I because $r(Y_2, x) \leqslant r(X, x) \leqslant \aleph_0$ for each $x \in R(Y_2)$.

Put $C_3 = \bigcap_{n=1}^{\infty} \bigcup_{f\in F_n} V_f$; so C_3 is homeomorphic to the Cantor set and $i|_{C_3}\colon C_3 \to I$ is one-to-one. Let $[p_1, q_1]$ be the minimal subarc of I (in the sense of inclusion) which contains $i(C_3)$. It is easy to see that $Y = [p_1, q_1] \cup \bigcup_{c\in C_3} [c, i(c)]$ is a dendroid which fulfils the assumptions of Lemma 4. Therefore Y is homeomorphic to the Cantor comb and $R(Y) \cup \{p_1, q_1\} = i(C_3) \subset I$.

Remark. Theorem 3 remains true without the assumption that $r(X, x) \leqslant \aleph_0$ for $x \in R(X) \cap I$. In the proof of this more general result one cannot use Theorem 2. The proof requires the use of Lemmas 7 and 8 and is somewhat similar to the proof of Theorem 2 (and to part of the proof of Theorem 3).

COROLLARY 3. *Each uncountable comb contains topologically the Cantor comb.*

By Lemmas 1, 2 and Theorems 2, 3 we obtain the following

THEOREM 4. *Let X be a dendroid. Then X has uncountably many end-points if and only if X contains topologically either the Cantor fan or the Cantor comb or the Gehman dendrite.*

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Embeddings, amalgamation and elementary equivalence: the representation of compact logics

by

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Abstract. Any logic L generates the L-embedding relation $\underset{L}{\to}$ just as first-order logic $L_{\omega\omega}$ generates the elementary embeddability relation. By abstracting from L, given a transitive relation $\to$ between structures we may ask whether there is a (perhaps unique) logic L such that $\to = \underset{L}{\to}$. We prove that if L is compact and $\to = \underset{L}{\to}$, then L is uniquely determined by $\to$: thus in particular $L_{\omega\omega}$ is uniquely determined by the elementary embeddability relation. We give necessary and sufficient conditions for the existence and uniqueness of a logic L such that $\to = \underset{L}{\to}$, in case $\to$ has a strong form of amalgamation property, called AP+. Upon restriction to countable structures of finite type there are exactly two nontrivial embedding relations with AP+.

0. Introduction. Given a logic L, say in the sense of [12], one defines the L-elementary embedding relation $\underset{L}{\to}$ just as for first-order logic $L_{\omega\omega}$ one defines the elementary embeddability relation (1.2(b), (c)). By abstracting from L, we may consider an arbitrary transitive relation $\to$ between structures which is preserved under isomorphism, reduct and renaming (1.1). Any such relation $\to$ generates an equivalence relation $\sim\, = \,\to^*$ between structures, by saying that $\mathfrak{A}\sim\mathfrak{B}$ iff $\mathfrak{A}$ and $\mathfrak{B}$ are connected by a finite path of arrows between structures of the same type (1.4). Conversely, any equivalence relation $\sim$ as defined in (1.3) generates an embedding relation $\to\, = \,\sim^*$, by saying that $\mathfrak{A}\to\mathfrak{B}$ iff the type $\tau(\mathfrak{A})$ of $\mathfrak{A}$ is contained in $\tau(\mathfrak{B})$ and some expansion of $\mathfrak{B}\upharpoonright\tau(\mathfrak{A})$ is $\sim$- equivalent to the diagram expansion $\mathfrak{A}_A$ of $\mathfrak{A}$ (1.4).

Given an abstract embedding relation $\to$ we consider the problem of *existence* and *uniqueness* of a logic L such that $\underset{L}{\to} = \to$. In Section 2 we give criteria for the *uniqueness* of L: Theorem 2.3 states that if $\underset{L}{\to} = \to$ and L is compact, then L is uniquely determined by $\to$: thus in particular $L_{\omega\omega}$ is uniquely determined by its own embeddability relation $\precsim$; the same holds for the logic with the cofinality ω quantifier (2.4). Theorem 2.6 establishes the following: if $\sim$ is an abstract equivalence relation, then $\sim\, = \,\equiv_L$ for at most one logic L, provided the pair $(\sim, \sim^*)$ has JEP: the latter means that whenever $\mathfrak{A}\sim\mathfrak{B}$ there is $\mathfrak{N}$ with $\mathfrak{A}\to\mathfrak{N}\leftarrow\mathfrak{B}$,