The number of zeros of polynomials in valuation rings of complete discretely valued fields

by

A. Schinzel (Warszawa)

To honour the hundredth birthday of Wacław Sierpiński

Abstract. Let K be a field complete with respect to a discrete valuation v and let I be the valuation ring, P the valuation ideal, R the residue field I/P.

In this paper we consider the number of zeros of a polynomial $f \in I[x]$ in I and express it in terms of the number of solutions of suitable systems of equations in several variables in R provided $\operatorname{char} R > \deg f$ or $\operatorname{char} R = 0$. The equations are uniform with respect to K and v and thus if $F \in Z[x, t]$ the result implies that for p large enough the number of solutions of $F(x, p) = 0$ in p-adic integers equals the number of solutions of $F(x, t) = 0$ in formal power series in t over the finite field of p elements.

§ 1. Introduction. Let K be a field complete with respect to a discrete valuation v and let I be the valuation ring, P the valuation ideal, R the residue field I/P and p an element of P with $v(p) = 1$.

It is an easy extension of a result of Nagell ([2], p. 349, see also [3], Theorem 53) that the number of zeros of a polynomial $f \in I[x]$ with the discriminant $\operatorname{disc} f \neq 0$ equals the number of solutions of the congruence

$$f(x) \equiv 0 (\operatorname{mod} P^{2\delta+1}).$$

where $\delta = v(\operatorname{disc} f)$.

In this paper under the assumption $\operatorname{char} R > \deg f$ or $\operatorname{char} R = 0$ we express the number of zeros of f in I in terms of the number of solutions of suitable systems of equations in several variables in R. In order to formulate the result we set, for every polynomial $A \in K[y_1, \dots, y_l]$,

$$A = \sum_{\mu=1}^{m} a_\mu \prod_{\lambda=1}^{l} y_\lambda^{\alpha_{\mu\lambda}}, \quad \text{the vectors } [\alpha_{\mu 1}, \dots, \alpha_{\mu l}] \text{ distinct},$$

$$v(A) := \min_{\mu\leqslant m} v(a_\mu),$$

$$\mathscr{K}A := \begin{cases} p^{-v(A)}A & \text{if } A \neq 0, \\ 0 & \text{if } A = 0, \end{cases}$$

and if $A \in \boldsymbol{I}[y_1, \ldots, y_l]$

$$\mathscr{L}A := \sum_{\mu=1}^{m} \bar{a}_\mu \prod_{\lambda=1}^{l} y_\lambda^{\alpha_{\mu\lambda}},$$

where the bar is the residue map. Moreover, we put

$$\boldsymbol{N}_+ := \boldsymbol{N} \cup \{0, \infty\} = \boldsymbol{N}_0 \cup \{\infty\} = \{0, 1, \ldots, \infty\}.$$

Now we can state

THEOREM 1. *For every $m \in \boldsymbol{N}$ there exist a system of forms $R_i(\boldsymbol{a})$ $(i \leqslant i^*)$ and polynomials $S_{jkl}(\boldsymbol{a}, y_1, \ldots, y_l)$ $(j \leqslant j^*,\ k \leqslant k_j,\ l \leqslant l_{jk})$ with integral coefficients, a decomposition*

$$\boldsymbol{N}_+^{i^*} = \bigcup_{j=1}^{j^*} \boldsymbol{X}_j$$

and $\boldsymbol{N}_0$-valued functions $\sigma_{jkl}(\boldsymbol{v})$ defined on $\boldsymbol{X}_j$ with the following property:

If $\operatorname{char}\boldsymbol{R} = 0$ *or* $\operatorname{char}\boldsymbol{R} > m$,

$$f(x) = \sum_{\mu=0}^{m} a_\mu x^{m-\mu} \in \boldsymbol{K}[x], \quad f \neq 0,\ \boldsymbol{a} = [a_0, \ldots, a_m],$$

$$\boldsymbol{v} = [v(R_1(\boldsymbol{a})), \ldots, v(R_{i^*}(\boldsymbol{a}))] \in \boldsymbol{X}_j$$

and

$$\tilde{S}_{jkl}(y_1, \ldots, y_l) = \mathscr{L}\mathscr{K}S_{jkl}(\boldsymbol{a}, p^{\sigma_{jk1}(\boldsymbol{v})}y_1, \ldots, p^{\sigma_{jkl}(\boldsymbol{v})}y_l),$$

then

$$\operatorname{card}\{\xi \in \boldsymbol{I}\colon f(\xi) = 0\} = \sum_{k=1}^{k_j} \operatorname{card}\{[\eta_1, \eta_2, \ldots] \in \boldsymbol{R}^{l_{jk}}\colon \bigwedge_{l=1}^{l_{jk}} \tilde{S}_{jkl}(\eta_1, \ldots, \eta_l) = 0\}.$$

The polynomials R_i, S_{jkl}, the sets $\boldsymbol{X}_j$ and the functions σ_{jkl} do not depend on the field $\boldsymbol{K}$, the valuation v or the element p.

The calculation of R_i, S_{jkl} etc., possible in principle for every m, is trivial for $m = 1$ and $m = 2$. At the end of the paper we give the result of the calculation for $m = 3$ and some comments on the cases $m = 4$, $m = 6$.

Theorem 1 easily implies

THEOREM 2. *For every $m \in \boldsymbol{N}$ there exist $c_1(m) \in \boldsymbol{N}$ and $c_2(m) \in \boldsymbol{N}$ such that, if $F \in \boldsymbol{Z}[x, t]$ is of degree m in x with the sum of the absolute values of the coefficients equal to, say, $l(F)$, then for all primes p satisfying*

$$p > c_1(m)\, l(F)^{c_2(m)}$$

we have

$$\operatorname{card}\{\xi \in \boldsymbol{Z}_p\colon F(\xi, p) = 0\} = \operatorname{card}\{\xi \in \boldsymbol{F}_p[[t]]\colon F(\xi, t) = 0\},$$

where $\boldsymbol{Z}_p$ is the ring of p-adic integers and $\boldsymbol{F}_p$ the field of p elements.

For the understanding of both theorems it is important to note that the values of a polynomial with integral coefficients for arguments in a field of positive characteristic are again in this field since a positive integer n is to be interpreted as $1+1+\ldots+1$ (n times).

The equality asserted in Theorem 2 restricted to primes p greater than a suitable primitive recursive function of the coefficients of F follows from a result of P. Cohen [1] (Corollary to Theorem 5.1). His theorem (Theorem 5.1) implies also that in the more general situation of Theorem 1 the solvability of $f(x) = 0$ in $\boldsymbol{I}$ is decidable in terms of $\boldsymbol{R}$ provided $\operatorname{char}\boldsymbol{R}$ is either zero or greater than a bound depending on f.

The proof of Theorem 1 is rather complicated and much notation is used. In addition to those already introduced the following symbols are used throughout:

$$\boldsymbol{N}_- := \boldsymbol{N}_0 \cup \{-\infty\},$$

$\mathop{\boldsymbol{P}}\limits_{i=1}^{n} Y_i$ for the Cartesian product $Y_1 \times Y_2 \times \ldots \times Y_n$,

$$Y^n := \mathop{\boldsymbol{P}}\limits_{i=1}^{n} Y.$$

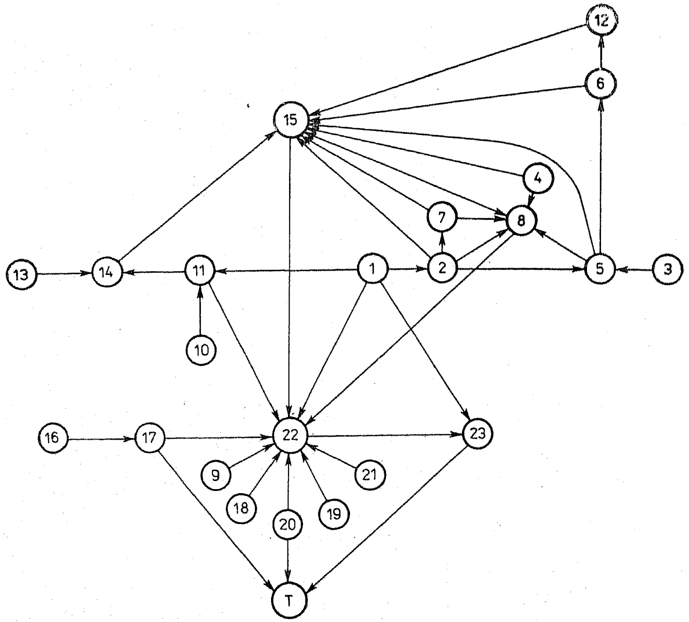
However, except for the sets $\{0, 1\}$, $\boldsymbol{N}$, $\boldsymbol{N}_0$, $\boldsymbol{N}_+$ and $\boldsymbol{N}_-$, no other set appears in the course of the proof with an exponent. Also the algebraic operation of raising into power with a simple (one letter) exponent is used rarely. Therefore, as a rule (with some exceptions), a simple superscript without parenthesis is to be understood as an index and not as an exponent. A superscript in a parenthesis means a differentiation.

For a given polynomial $f = \sum_{\mu=0}^{m} a_\mu x^{m-\mu}$, $*f$ is the vector $[a_0, \ldots, a_m]$. Thus $*f$ is determined by f up to a sequence of zeros preceding the leading coefficient. The length of the sequence will be clear from each context. Whenever possible without danger of confusion we shall write f instead of $*f$, also $f^{(n)}$ instead of $*f^{(n)}$ and $\bar{f}$ instead of $\mathscr{L}f$. Ordinary capital letters except Π, Σ and occasionally E denote polynomials in several variables, small bold face letters vectors, capital bold face letters (except $\boldsymbol{P}$) sets; script capital letters will denote operations; for two polynomials f, g $\operatorname{res}(f, g)$ is their resultant ([1]). Finally we accept the usual convention: $\deg 0 = -\infty$ and for a vector $\boldsymbol{a} = [a_0, \ldots, a_m]$ we set

$$\boldsymbol{a}^{(n)} = n!\left[0, \ldots, 0, \binom{m}{n}a_0, \ldots, \binom{n}{n}a_{m-n}\right].$$

For the convenience of the reader we give a flow chart of the proof of Theorem 1. The numbers denote lemmata, the arrows implications; T denotes the theorem.

([1]) res $(f, 0) = 1$ if $f = \text{const} \neq 0$ otherwise res $(f, 0) = 0$.



§ 2. Lemmata.

Definition 1. $C_l(m_1, m_2, \dots, m_k)$ is the class of all polynomials

$$A \in \boldsymbol{Z}[\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_k, y_1, y_2, \dots, y_l]$$

that are homogeneous in each vector of variables $\boldsymbol{x}_i = [x_{i0}, \dots, x_{im_i}]$ separately and isobaric in all the variables jointly, where the weight of x_{ij} is j and the weight of y_i is 1. The degree of A with respect to $\boldsymbol{x}_i$ is denoted by $\deg^i A$ and the common weight of all terms of A is denoted $w(A)$.

Lemma 1. *If* $A \in C_l(m_1, m_2, \dots, m_k)$, $C_i \in C_1(n_1, \dots, n_r)$ $(1 \leqslant i \leqslant k)$ *C_i depend formally on the same vectors of variables* $\boldsymbol{x}_1, \dots, \boldsymbol{x}_r$ *besides* $y_1 = y$ *and for all* i $\deg_y C_i \leqslant m_i$, *then for all vectors* $[p_1, \dots, p_k] \in N_0^k$ *we have*

$$B := A(*C_1^{(p_1)}, \dots, *C_k^{(p_k)}, y_1, \dots, y_l) \in C_l(n_1, \dots, n_r)$$

where C_i is differentiated with respect to y and $C_i^{(p_i)}$ treated as a polynomial in y. Moreover

$$\deg^q B = \sum_{i=1}^{k} \deg^i A \deg^q C_i \quad (1 \leqslant q \leqslant r), \tag{1}$$

$$w(B) = w(A) + \sum_{i=1}^{k} \deg^i A\big(w(C_i) - m_i - p_i\big). \tag{2}$$

Proof. Let us consider a typical monomial of B:

$$M(\boldsymbol{x}_1, \dots, \boldsymbol{x}_r, y_1, \dots, y_k) = m \prod_{i,s} c_{is}^{\alpha_{is}} \prod_{j=1}^{l} y_j^{\alpha_j},$$

where c_{is} is the coefficient of $x^{\deg_y C_i - s}$ in C_i and $m \neq 0$. Since $C_i \in C_1(n_1, \dots, n_r)$, we have $\deg^q c_{is} = \deg^q C_i$, $w(c_{is}) = w(C_i) - \deg_y C_i + s$. Since $A \in C_l(m_1, \dots, m_k)$, we have for each $i \leqslant k$

$$\sum_s \alpha_{is} = \deg^i A. \tag{3}$$

Hence for each $q \leqslant r$

$$\deg^q M = \sum_{i,s} \alpha_{is} \deg^q C_i = \sum_{i=1}^{k} \deg^i A \deg^q C_i,$$

which proves (1).

Now consider the weight of M. It equals

$$w(M) = \sum_{i,s} \alpha_{is}\big(w(C_i) - \deg_y C_i + s\big) + \sum_{j=1}^{l} \alpha_j.$$

The variable x_{is} occurs in $C_i^{(p_i)}$ in the coefficient of $y^{\deg_y C_i - p_i - s}$ rather than in that of $y^{m_i - s}$. Since $A \in C_l(m_1, \dots, m_k)$ we get

$$\sum_{i,s} \alpha_{is}(s + m_i + p_i - \deg_y C_i) + \sum_{j=1}^{l} \alpha_j = w(A).$$

Hence by (3)

$$w(M) = w(A) + \sum_{i,s} \alpha_{is}\big(w(C_i) - m_i - p_i\big) = w(A) + \sum_i \deg^i A\big(w(C_i) - m_i - p_i\big),$$

which proves (2).

Definition 2. For a field $\boldsymbol{L}$ and $\alpha \in \boldsymbol{L} \cup \{\infty\}$, let

$$\mathrm{sg}_{\boldsymbol{L}} \alpha := \begin{cases} 1 & \text{if } \alpha \neq 0, \\ 0 & \text{if } \alpha = 0. \end{cases}$$

Definition 3. For a given subset $\boldsymbol{M}$ of N_-^k $\Omega(\boldsymbol{M})$ is the class of all operations $\mathscr{A}$ on polynomials with coefficients in a field such that for every vector $[m_1, \dots, m_k] \in N_0^k$ there exist polynomials $A_i, B_j \in C_0(m_1, \dots, m_k)$, $C_j \in C_1(m_1, \dots, m_k)$ $(i \leqslant i_0,\ j \leqslant j_0)$ and a decomposition

$$\{0, 1\}^{i_0} = \bigcup_{j=1}^{j_0} \boldsymbol{S}_j$$

with the following property:

If $\boldsymbol{L}$ is a field, $f_i \in \boldsymbol{L}[x]$, $\deg f_i \leqslant m_i$ $(1 \leqslant i \leqslant k)$,

$$[\deg f_1, \dots, \deg f_k] \in \boldsymbol{M},$$

$$[\mathrm{sg}_{\boldsymbol{L}} A_1(f_1, \dots, f_k), \dots, \mathrm{sg}_{\boldsymbol{L}} A_i(f_1, \dots, f_k)] \in \boldsymbol{S}_j,$$

then $B_j(f_1, \ldots, f_k) \neq 0$ and

$$\mathscr{A}(f_1, \ldots, f_k) = \frac{C_j(f_1, \ldots, f_k, x)}{B_j(f_1, \ldots, f_k)}.$$

If for some integers $d_0, \ldots, d_k$ and all $[m_1, \ldots, m_k] \in N_0^k$, $j \leqslant j_0$ we have either $C_j = 0$ or

$$\deg^{\varkappa} C_j - \deg^{\varkappa} B_j = d_{\varkappa} \quad (1 \leqslant \varkappa \leqslant k)$$

and

$$w(C_j) - w(B_j) = d_0 + \sum_{\varkappa=1}^{k} d_{\varkappa} m_{\varkappa},$$

then we write

$$\mathscr{A} \in \Omega(\boldsymbol{M}; d_0, \ldots, d_k).$$

Remark. As an immediate consequence of Definition 2 we have

$$\Omega\Big(\bigcup_{\mu=1}^{\infty} \boldsymbol{M}_\mu\Big) = \bigcap_{\mu=1}^{\infty} \Omega(\boldsymbol{M}_\mu), \tag{4}$$

$$\Omega\Big(\bigcup_{\mu=1}^{\infty} \boldsymbol{M}_\mu; d_0, \ldots, d_k\Big) = \bigcap_{\mu=1}^{\infty} \Omega(\boldsymbol{M}_\mu; d_0, \ldots, d_k) \tag{5}$$

provided $\bigcup_{\mu=1}^{\infty} \boldsymbol{M}_\mu \subset \boldsymbol{N}_-^k$.

Lemma 2. *Let $\boldsymbol{M} \subset \boldsymbol{N}_-^k$, $\boldsymbol{M}_0 \subset \boldsymbol{N}_-^l$, $\mathscr{A}_\lambda \in \Omega(\boldsymbol{M})$ $(\lambda = 1, 2, \ldots, l)$, $\mathscr{A}_0 \in \Omega(\boldsymbol{M}_0)$ and assume that for every field $\boldsymbol{L}$ the condition*

$$f_i \in \boldsymbol{L}[x], \quad [\deg f_1, \ldots, \deg f_k] \in \boldsymbol{M}$$

implies

$$[\deg \mathscr{A}_1(f_1, \ldots, f_k), \ldots, \deg \mathscr{A}_l(f_1, \ldots, f_k)] \in \boldsymbol{M}_0.$$

Then

$$\mathscr{A}_0(\mathscr{A}_1, \ldots, \mathscr{A}_l) \in \Omega(\boldsymbol{M}).$$

Moreover, if $\mathscr{A}_\lambda \in \Omega(\boldsymbol{M}; d_{\lambda 0}, \ldots, d_{\lambda k})$ $(1 \leqslant \lambda \leqslant l)$, $\mathscr{A}_0 \in \Omega(\boldsymbol{M}_0; d_{00}, \ldots, d_{0l})$ then

$$\mathscr{A}_0(\mathscr{A}_1, \ldots, \mathscr{A}_l) \in \Omega\Big(\boldsymbol{M}; d_{00} + \sum_{\lambda=1}^{l} d_{0\lambda} d_{\lambda 0}, \sum_{\lambda=1}^{l} d_{0\lambda} d_{\lambda 1}, \ldots, \sum_{\lambda=1}^{l} d_{0\lambda} d_{\lambda k}\Big).$$

Proof. Take k nonnegative integers $m_1, \ldots, m_k$. Since $\mathscr{A}_\lambda \in \Omega(\boldsymbol{M})$ $(\lambda = 1, 2, \ldots, l)$, there exist polynomials $A_i^\lambda, B_j^\lambda \in \boldsymbol{C}_0(m_1, \ldots, m_k)$, $C_j^\lambda \in \boldsymbol{C}_1(m_1, \ldots, m_k)$ $(i \leqslant i_\lambda, j \leqslant j_\lambda)$ and a decomposition

$$\{0, 1\}^{i_\lambda} = \bigcup_{j=1}^{j_\lambda} \boldsymbol{S}_j^\lambda$$

with the following property:

(6) If $\boldsymbol{L}$ is a field, $f_i \in \boldsymbol{L}[x]$, $\deg f_i \leqslant m_i$ $(1 \leqslant i \leqslant k)$, $[\deg f_1, \ldots, \deg f_k] \in \boldsymbol{M}$ and

$$[\mathrm{sg}_{\boldsymbol{L}} A_1^\lambda(f_1, \ldots, f_k), \ldots, \mathrm{sg}_{\boldsymbol{L}} A_{i_\lambda}^\lambda(f_1, \ldots, f_k)] \in \boldsymbol{S}_j^\lambda \quad (j \leqslant j_\lambda)$$

then $B_j^\lambda(f_1, \ldots, f_k) \neq 0$ and

$$\mathscr{A}_\lambda(f_1, \ldots, f_k) = \frac{C_j^\lambda(f_1, \ldots, f_k, x)}{B_j^\lambda(f_1, \ldots, f_k)}.$$

Put $c_\lambda = \max\limits_{1 \leqslant j \leqslant j_\lambda} \deg_x C_j^\lambda$.

Since $\mathscr{A}_0 \in \Omega(\boldsymbol{M}_0)$, there exist polynomial $A_i^0, B_i^0 \in \boldsymbol{C}_0(c_1, \ldots, c_l)$, $C_i^0 \in \boldsymbol{C}_1(c_1, \ldots, c_l)$ and a decomposition

$$\{0, 1\}^{i_0} = \bigcup_{j=1}^{j_0} \boldsymbol{S}_j^0$$

with the following property:

(7) If $g_i \in \boldsymbol{L}[x]$, $\deg g_i \leqslant c_i$ $(1 \leqslant i \leqslant l)$, $[\deg g_1, \ldots, \deg g_l] \in \boldsymbol{M}_0$ and

$$[\mathrm{sg}_{\boldsymbol{L}} A_1^0(g_1, \ldots, g_l), \ldots, \mathrm{sg}_{\boldsymbol{L}} A_{i_0}^0(g_1, \ldots, g_l)] \in \boldsymbol{S}_j^0 \quad (j \leqslant j_0),$$

then $B_j^0(g_1, \ldots, g_l) \neq 0$ and

$$\mathscr{A}_0(g_1, \ldots, g_l) = \frac{C_j^0(g_1, \ldots, g_l, x)}{B_j^0(g_1, \ldots, g_l)}.$$

Let us order the Cartesian product $\mathop{\boldsymbol{P}}\limits_{\lambda=1}^{l} \{1, 2, \ldots, j_\lambda\}$ into a sequence, denote the νth term of this sequence by $[j_\nu^1, \ldots, j_\nu^l]$ $(1 \leqslant \nu \leqslant j_1 j_2 \ldots j_k = n)$ and take $m = \sum_{\lambda=1}^{l} i_\lambda$. Further, put

$$A_i = \begin{cases} A_\mu^\lambda & \text{if } i = \sum\limits_{\varkappa < \lambda} i_\varkappa + \mu,\ 1 \leqslant \mu \leqslant i_\lambda, \\ A_\varrho^0(*C_{j_\nu^1}^1, \ldots, *C_{j_\nu^l}^l) & \text{if } i - m = i_0(\nu - 1) + \varrho,\ 1 \leqslant \varrho \leqslant i_0; \end{cases} \tag{8}$$

and, if $j = j_0(\nu - 1) + \sigma$, $1 \leqslant \nu \leqslant n$, $1 \leqslant \sigma \leqslant j_0$,

$$B_j = B_\sigma^0(*C_{j_\nu^1}^1, \ldots, *C_{j_\nu^l}^l) \prod_{\lambda=1}^{l} (B_{j_\nu^\lambda}^\lambda)^{\deg^\lambda C_\sigma^0}, \tag{9}$$

$$C_j = C_\sigma^0(*C_{j_\nu^1}^1, \ldots, *C_{j_\nu^l}^l, x) \prod_{\lambda=1}^{l} (B_{j_\nu^\lambda}^\lambda)^{\deg^\lambda C_\sigma^0}, \tag{10}$$

$$\boldsymbol{S}_j = \mathop{\boldsymbol{P}}\limits_{\lambda=1}^{l} \boldsymbol{S}_{j_\nu^\lambda}^\lambda \times \{0, 1\}^{i_0(\nu-1)} \times \boldsymbol{S}_\sigma^0 \times \{0, 1\}^{i_0(n-\nu)}. \tag{11}$$

We have by Lemma 1 $A_i,\ B_j \in C_0(m_1, \dots, m_k)$, $C_j \in C_1(m_1, \dots, m_k)$ $(i \leqslant m+i_0 n,\ j \leqslant j_0 n)$; moreover, the sets S_j are disjoint and

$$\{0, 1\}^{m+i_0 n} = \bigcup_{j=1}^{j_0 n} S_j .$$

Assume now that $f_\varkappa \in L[x]$, $\deg f_\varkappa \leqslant m_\varkappa$ $(1 \leqslant \varkappa \leqslant k)$ and

(12) $$[\mathrm{sg}_L A_1(f_1, \dots, f_k), \dots, \mathrm{sg}_L A_{m+i_0 n}(f_1, \dots, f_k)] \in S_j ,$$

where $j = j_0(\nu-1)+\sigma$, $1 \leqslant \nu \leqslant n$, $1 \leqslant \sigma \leqslant j_0$. Then by the definition of A_i

$$[\mathrm{sg}_L A_1^\lambda(f_1 \dots, f_k), \dots, \mathrm{sg}_L A_{i_\lambda}^\lambda(f_1, \dots, f_k)] \in S^\lambda_{j^\lambda_\nu} .$$

Therefore by (6)

(13) $$B^\lambda_{j^\lambda_\nu}(f_1, \dots, f_k) \neq 0$$

and

(14) $$\mathscr{A}_\lambda(f_1, \dots, f_k) = \frac{C^\lambda_{j^\lambda_\nu}(f_1, \dots, f_k, x)}{B^\lambda_{j^\lambda_\nu}(f_1, \dots, f_k)} := g_{\lambda\nu}(x).$$

Now

(15) $$\deg g_{\lambda\nu} \leqslant \deg_x C^\lambda_{j^\lambda_\nu} \leqslant c_\lambda$$

and by the assumption of the lemma

$$[\deg g_{1\nu}, \dots, \deg g_{l\nu}] \in M_0 .$$

Moreover, since A_i^0 are homogeneous in each vector of variables separately,

$$\mathrm{sg}_L A_i^0(g_{1\nu}, \dots, g_{l\nu}) = \mathrm{sg}_L A_i^0(*C^1_{j^1_\nu}(f_1, \dots, f_k, x), \dots, *C^l_{j^l_\nu}(f_1, \dots, f_k, x))$$

and by (8), (11) and (12)

$$[\mathrm{sg}_L A_1^0(g_{1\nu}, \dots, g_{l\nu}), \dots, \mathrm{sg}_L A_{i_0}^0(g_{1\nu}, \dots, g_{l\nu})] \in S^0_\sigma .$$

Now by (7) and (14)

(16) $$B^0_\sigma(g_{1\nu}, \dots, g_{l\nu}) \neq 0$$

and

(17) $$\mathscr{A}_0(g_{1\nu}, \dots, g_{l\nu}) = \frac{C^0_\sigma(g_{1\nu}, \dots, g_{l\nu}, x)}{B^0_\sigma(g_{1\nu}, \dots, g_{l\nu})} .$$

Since B^0_σ and C^0_σ are homogeneous in each vector of variables separately

$$B^0_\sigma(g_{1\nu}, \dots, g_{l\nu}) = B^0_\sigma(*C^1_{j^1_\nu}(f_1, \dots, f_k, x), \dots, *C^l_{j^l_\nu}(f_1, \dots, f_k; x)) \prod_{\lambda=1}^{l} B^\lambda_{j^\lambda_\nu}(f_1, \dots, f_k)^{-\deg^\lambda B^0_\sigma} ,$$

$$C^0_\sigma(g_{1\nu}, \dots, g_{l\nu}) = C^0_\sigma(*C^1_{j^1_\nu}(f_1, \dots, f_k, x), \dots, *C^l_{j^l_\nu}(f_1, \dots, f_k, x)) \prod_{\lambda=1}^{l} B^\lambda_{j^\lambda_\nu}(f_1, \dots, f_k)^{-\deg^\lambda C^0_\sigma}$$

and inequality (16) implies that

$$B^0_\sigma(*C^1_{j^1_\nu}(f_1, \dots, f_k, x), \dots, *C^l_{j^l_\nu}(f_1, \dots, f_k, x)) \neq 0 .$$

Hence also by (9) and (13)

$$B_j(f_1, \dots, f_k) \neq 0$$

and by (10), (14) and (17)

$$\mathscr{A}_0(\mathscr{A}_1(f_1, \dots, f_k), \dots, \mathscr{A}_l(f_1, \dots, f_k)) = \frac{C_j(f_1, \dots, f_k, x)}{B_j(f_1, \dots, f_k)} .$$

This completes the proof of the first part of the lemma. In order to prove the second part we use formulae (9) and (10) and the second part of Lemma 1. If $j = j_0(\nu-1)+\sigma$, $1 \leqslant \nu \leqslant n$, $1 \leqslant \sigma \leqslant j_0$ we have for each $\varkappa \leqslant k$

$$\begin{aligned}\deg^\varkappa C_j - \deg^\varkappa B_j &= \sum_{\lambda=1}^{l} (\deg^\lambda C^0_\sigma - \deg^\lambda B^0_\sigma) \deg^\varkappa C^\lambda_{j^\lambda_\nu} + \sum_{\lambda=1}^{l} (\deg^\lambda B^0_\sigma - \deg^\lambda C^0_\sigma) \deg^\varkappa B^\lambda_{j^\lambda_\nu} \\ &= \sum_{\lambda=1}^{l} (\deg^\lambda C^0_\sigma - \deg^\lambda B^0_\sigma)(\deg^\varkappa C^\lambda_{j^\lambda_\nu} - \deg^\varkappa B^\lambda_{j^\lambda_\nu}) = \sum_{\lambda=1}^{l} d_{0\lambda} d_{\lambda\varkappa} .\end{aligned}$$

Moreover

$$\begin{aligned} w(C_j) - w(B_j) &= w(C^0_\sigma) - w(B^0_\sigma) + \sum_{\lambda=1}^{l} (\deg^\lambda C^0_\sigma - \deg^\lambda B^0_\sigma)(w(C^\lambda_{j^\lambda_\nu}) - c_\lambda) + \\ &\quad + \sum_{\lambda=1}^{l} (\deg^\lambda B^0_\sigma - \deg^\lambda C^0_\sigma) w(B^\lambda_{j^\lambda_\nu}) \\ &= d_{00} + \sum_{\lambda=1}^{l} (\deg^\lambda C^0_\sigma - \deg^\lambda B^0_\sigma)(w(C^\lambda_{j^\lambda_\nu}) - w(B^\lambda_{j^\lambda_\nu})) \\ &= d_{00} + \sum_{\lambda=1}^{l} d_{0\lambda}(d_{\lambda 0} + \sum_{\varkappa=1}^{k} d_{\lambda\varkappa} m_\varkappa) \\ &= d_{00} + \sum_{\lambda=1}^{l} d_{0\lambda} d_{\lambda 0} + \sum_{\varkappa=1}^{k} m_\varkappa (\sum_{\lambda=1}^{l} d_{0\lambda} d_{\lambda\varkappa}) .\end{aligned}$$

LEMMA 3. *If* $\mathscr{A}_\lambda \in \Omega(M, d_0, \dots, d_k)$ $(\lambda = 1, \dots, l)$, *then*

$$\mathscr{A}_1 + \mathscr{A}_2 + \dots + \mathscr{A}_l \in \Omega(M, d_0, \dots, d_k) .$$

Proof — similar to that of Lemma 2. The crucial formulae (8)–(11) are replaced by

$$A_i = A^\lambda_\mu \quad \text{for} \quad i = \sum_{\varkappa<\lambda} i_\varkappa + \mu,\ 1 \leqslant \mu \leqslant i_\lambda,$$

$$B_j = \prod_{\lambda=1}^{l} B^\lambda_{j^\lambda_\nu}, \quad C_j = \Big(\sum_{\lambda=1}^{l} \frac{C^\lambda_{j^\lambda_\nu}}{B^\lambda_{j^\lambda_\nu}}\Big) \prod_{\lambda=1}^{l} B^\lambda_{j^\lambda_\nu}, \quad S_j = \mathop{\boldsymbol{P}}_{\lambda=1}^{l} S^\lambda_{j^\lambda_\nu}.$$

Since the sum of homogeneous (resp. isobaric) polynomials of the same degree (resp. weight) is a homogeneous (resp. isobaric) polynomial of the said degree (resp. weight), we have

$$\deg^\varkappa C_j - \deg^\varkappa B_j = d_\varkappa,$$

$$w(C_j) - w(B_j) = d_0 + \sum_{\varkappa=1}^{k} d_\varkappa m_\varkappa$$

and

$$\mathscr{A}_1 + \mathscr{A}_2 + \ldots + \mathscr{A}_l \in \Omega(\boldsymbol{M}, d_0, \ldots, d_k).$$

LEMMA 4. *For every k the operation: $[f_1, \ldots, f_k] \to f_1 \ldots f_k$ belongs to $\Omega(N^k_-;\ 0, 1, \ldots, 1)$.*

Proof. It is enough to take in Definition 2, for arbitrary $m_1, \ldots, m_k$, $i_0 = 0$, $j_0 = 1$, $B_1 = 1$

$$C_1(x_1, \ldots, x_k, x) = \prod_{i=1}^{k} \sum_{j=0}^{m_i} x_{ij} x^{m_i - j}, \quad \text{where} \quad x_i = [x_{i0}, \ldots, x_{im_i}].$$

We have

$$\deg^i C_1 - \deg^i B_1 = 1, \quad w(C_1) - w(B_1) = \sum_{i=1}^{k} m_i.$$

LEMMA 5. *The operations of taking the partial quotient $\mathscr{Q}(f, g)$ and the remainder $\mathscr{R}(f, g)$ from the division of f by g belong to $\Omega(N_- \times N_0;\ 0, 1, -1)$ and $\Omega(N_- \times N_0;\ 0, 1, 0)$ respectively.*

Proof. Let us consider the following operations:

$$\mathscr{A}_1(f, g) = \begin{cases} \dfrac{a}{b} x^{\deg f - \deg g}, & \text{where } a, b \text{ are the leading coefficients of } f, g \\ & \text{respectively if } \deg f \geqslant \deg g, \\ 0 & \text{if } \deg f < \deg g; \end{cases}$$

$$\mathscr{A}_2(f, g) = f - \mathscr{A}_1(f, g) g.$$

It is clear that $\mathscr{A}_1 \in \Omega(N_- \times N_0;\ 0, 1, -1)$, $\mathscr{A}_2 \in \Omega(N_- \times N_0;\ 0, 1, 0)$. We take $M_\mu = \{[d, e] \in N_- \times N_0\colon d < \mu\}$ $(\mu = 0, 1, 2, \ldots)$,

$$M^1_\mu = \{[d, e] \in M\colon d \geqslant e\}, \quad M^2_\mu = \{[d, e] \in M\colon d < e\},$$

and we shall prove by induction on μ that

$$\mathscr{Q} \in \Omega(M_\mu;\ 0, 1, -1), \quad \mathscr{R} \in \Omega(M_\mu;\ 0, 1, 0).$$

For $\mu = 0$, $\mathscr{Q}(f, g) = \mathscr{R}(f, g) = 0$; hence the statement is true. Assume now that it is true for some μ. We have

$$\mathscr{Q}(f, g) = \mathscr{A}_1(f, g) + \mathscr{Q}(\mathscr{A}_2(f, g), g),$$

(18)

$$\mathscr{R}(f, g) = \mathscr{R}(\mathscr{A}_2(f, g), g).$$

Denote by $\mathscr{I}_2$ the operation $[f, g] \to g$. Clearly $\mathscr{I}_2 \in \Omega(\boldsymbol{M};\ 0, 0, 1)$. Moreover, if

$$f, g \in \boldsymbol{L}[x], \quad [\deg f, \deg g] \in \boldsymbol{M}^1_{\mu+1},$$

then

$$[\deg \mathscr{A}_2(f, g), \deg \mathscr{I}_2(f, g)] \in \boldsymbol{M}_\mu$$

and by the inductive assumption

$$\mathscr{Q} \in \Omega(M_\mu;\ 0, 1, -1), \quad \mathscr{R} \in \Omega(M_\mu;\ 0, 1, 0).$$

By Lemma 2

$$\mathscr{Q}(\mathscr{A}_2, \mathscr{I}_2) \in \Omega(\boldsymbol{M}^1_{\mu+1};\ 0, 1, -1), \quad \mathscr{R}(\mathscr{A}_2, \mathscr{I}_2) \in \Omega(\boldsymbol{M}^1_{\mu+1};\ 0, 1, 0).$$

By Lemma 3

$$\mathscr{A}_1 + \mathscr{Q}(\mathscr{A}_2, \mathscr{I}_2) \in \Omega(\boldsymbol{M}^1_{\mu+1};\ 0, 1, -1)$$

and in virtue of (18)

$$\mathscr{Q} \in \Omega(\boldsymbol{M}^1_{\mu+1};\ 0, 1, -1), \quad \mathscr{R} \in \Omega(\boldsymbol{M}^1_{\mu+1};\ 0, 1, 0).$$

On the other hand, if $[\deg f, \deg g] \in \boldsymbol{M}^2_{\mu+1}$ we have

$$\mathscr{Q}(f, g) = 0, \quad \mathscr{R}(f, g) = f;$$

hence

$$\mathscr{Q} \in \Omega(\boldsymbol{M}^2_{\mu+1};\ 0, 1, -1), \quad \mathscr{R} \in \Omega(\boldsymbol{M}^2_{\mu+1};\ 0, 1, 0).$$

Since $\boldsymbol{M}_{\mu+1} = \boldsymbol{M}^1_{\mu+1} \cup \boldsymbol{M}^2_{\mu+1}$, the inductive assertion follows from (5). Another application of (5) gives the lemma.

DEFINITION 4. For two polynomials f, g we set

$$\mathscr{E}_0(f, g) := f, \quad \mathscr{E}_1(f, g) := g,$$

$$\mathscr{E}_{k+1}(f, g) := \begin{cases} \mathscr{E}_{k-1}(f, g) & \text{if } \mathscr{E}_k(f, g) = 0, \\ \mathscr{R}(\mathscr{E}_{k-1}(f, g), \mathscr{E}_k(f, g)) & \text{if } \mathscr{E}_k(f, g) \neq 0. \end{cases}$$

LEMMA 6. *$\mathscr{E}_k \in \Omega(N^2_-;\ 0, 1, 0)$ if k is even and $\mathscr{E}_k \in \Omega(N^2_-;\ 0, 0, 1)$ if k is odd.*

Proof — by induction on k. For $k = 0$ or 1 the assertion is obvious. For $k = 2$ we put $\boldsymbol{M}_1 = N_- \times \{-\infty\}$, $\boldsymbol{M}_2 = N_- \times N_0$. We have $\mathscr{E}_2 \in \Omega(\boldsymbol{M}_1;\ 0, 1, 0)$ and by Lemma 5 $\mathscr{E}_2 \in \Omega(\boldsymbol{M}_2;\ 0, 1, 0)$, hence by (5) $\mathscr{E}_2 \in \Omega(N^2_-;\ 0, 1, 0)$. Assume now that the lemma is true for all $k \leqslant l$ $(l \geqslant 2)$. Then

$$\mathscr{E}_{l+1} = \mathscr{E}_2(\mathscr{E}_{l-1}, \mathscr{E}_l)$$

and by Lemma 2

$$\mathscr{E}_{l+1} \in \begin{cases} \Omega(N_-^2;\ 0,1,0) & \text{if } l \text{ is odd}, \\ \Omega(N_-^2;\ 0,0,1) & \text{if } l \text{ is even}. \end{cases}$$

LEMMA 7. *For every* $k \geqslant 1$ *the operation on polynomials*

$$\mathscr{D}_k\colon [f_1, \ldots, f_k] \to \begin{cases} 0 & \textit{if } f_1 = f_2 = \ldots = f_k = 0, \\ (f_1, \ldots, f_k) & \textit{otherwise} \end{cases}$$

belongs to $\Omega(N_-^k)$. *In the case* $k = 1, f_1 \neq 0$, *by* (f_1) *we mean* f_1 *divided by its leading coefficient.*

Proof — by induction on k. For $k = 1$ we take $M_0 = \{-\infty\}$, $M_\mu = \{\mu - 1\}$ $(\mu = 1, 2, \ldots)$. Clearly $\mathscr{D}_1 \in \Omega(M_\mu)$ for all μ and by (4) $\mathscr{D}_1 \in \Omega(\bigcup_{\mu=0}^{\infty} M_\mu) = \Omega(N_-)$.

For $k = 2$ we take $M_0 = N_-^2 \setminus N_0^2$,

$$M_\mu = N \times \{-\infty, 0, 1, \ldots, \mu\} \quad (\mu = 1, 2, \ldots).$$

Clearly $\mathscr{D}_2 \in \Omega(M_0)$ and if $[\deg f, \deg g] \in M_\mu$ we have

$$\mathscr{D}_2(f, g) = \mathscr{D}_2(\mathscr{E}_{\mu+1}(f, g), \mathscr{E}_{\mu+2}(f, g)),$$

$$[\deg \mathscr{E}_{\mu+1}(f, g), \deg \mathscr{E}_{\mu+2}(f, g)] \in M_0.$$

It follows from Lemma 2 that $\mathscr{D}_2 \in \Omega(M_\mu)$ and from (4) that $\mathscr{D}_2 \in \Omega(N_-^2)$.

Assume now that the lemma is true for the operations $\mathscr{D}_{k-1}$ $(k \geqslant 3)$. We have

$$\mathscr{D}_k(f_1, \ldots, f_k) = \mathscr{D}_2(\mathscr{D}_{k-1}(f_1, \ldots, f_{k-1}), \mathscr{D}_{k-1}(f_2, \ldots, f_k))$$

and the lemma follows by an application of Lemma 2 (cf. [4]).

LEMMA 8. *For every* $n \geqslant 1$ *the operation on polynomials*

$$\mathscr{O}_n(f) := \begin{cases} \dfrac{(f, \ldots, f^{(n-1)})(f, \ldots, f^{(n+1)})}{(f, \ldots, f^{(n)})^2} & \textit{if } f \neq 0, \\ 0 & \textit{if } f = 0 \end{cases}$$

belongs to $\Omega(N_-)$.

Proof. The operation $f \to f^{(n)}$ is in $\Omega(N_0)$ for every n. Hence by Lemmata 2 and 7 the operation $f \to (f, \ldots, f^{(n)})$ is in $\Omega(N_0)$, and by Lemmata 2 and 4 the operations

$$f \to (f, \ldots, f^{(n-1)})(f, \ldots, f^{(n-1)})$$

and

$$f \to (f, \ldots, f^{(n)})^2$$

are in $\Omega(N_0^2)$. Moreover, if $\deg f \in N_0$, then $\deg (f, \ldots, f^{(n)}) \in N_0$. Hence by Lemmata 2 and 5 the operation

$$f \to \mathscr{Q}((f, \ldots, f^{(n-1)})(f, \ldots, f^{(n+1)}), (f, \ldots, f^{(n)})^2)$$

is in $\Omega(N_0)$. We proceed to show that

$$(19) \qquad (f, \ldots, f^{(n)})^2 \mid (f, \ldots, f^{(n-1)})(f, \ldots, f^{(n+1)}).$$

Indeed, let L be the coefficients field of f, ξ an element of $\hat{L}$, the algebraic closure of L, and

$$v_k = \operatorname{ord}_{x-\xi} f^{(k)}(x).$$

It is enough to show that for every ξ

$$\min\{v_0, \ldots, v_{n-1}\} + \min\{v_0, \ldots, v_{n+1}\} \geqslant 2 \min\{v_0, \ldots, v_n\}.$$

Clearly $v_{n+1} \geqslant v_n - 1$; hence the above inequality holds unless

$$(20) \qquad v_n = \min\{v_0, \ldots, v_{n-1}\} \quad \text{and} \quad v_{n+1} = v_n - 1.$$

However, conditions (20) are impossible. They imply

$v_n = v_k$ for some $k \leqslant n$ and either $\operatorname{char} L = 0$ or
$v_n \not\equiv 0 \bmod \operatorname{char} L$.

But then $v_{k+1} = v_k - 1$, $k < n-1$ and $v_k \neq \min\{v_0, \ldots, v_{n-1}\}$, a contradiction.

Thus (19) holds,

$$\mathscr{Q}((f, \ldots, f^{(n-1)})(f, \ldots, f^{(n+1)}), (f, \ldots, f^{(n)})^2) = \mathscr{O}_n(f)$$

and $\mathscr{O}_n \in \Omega(N_0)$. Since clearly $\mathscr{O}_n \in \Omega(\{-\infty\})$, we get by (4) $\mathscr{O}_n \in \Omega(N_-)$.

LEMMA 9. *For every field* L *and every polynomial* $f \in L[x]$ *satisfying* $f \neq 0$ *and* $\operatorname{char} L = 0$ *or* $\deg f < \operatorname{char} L$ *we have*

$$\mathscr{O}_n(f) = \prod_{(x-\xi)^n \| f(x),\ \xi \in \hat{L}} (x - \xi)$$

where $\hat{L}$ *is the algebraic closure of* L.

Proof. Let

$$f(x) = \text{const} \prod_{\xi \in \hat{L}} (x - \xi)^{\alpha(\xi)}.$$

Then

$$(f, \ldots, f^{(k)}) = \prod_{\xi \in \hat{L}} (x - \xi)^{\max\{\alpha(\xi) - k, 0\}}.$$

Since

$$\max\{\alpha(\xi) - n + 1, 0\} + \max\{\alpha(\xi) - n - 1, 0\} - 2\max\{\alpha(\xi) - n, 0\} = \begin{cases} 1 & \text{if } \alpha(\xi) = n, \\ 0 & \text{otherwise}, \end{cases}$$

we get

$$\mathscr{O}_n(f) = \prod_{\alpha(\xi) = n} (x - \xi).$$

LEMMA 10. *Let* $\tilde{K}$ *be a field with a discrete valuation* $\tilde{v}$, $\tilde{I}$ *its valuation ring,* $\tilde{R}$ *its residue field and the bar the residue map. If* A *is a polynomial with integral coefficients,* $h_1, \ldots, h_l \in \tilde{I}[x]$ *and* $A(h_1, \ldots, h_l)$ *is defined, then*

$$\operatorname{sg}_{\tilde{R}} A(\bar{h}_1, \ldots, \bar{h}_l) = 1 - \operatorname{sg}_Q \tilde{v}(A(h_1, \ldots, h_l)).$$

Proof. If $\tilde{v}(A(\boldsymbol{h}_1, \ldots, \boldsymbol{h}_l)) > 0$ we have

$$\overline{A(\boldsymbol{h}_1, \ldots, \boldsymbol{h}_l)} = A(\bar{\boldsymbol{h}}_1, \ldots, \bar{\boldsymbol{h}}_l) = 0$$

and

$$\operatorname{sg}_{\tilde{\mathbf{R}}} A(\bar{\boldsymbol{h}}_1, \ldots, \bar{\boldsymbol{h}}_l) = 0 = 1 - \operatorname{sg}_{\mathbf{Q}} \tilde{v}(A(\boldsymbol{h}_1, \ldots, \boldsymbol{h}_l)).$$

If $\tilde{v}(A(\boldsymbol{h}_1, \ldots, \boldsymbol{h}_l)) = 0$ we have

$$\overline{A(\boldsymbol{h}_1, \ldots, \boldsymbol{h}_l)} = A(\bar{\boldsymbol{h}}_1, \ldots, \bar{\boldsymbol{h}}_l) \neq 0$$

and

$$\operatorname{sg}_{\tilde{\mathbf{R}}} A(\bar{\boldsymbol{h}}_1, \ldots, \bar{\boldsymbol{h}}_l) = 1 = 1 - \operatorname{sg}_{\mathbf{Q}} \tilde{v}(A(\boldsymbol{h}_1, \ldots, \boldsymbol{h}_l)).$$

DEFINITION 5. For a subset M of N_-^k, $\Omega^*(M)$ is the class of all operations $\mathscr{B}$ on polynomials with coefficients in a valuation ring such that for every vector $[m_1, \ldots, m_k] \in N_0^k$ there exist polynomials $F_i, G_j \in C_0(m_1, \ldots, m_k)$ and

$$H_j \in C_1(m_1, \ldots, m_k) \quad (i \leqslant i_0, j \leqslant j_0)$$

and a decomposition

$$N_+^{i_0} = \bigcup_{j=1}^{j_0} T_j$$

with the following property.

If $\tilde{\boldsymbol{K}}$ is a field with a discrete valuation $\tilde{v}$, $\tilde{\boldsymbol{I}}$ its valuation ring and $\tilde{\mathscr{L}}$ the residue map for polynomials over $\tilde{\boldsymbol{I}}$, $f_\varkappa \in \boldsymbol{I}[x]$, $\deg f_\varkappa \leqslant m_\varkappa$ $(1 \leqslant \varkappa \leqslant k)$, $[\deg f_1, \ldots, \deg f_k] \in M$, $[\tilde{v}(F_1(f_1, \ldots, f_k)), \ldots, \tilde{v}(F_{i_0}(f_1, \ldots, f_k))] \in T_j$ then

$$G_j(f_1, \ldots, f_k) \neq 0, \quad \frac{H_j(f_1, \ldots, f_k, x)}{G_j(f_1, \ldots, f_k)} \in \tilde{I}]x]$$

and

$$\mathscr{B}(f_1, \ldots, f_k) = \tilde{\mathscr{L}} \frac{H_j(f_1, \ldots, f_k, x)}{G_j(f_1, \ldots, f_k)}.$$

LEMMA 11. *If $\mathscr{B}_\lambda \in \Omega^*(M)$ $(\lambda = 1, \ldots, l)$, $\mathscr{A} \in \Omega(N_-^l)$ then*

$$\mathscr{A}(\mathscr{B}_1, \ldots, \mathscr{B}_l) \in \Omega^*(M).$$

Proof. Let $M \subset N^k$ and take $[m_1, \ldots, m_k] \in N_0^k$. Since $\mathscr{B}_\lambda \in \Omega^*(M)$, there exist polynomials $F_i^\lambda, G_j^\lambda \in C_0(m_1, \ldots, m_k)$, $H_j^\lambda \in C_1(m_1, \ldots, m_k)$ $(1 \leqslant i \leqslant i_\lambda, 1 \leqslant j \leqslant j_\lambda)$ and a decomposition

$$N_+^{i_\lambda} = \bigcup_{j=1}^{j_\lambda} T_j^\lambda$$

with the following property.

If, in the notation of Definition 5, $f_\varkappa \in \boldsymbol{K}[x]$, $\deg f_\varkappa \leqslant m_\varkappa$ $(\varkappa \leqslant k)$, $[\deg f_1, \ldots$ $\ldots, \deg f_k] \in M$,

$$[\tilde{v}(F_1^\lambda(f_1, \ldots, f_k)), \ldots, \tilde{v}(F_{i_\lambda}^\lambda(f_1, \ldots, f_k))] \in T_j^\lambda,$$

then

$$G_j^\lambda(f_1, \ldots, f_k) \neq 0, \quad \frac{H_j^\lambda(f_1, \ldots, f_k, x)}{G_j^\lambda(f_1, \ldots, f_k)} \in \tilde{\boldsymbol{I}}[x] \tag{21}$$

and

$$\mathscr{B}_j(f_1, \ldots, f_k) = \tilde{\mathscr{L}} \frac{H_j^\lambda(f_1, \ldots, f_k, x)}{G_j^\lambda(f_1, \ldots, f_k)}.$$

Let $\chi_\lambda = \max_{j \leqslant j_\lambda} \deg_x H_j^\lambda$. Since $\mathscr{A} \in \Omega(N_-^l)$, there exist polynomials

$$A_i, B_j \in C_0(\chi_1, \ldots, \chi_l), \quad C_j \in C_1(\chi_1, \ldots, \chi_l) \quad (i \leqslant i_0, j \leqslant j_0)$$

and a decomposition

$$\{0, 1\}^{i_0} = \bigcup_{j=1}^{j_0} \boldsymbol{S}_j$$

with the following property:

(22) If $h_1, \ldots, h_l \in \tilde{\boldsymbol{R}}[x]$, $\deg h_\lambda \leqslant \chi_\lambda$ $(1 \leqslant \lambda \leqslant l)$ and $[\operatorname{sg}_{\tilde{\mathbf{R}}} A_1(\boldsymbol{h}_1, \ldots, \boldsymbol{h}_l), \ldots, \operatorname{sg}_{\tilde{\mathbf{R}}} A_{i_0}(\boldsymbol{h}_1, \ldots, \boldsymbol{h}_l)] \in \boldsymbol{S}_j$, then $B_j(\boldsymbol{h}_1, \ldots, \boldsymbol{h}_l) \neq 0$ and

$$\mathscr{A}(h_1, \ldots, h_l) = \frac{C_j(\boldsymbol{h}_1, \ldots, \boldsymbol{h}_l, x)}{B_j(\boldsymbol{h}_1, \ldots, \boldsymbol{h}_l)}.$$

Let us order the Cartesian product $\mathop{\boldsymbol{P}}_{\lambda=1}^{l} \{1, 2, \ldots, j_\lambda\}$ into a sequence, call the νth term of this sequence $[j_\nu^1, \ldots, j_\nu^l]$ $(1 \leqslant \nu \leqslant j_1 \ldots j_l = n)$ and take $m = \sum_{\lambda=1}^{l} i_\lambda$,

$$F_i = \begin{cases} F_\mu^\lambda & \text{if } i = \sum_{\varkappa < \lambda} i_\varkappa + \mu;\ 0 < \mu \leqslant i_\lambda, \\ A_\varrho(\boldsymbol{H}_{j_\nu^1}^1, \ldots, \boldsymbol{H}_{j_\nu^l}^l) & \text{if } i - m = 2i_0(\nu - 1) + 2\varrho - 1,\ 1 \leqslant \nu \leqslant n,\ 1 \leqslant \varrho \leqslant i_0, \\ \prod_{\lambda=1}^{l} (G_{j_\nu^\lambda}^\lambda)^{\deg^\lambda A_\varrho} & \text{if } i - m = 2i_0(\nu - 1) + 2\varrho,\ 1 \leqslant \nu \leqslant n,\ 1 \leqslant \varrho \leqslant i_0; \end{cases} \tag{23}$$

moreover, if $j = j_0(\nu - 1) + \sigma$, $1 \leqslant \nu \leqslant n$, $1 \leqslant \sigma \leqslant j_0$

$$\begin{aligned} G_j &= B_\sigma(\boldsymbol{H}_{j_\nu^1}^1, \ldots, \boldsymbol{H}_{j_\nu^l}^l) \prod_{\lambda=1}^{l} (G_{j_\nu^\lambda})^{\deg \lambda C_\sigma}, \\ H_j &= C_\sigma(\boldsymbol{H}_{j_\nu^1}^1, \ldots, \boldsymbol{H}_{j_\nu^l}^l, x) \prod_{\lambda=1}^{l} (G_{j_\nu^\lambda})^{\deg^\lambda B_\sigma}, \end{aligned} \tag{24}$$

$$T_j = \mathop{\boldsymbol{P}}_{\lambda=1}^{l} T_{j_\nu^\lambda}^\lambda \times N_+^{2i_0(\nu-1)} \times \tau^{-1}(S_\sigma) \times N_+^{2i_0(n-\nu)}, \tag{25}$$

where the transformation $\tau\colon N_+^{2i_0} \to \{0, 1\}^{i_0}$ is given by the formula

$$(26)\qquad \tau(n_1, \ldots, n_{2i_0}) = \begin{cases} [1-\operatorname{sg}_{\mathbf{Q}}(n_1-n_2),\ 1-\operatorname{sg}_{\mathbf{Q}}(n_3-n_4), \ldots, 1-\operatorname{sg}_{\mathbf{Q}}(n_{2i_0-1}-n_{2i_0})] \\ \qquad\qquad \text{if } \sum_{\varrho=1}^{i_0} n_{2\varrho} < \infty, \\ [0, \ldots, 0] \quad \text{if } \sum_{\varrho=1}^{i_0} n_{2\varrho} = \infty . \end{cases}$$

We have $F_i, G_j \in \boldsymbol{C}_0(m_1, \ldots, m_k)$ and $H_j \in \boldsymbol{C}_1(m_1, \ldots, m_k)$ in virtue of Lemma 1; moreover, the sets $\boldsymbol{T}_j$ are disjoint and

$$N_+^{m+2i_0n} = \bigcup_{j=1}^{j_0n} \boldsymbol{T}_j .$$

Assume now $f_\varkappa \in \tilde{K}[x]$, $\deg f_\varkappa \leqslant m_\varkappa$ $(1 \leqslant \varkappa \leqslant k)$, $[\deg f_1, \ldots, \deg f_k] \in \boldsymbol{M}$ and

$$(27)\qquad [\tilde{v}(F_1(f_1, \ldots, f_k)), \ldots, \tilde{v}(F_{m+2i_0n}(f_1, \ldots, f_k))] \in \boldsymbol{T}_j ,$$

where $j = j_0(\nu-1)+\sigma$, $1 \leqslant \nu \leqslant n$, $1 \leqslant \sigma \leqslant j_0$. Then by (23) and (25)

$$[\tilde{v}(F_1^\lambda(f_1, \ldots, f_k)), \ldots, \tilde{v}(F_{i_\lambda}^\lambda(f_1, \ldots, f_k))] \in \boldsymbol{T}_{j_\nu^\lambda}^\lambda \qquad (1 \leqslant \lambda \leqslant l).$$

Therefore, by (21)

$$G_{j_\nu^\lambda}^\lambda(f_1, \ldots, f_k) \neq 0, \qquad \frac{H_{j_\nu^\lambda}^\lambda(f_1, \ldots, f_k, x)}{G_{j_\nu^\lambda}^\lambda(f_1, \ldots, f_k)} := h_{\lambda\nu} \in \tilde{I}[x]$$

and

$$(28)\qquad \mathscr{B}_\lambda(f_1, \ldots, f_k) = \mathscr{Z}\, \frac{H_{j_\nu^\lambda}^\lambda(f_1, \ldots, f_k, x)}{G_{j_\nu^\lambda}^\lambda(f_1, \ldots, f_k)} = \bar{h}_{\lambda\nu} .$$

Now $\deg \bar{h}_{\lambda\nu} \leqslant \deg_x H_{j_\nu^\lambda}^\lambda \leqslant \chi_\lambda$. Moreover, since A_ϱ are homogeneous in each vector of variables separately, we have by (23) for each $\varrho \leqslant i_0$

$$\begin{aligned} &\tilde{v}(A_\varrho(\boldsymbol{h}_{1\nu}, \ldots, \boldsymbol{h}_{l\nu})) \\ &= \tilde{v}(A_\varrho(*H_{j_\nu^1}^1(f_1, \ldots, f_k, x), \ldots, *H_{j_\nu^l}^l(f_1, \ldots, f_k, x))) - \sum_{\lambda=1}^{l} \deg^\lambda A_\varrho\, \tilde{v}(G_{j_\nu^\lambda}^\lambda(f_1, \ldots, f_k)) \\ &= \tilde{v}(F_{m+2i_0(\nu-1)+2\varrho-1}(f_1, \ldots, f_k)) - \tilde{v}(F_{m+2i_0(\nu-1)+2\varrho}(f_1, \ldots, f_k)). \end{aligned}$$

Thus (25)–(27) imply

$$[1-\operatorname{sg}_{\mathbf{Q}}\tilde{v}(A_1(\bar{\boldsymbol{h}}_{1\nu}, \ldots, \bar{\boldsymbol{h}}_{l\nu})), \ldots, 1-\operatorname{sg}_{\mathbf{Q}}\tilde{v}(A_{i_0}(\bar{\boldsymbol{h}}_{1\nu}, \ldots, \bar{\boldsymbol{h}}_{l\nu}))] \in \boldsymbol{S}_\sigma$$

and by Lemma 10 and (28)

$$[\operatorname{sg}_{\bar{\boldsymbol{K}}} A_1(\bar{\boldsymbol{h}}_{1\nu}, \ldots, \bar{\boldsymbol{h}}_{l\nu}), \ldots, \operatorname{sg}_{\bar{\boldsymbol{K}}} A_{i_0}(\bar{\boldsymbol{h}}_{1\nu}, \ldots, \bar{\boldsymbol{h}}_{l\nu})] \in \boldsymbol{S}_\sigma .$$

By (22) with $h_l = \bar{h}_{l\nu}$ we have

$$B_\sigma(\bar{\boldsymbol{h}}_{1\nu}, \ldots, \bar{\boldsymbol{h}}_{l\nu}) \neq 0$$

and

$$\mathscr{A}(\bar{\boldsymbol{h}}_{1\nu}, \ldots, \bar{\boldsymbol{h}}_{l\nu}) = \frac{C_\sigma(\bar{\boldsymbol{h}}_{1\nu}, \ldots, \bar{\boldsymbol{h}}_{l\nu}, x)}{B_\sigma(\bar{\boldsymbol{h}}_{1\nu}, \ldots, \bar{\boldsymbol{h}}_{l\nu})} .$$

Since B_σ and C_σ are polynomials with integral coefficients homogeneous in each vector of variables separately, we have by (28)

$$B_\sigma(\bar{\boldsymbol{h}}_{1\nu}, \ldots, \bar{\boldsymbol{h}}_{l\nu}) = \mathscr{Z}\, \frac{B_\sigma(*H_{j_\nu^1}^1(f_1, \ldots, f_k, x), \ldots, *H_{j_\nu^l}^l(f_1, \ldots, f_k, x))}{\prod_{\lambda=1}^{l} G_{j_\nu^\lambda}^\lambda(f_1, \ldots, f_k)^{\deg^\lambda B_\sigma}}$$

$$C_\sigma(\bar{\boldsymbol{h}}_{1\nu}, \ldots, \bar{\boldsymbol{h}}_{l\nu}, x) = \mathscr{Z}\, \frac{C_\sigma(*H_{j_\nu^1}^1(f_1, \ldots, f_k, x), \ldots, *H_{j_\nu^l}^l(f_1, \ldots, f_k, x))}{\prod_{\lambda=1}^{l} G_{j_\nu^\lambda}^\lambda(f_1, \ldots, f_k)^{\deg^\lambda C_\sigma}}$$

and by (24)

$$\mathscr{A}(\mathscr{B}_1(f_1, \ldots, f_k), \ldots, \mathscr{B}_l(f_1, \ldots, f_k)) = \mathscr{Z}\, \frac{H_j(f_1, \ldots, f_k, x)}{G_j(f_1, \ldots, f_k)} .$$

Lemma 12. *In the notation of Definition 5, let $\tilde{\mathscr{K}}$ be the analogue of $\mathscr{K}$ defined by means of an element $\tilde{p} \in \tilde{K}$ with $\tilde{v}(\tilde{p}) = 1$. Then the operation $\tilde{\mathscr{M}}$ on polynomials $f \in \tilde{K}[x]$ defined by the formula*

$$\tilde{\mathscr{M}}(f) := \begin{cases} \mathscr{D}_1 \mathscr{Z} \tilde{\mathscr{K}} f & \text{if } f \neq 0, \\ 1 & \text{if } f = 0 \end{cases}$$

belongs to $\Omega^*(N_-)$ and for every operation $\mathscr{A} \in \Omega(N_-^2)$ the operation $\tilde{\mathscr{M}}\mathscr{A}$ belongs to $\Omega^*(N_-^2)$.

Proof. It is sufficient to prove the second part of the lemma since the operation $\mathscr{I}_1\colon [f, g] \to f$ belongs to $\Omega(N_-^2)$ and the first part follows from the second on substituting $\mathscr{A} = \mathscr{I}_1$. Take two nonnegative integers m, n. By Lemma 6 there exist polynomials $A_i, B_j \in \boldsymbol{C}_0(m, n)$, $C_j \in \boldsymbol{C}_1(m, n)$ $(i \leqslant i_0, j \leqslant j_0)$ and a decomposition

$$\{0, 1\}^{i_0} = \bigcup_{j=1}^{j_0} \boldsymbol{S}_j$$

such that if $f, g \in \tilde{K}[x]$, $\deg f \leqslant m$, $\deg g \leqslant n$ and

$$[\operatorname{sg}_{\bar{\boldsymbol{K}}} A_1(f, g), \ldots, \operatorname{sg}_{\bar{\boldsymbol{K}}} A_{i_0}(f, g)] \in \boldsymbol{S}_j ,$$

then $B_j(f, g) \neq 0$ and

$$\mathscr{A}(f, g) = \frac{C_j(f, g, x)}{B_j(f, g)} .$$

Let $c = \max_{1 \leqslant j \leqslant j_0} \deg_x C_j$, $C_\mu(x_1, x_2, x) = \sum_{\nu=0}^{c} C_{\mu\nu}(x_1, x_2)\, x^{c-\nu}$;

$$(29)\qquad F_i = \begin{cases} A_i & \text{if } i \leqslant i_0, \\ C_{\mu\nu} & \text{if } i - i_0 - 1 = (\mu-1)(c+1)+\nu,\ 1 \leqslant \mu \leqslant j_0,\ 0 \leqslant \nu \leqslant c \end{cases}$$

and if $j=(\mu-1)(c+2)+\nu+1$, $1\leqslant\mu\leqslant j_0$, $0\leqslant\nu\leqslant c+1$,

$$G_j=\begin{cases}C_{\mu\nu} & \text{if } 0\leqslant\nu\leqslant c,\\ 1 & \text{if } \nu=c+1,\end{cases}\qquad H_j=\begin{cases}C_{\mu} & \text{if } 0\leqslant\nu\leqslant c,\\ 1 & \text{if } \nu=c+1,\end{cases}\tag{30}$$

$$T_j=\tau^{-1}(S_\mu)\times N_+^{(c+1)(\mu-1)}\times Y_\nu\times N_+^{(c+1)(j_0-\mu)},\tag{31}$$

where the transformation $\tau\colon N_+^{i_0}\to\{0,1\}^{i_0}$ is given by the formula

$$\tau(n_1,\dots,n_{i_0})=[\mathrm{sg}_{\mathbf{Q}}n_1^{-1},\dots,\mathrm{sg}_{\mathbf{Q}}n_{i_0}^{-1}]\tag{32}$$

and

$$\begin{aligned}Y_\nu&=\{[n_0,\dots,n_c]\in N_+^{c+1}\colon \min\{n_0,\dots,n_{\nu-1}\}>n_\nu=\min\{n_0,\dots,n_c\}\},\\ Y_{c+1}&=[\infty,\dots,\infty].\end{aligned}\tag{33}$$

The sets T_j are clearly disjoint and

$$N_+^{i_0+j_0(c+1)}=\bigcup_{j=1}^{j_0(c+2)}T_j.$$

Suppose now that $f,g\in\tilde{K}[x]$, $\deg f\leqslant m$, $\deg g\leqslant n$ and

$$[\tilde v(F_1(f,g)),\dots,\tilde v(F_{i_0+j_0(c+1)}(f,g))]\in T_j,\tag{34}$$

where $j=(\mu-1)(c+2)+\nu+1$, $1\leqslant\mu\leqslant j_0$, $0\leqslant\nu\leqslant c+1$. Then by (30) and (31)

$$[\mathrm{sg}_{\mathbf{Q}}\tilde v^{-1}(F_1(f,g)),\dots,\mathrm{sg}_{\mathbf{Q}}\tilde v^{-1}(F_{i_0}(f,g))]\in S_\mu$$

and, since $\mathrm{sg}_{\mathbf{Q}}\tilde v^{-1}(a)=\mathrm{sg}_{\tilde K}a$ for all $a\in\tilde K$, we get by (29)

$$[\mathrm{sg}_{\tilde K}A_1(f,g),\dots,\mathrm{sg}_{\tilde K}A_{i_0}(f,g)]\in S_\mu.$$

Hence $B_\mu(f,g)\neq 0$ and

$$\mathscr{A}(f,g)=\frac{C_\mu(f,g,x)}{B_\mu(f,g)}.\tag{35}$$

Moreover, by (31) and (34)

$$[\tilde v(F_{i_0+(\mu-1)(c+1)+1}(f,g)),\dots,\tilde v(F_{i_0+(\mu-1)(c+1)+c+1}(f,g))]\in Y_\nu;$$

hence if $\nu\leqslant c$ then by (29) and (33)

$$\begin{aligned}\min\{\tilde v(C_{\mu0}(f,g)),\dots,\tilde v(C_{\mu\nu-1}(f,g))\}&>\tilde v(C_{\mu\nu}(f,g))\\ &=\min\{\tilde v(C_{\mu0}(f,g)),\dots,\tilde v(C_{\mu c}(f,g))\},\end{aligned}$$

and if $\nu=c+1$ then

$$\tilde v(C_{\mu0}(f,g))=\dots=\tilde v(C_{\mu c}(f,g))=\infty.\tag{36}$$

Therefore, if $\nu\leqslant c$ we get $\tilde v(C_\mu(f,g,x))=\tilde v(C_{\mu\nu}(f,g))$ and from (35)

$$\mathscr{L}\tilde{\mathscr{K}}\mathscr{A}(f,g)=\tilde{\mathscr{L}}\frac{\tilde p^{-\tilde v(C_{\mu\nu}(f,g))}C_\mu(f,g,x)}{\tilde p^{-\tilde v(B_\mu(f,g))}B_\mu(f,g)}.$$

Moreover, the leading coefficient of the above polynomial equals

$$\tilde{\mathscr{L}}\frac{\tilde p^{-\tilde v(C_{\mu\nu}(f,g))}C_{\mu\nu}(f,g)}{\tilde p^{-\tilde v(B_\mu(f,g))}B_\mu(f,g)};$$

hence

$$\tilde{\mathscr{M}}\mathscr{A}(f,g)=\mathscr{D}_1\tilde{\mathscr{L}}\tilde{\mathscr{K}}\mathscr{A}(f,g)=\tilde{\mathscr{L}}\frac{C_\mu(f,g,x)}{C_{\mu\nu}(f,g)}=\tilde{\mathscr{L}}\frac{H_j}{G_j}.$$

If $\nu=c+1$ we have by (35) and (36) $\mathscr{E}_\varkappa(f,g)=0$ and by (30)

$$\tilde{\mathscr{M}}\mathscr{A}(f,g)=1=\tilde{\mathscr{L}}\frac{H_j}{G_j}.$$

The proof is complete.

LEMMA 13. *Let* $f,g\in I[x]$, $\mathrm{char}\,R=0$ *or* $\mathrm{char}\,R>\max\{\deg f,\deg g\}$, $\xi\in I$, $\mathscr{E}_{\lambda-1}(f,g)\mathscr{E}_\lambda(f,g)\neq 0$, $h_\lambda=\mathscr{K}\mathscr{E}_\lambda(f,g)$

$$(x-\xi)^{d_\lambda}\,\|\,\bar h_\lambda(x),\qquad h_\lambda^+\equiv(x-\xi)^{d_\lambda}\bmod P,\qquad h_\lambda^+\,|\,h_\lambda\tag{37}$$

where h_λ^+ *is monic and hence determined uniquely. If*

$$r_\lambda=\mathrm{res}(h_{\lambda-1}^+,\mathscr{E}_\lambda(f,g))\neq 0,\qquad e_\lambda=v(\mathscr{E}_\lambda(f,g))$$

then $r_{\lambda+1}\neq 0$,

$$v(r_\lambda)=e_\lambda d_{\lambda-1}-e_{\lambda-1}d_\lambda+v(r_{\lambda+1})\tag{38}$$

and

$$p^{-v(r_\lambda)}r_\lambda\equiv(-1)^{d_{\lambda-1}d_\lambda}\left(\frac{h_\lambda^{(d_\lambda)}(\xi)}{d_\lambda!}\right)^{d_{\lambda-1}}\left(\frac{h_{\lambda-1}^{(d_{\lambda-1})}(\xi)}{d_{\lambda-1}!}\right)^{d_\lambda}p^{-v(r_{\lambda+1})}r_{\lambda+1}\bmod P.\tag{39}$$

Proof. Let $h_\lambda=h_\lambda^+h_\lambda^-$. We have by (37)

$$h_\lambda^{(d_\lambda)}(\xi)\equiv\sum_{\nu=0}^{d_\lambda}\binom{d_\lambda}{\nu}h_\lambda^{+(\nu)}(\xi)h_\lambda^{-(d_\lambda-\nu)}(\xi)\equiv d_\lambda!\,h_\lambda^-(\xi)\not\equiv 0\bmod P.\tag{40}$$

Moreover,

$$\begin{aligned}r_\lambda&=\mathrm{res}(h_{\lambda-1}^+,p^{e_\lambda}h_\lambda)=p^{e_\lambda d_{\lambda-1}}\mathrm{res}(h_{\lambda-1}^+,h_\lambda)\\ &=p^{e_\lambda d_{\lambda-1}}\mathrm{res}(h_{\lambda-1}^+,h_\lambda^+)\mathrm{res}(h_{\lambda-1}^+,h_\lambda^-)\\ &=p^{e_\lambda d_{\lambda-1}}(-1)^{d_{\lambda-1}d_\lambda}\mathrm{res}(h_\lambda^+,h_{\lambda-1}^+)\mathrm{res}(h_{\lambda-1}^+,h_\lambda^-)\\ &=(-1)^{d_{\lambda-1}d_\lambda}p^{e_\lambda d_{\lambda-1}}\frac{\mathrm{res}(h_\lambda^+,h_{\lambda-1})}{\mathrm{res}(h_\lambda^+,h_{\lambda-1}^-)}\mathrm{res}(h_{\lambda-1}^+,h_\lambda^-).\end{aligned}$$

Now, by (37) and (40) we have

$$\begin{aligned}\mathrm{res}(h_\lambda^+,h_{\lambda-1}^-)&\equiv\mathrm{res}((x-\xi)^{d_\lambda},h_{\lambda-1}^-)\equiv\mathrm{res}(x-\xi,h_{\lambda-1}^-)^{d_\lambda}\\ &\equiv\left(\frac{h_{\lambda-1}^{(d_{\lambda-1})}(\xi)}{d_{\lambda-1}!}\right)^{d_\lambda}\bmod P,\end{aligned}\tag{41}$$

$$(42)\qquad \operatorname{res}(h_{\lambda-1}^{+}, h_{\lambda}^{-}) \equiv \operatorname{res}\big((x-\xi)^{d_{\lambda-1}}, h_{\lambda}^{-}\big) \equiv \operatorname{res}(x-\xi, h_{\lambda}^{-})^{d_{\lambda-1}} \equiv (h_{\lambda}^{-}(\xi))^{d_{\lambda-1}} \equiv \left(\frac{h_{\lambda}^{(d_\lambda)}(\xi)}{d_\lambda!}\right)^{d_{\lambda-1}} \bmod \boldsymbol{P}.$$

Finally, $\mathscr{E}_{\lambda+1}(f, g) \equiv \mathscr{E}_{\lambda-1}(f, g) \bmod h_{\lambda}^{+}$, and thus

$$\operatorname{res}(h_{\lambda}^{+}, h_{\lambda-1}) = p^{-e_{\lambda-1}d_\lambda}\operatorname{res}(h_{\lambda}^{+}, \mathscr{E}_{\lambda-1}(f, g)) = p^{-e_{\lambda-1}d_\lambda}\operatorname{res}(h_{\lambda}^{+}, \mathscr{E}_{\lambda+1}(f, g)).$$

Since the extreme right-hand sides of (41) and (42) are prime to p, we get Lemma 13.

LEMMA 14. *Let f, g, ξ and λ satisfy the assumptions of Lemma 13 for all $\lambda<l$, let d_λ have the meaning of that lemma and $E_\lambda = \mathscr{E}_\lambda(f, g)$. Assume that $d_0 = 1$ and l is the least nonnegative integer such that $E_{l+1} = 0$. Then if $d_l>0$ we have $g(\xi) = 0$, if $d_l = 0$ we have $g(\xi) \neq 0$,*

$$(43)\qquad g(\xi) = \prod_{\lambda=1}^{l} (-1)^{d_{\lambda-1}d_\lambda}\left(\frac{E_{\lambda}^{(d_\lambda)}(\xi)}{d_\lambda!}\right)^{d_{\lambda-1}}\left(\frac{E_{\lambda-1}^{(d_{\lambda-1})}(\xi)}{d_{\lambda-1}!}\right)^{-d_\lambda} \bmod \boldsymbol{P}^{v(g(\xi))+1}$$

and $v(E_{\lambda}^{(d_\lambda)}(\xi)) = v(E_\lambda)$ $(\lambda = 0, 1, \ldots, l)$.

Proof. If $d_l>0$ we have $\deg h_l^{+}>0$. Moreover, from $h_l^{+}|E_l$ it follows that $h_l^{+}|(f, g), h_l^{+}|h_0^{+}$. But $h_0^{+} = x-\xi$, whence $h_l^{+} = x-\xi$ and $g(\xi) = 0$.

If $d_l = 0$ then for every $\lambda\leqslant l$ we have $\operatorname{res}(h_{\lambda-1}^{+}, E_\lambda) \neq 0$. Indeed, $(h_{\lambda-1}^{+}, E_\lambda)|E_l$, and thus $\operatorname{res}(h_{\lambda-1}^{+}, E_\lambda) = 0$ would imply $d_l>0$. Using Lemma 13, we get (38) and (39) for all $\lambda\leqslant l$. Summing or multiplying over λ, we obtain

$$\sum_{\lambda=1}^{l} v(r_\lambda) = \sum_{\lambda=1}^{l}(e_\lambda d_{\lambda-1} - e_{\lambda-1}d_\lambda) + \sum_{\lambda=1}^{l} v(r_{\lambda+1}).$$

$$\prod_{\lambda=1}^{l} p^{-v(r_\lambda)}r_\lambda \equiv \prod_{\lambda=1}^{l} (-1)^{d_{\lambda-1}d_\lambda}\left(\frac{h_{\lambda}^{(d_\lambda)}(\xi)}{d_\lambda!}\right)^{d_{\lambda-1}}\left(\frac{h_{\lambda-1}^{(d_{\lambda-1})}(\xi)}{d_{\lambda-1}!}\right)^{-d_\lambda} \prod_{\lambda=1}^{l} p^{-v(r_{\lambda+1})}r_{\lambda+1} \bmod \boldsymbol{P}.$$

Since $r_{l+1} = \operatorname{res}(1, 0) = 1$, it follows that

$$v(r_1) = \sum_{\lambda=1}^{l}(e_\lambda d_{\lambda-1} - e_{\lambda-1}d_\lambda),$$

$$p^{-v(r_1)}r_1 \equiv \prod_{\lambda=1}^{l} (-1)^{d_{\lambda-1}d_\lambda}\left(\frac{h_{\lambda}^{(d_\lambda)}(\xi)}{d_\lambda!}\right)^{d_{\lambda-1}}\left(\frac{h_{\lambda-1}^{(d_{\lambda-1})}(\xi)}{d_{\lambda-1}!}\right)^{-d_\lambda} \bmod \boldsymbol{P}.$$

Since $h_{\lambda}^{(d_\lambda)}(\xi) = p^{-e_\lambda}E_{\lambda}^{(d_\lambda)}(\xi)$, we get

$$r_1 \equiv \prod_{\lambda=1}^{l} (-1)^{d_{\lambda-1}d_\lambda}\left(\frac{E_{\lambda}^{(d_\lambda)}(\xi)}{d_\lambda!}\right)^{d_{\lambda-1}}\left(\frac{E_{\lambda-1}^{(d_{\lambda-1})}(\xi)}{d_{\lambda-1}!}\right)^{d_\lambda} \bmod \boldsymbol{P}^{v(r_1)+1}.$$

However, $r_1 = \operatorname{res}(x-\xi, g(x)) = g(\xi)$; thus $g(\xi) \neq 0$ and (43) follows. The last statement of the lemma is a direct consequence of (40).

DEFINITION 6. The operation $\mathscr{M}$ on polynomials $f \in K[x]$ is defined by the formula

$$\mathscr{M}f := \begin{cases}\mathscr{D}_1\mathscr{L}\mathscr{K}f & \text{if } f\neq 0,\\ 1 & \text{if } f = 0.\end{cases}$$

LEMMA 15. *For arbitrary nonnegative integers $m, \alpha, n_1, \ldots, n_k$ less than $\operatorname{char}\boldsymbol{R}$ unless $\operatorname{char}\boldsymbol{R} = 0$ there exist polynomials $F_i, G_{j\nu}, H_{j\nu}$ $(i\leqslant i_0, j\leqslant j_0, \nu\leqslant\nu_0)$, $K_{j\nu\varkappa}, L_{j\nu\varkappa}$ $(\varkappa\leqslant k, j\leqslant j_0, \nu\leqslant\nu_0)$ and a decomposition*

$$N_{+}^{i_0} = \bigcup_{j=1}^{j_0} \boldsymbol{T}_j$$

independent of K, v and p with the following properties:

$$(44)\qquad F_i, G_{j\nu} \in \boldsymbol{C}_0(m, m, n_1, \ldots, n_k), \quad H_{j\nu} \in \boldsymbol{C}_1(m, m, n_1, \ldots, n_k),$$

$$(45)\qquad K_{j\nu\varkappa}, L_{j\nu\varkappa} \in \boldsymbol{C}_1(m, n_\varkappa) \text{ and either } L_{j\nu\varkappa} = 0 \text{ or } \deg^1 L_{j\nu\varkappa} = \deg^1 K_{j\nu\varkappa},\ \deg^2 L_{j\nu\varkappa} = \deg^2 K_{j\nu\varkappa}+1,\ w(L_{j\nu\varkappa}) = w(K_{j\nu\varkappa})+n_\varkappa.$$

If $f_1, f_2, g_1, \ldots, g_k \in \boldsymbol{I}[x]$, $\deg f_\sigma\leqslant m$ $(\sigma = 1, 2)$, $\deg g_\varkappa\leqslant n_\varkappa$ $(\varkappa\leqslant k)$

$$(46)\qquad \mathcal{O}_\alpha\mathscr{M}f_2|\mathscr{M}f_1, \left(\mathcal{O}_\alpha\mathscr{M}f_2, \frac{\mathscr{M}f_1}{\mathcal{O}_\alpha\mathscr{M}f_2}\right) = 1$$

and

$$[v(F_1(f_1, f_2, g_1, \ldots, g_k)), \ldots, v(F_{i_0}(f_1, f_2, g_1, \ldots, g_k))] \in \boldsymbol{T}_j$$

then

$$(47)\qquad G_j(f_1, f_2, g_1, \ldots, g_k) \neq 0, \quad \frac{H_j(f_1, f_2, g_1, \ldots, g_k, x)}{G_j(f_1, f_2, g_1, \ldots, g_k)} \in \boldsymbol{I}[x],$$

$$(48)\qquad \mathcal{O}_\alpha\mathscr{M}f_2(x) = \prod_{\nu=1}^{\nu_0} \mathscr{L}\frac{H_{j\nu}(f_1, f_2, g_1, \ldots, g_k, x)}{G_{j\nu}(f_1, f_2, g_1, \ldots, g_k)}$$

and if $\xi \in \boldsymbol{I}$

$$f_1(\xi) = 0, \quad \overline{\mathscr{K}H_j(f_1, f_2, g_1, \ldots, g_k, x)|_{x=\xi}} = 0$$

then for all $\varkappa\leqslant k$

$$(49)\qquad K_{j\nu\varkappa}(f_1, g_\varkappa, \xi) \neq 0, \quad \frac{L_{j\nu\varkappa}(f_1, g_\varkappa, \xi)}{K_{j\nu\varkappa}(f_1, g_\varkappa, \xi)} \in \boldsymbol{I}$$

and

$$(50)\qquad g_\varkappa(\xi) \equiv \frac{L_{j\nu\varkappa}(f_1, g_\varkappa, \xi)}{K_{j\nu\varkappa}(f_1, g_\varkappa, \xi)} \bmod \boldsymbol{P}^{v(g_\varkappa(\xi))+1}$$

$(L_{j\nu\varkappa}(f, g_\varkappa, \xi) = 0$ implies $g_\varkappa(\xi) = 0)$. Moreover

$$v(L_{j\nu\varkappa}(f_1, g_\varkappa, \xi)) = v(L_{j\nu\varkappa}(f_1, g_\varkappa, x)),$$
$$v(K_{j\nu\varkappa}(f_1, g_\varkappa, \xi)) = v(K_{j\nu\varkappa}(f_1, g_\varkappa, x)).$$

Proof. Let M be the set of all pairs $[\varkappa, \lambda]$, where $1 \leqslant \varkappa \leqslant k$, $0 \leqslant \lambda \leqslant n_\varkappa + 2$, Δ the set of all integer-valued functions δ on M such that $0 \leqslant \delta(\varkappa, \lambda) \leqslant n_\varkappa$ and $\delta(\varkappa, 0) = 1$. By Lemmata 4, 11 and 12 for each $\delta \in \Delta$ the operation

$$[f_1, g_1, \ldots, g_k] \to \prod_{\substack{[\varkappa,\lambda]\in M \\ \delta(\varkappa,\lambda)=0}} \tilde{\mathcal{M}}\mathcal{E}_\lambda(f_1, g_\varkappa)$$

belongs to $\Omega^*(N_-^{k+1})$. By Lemmata 8, 11 and 12 we have $\mathcal{O}_\alpha \tilde{\mathcal{M}} \in \Omega^*(N_-)$. By Lemmata 7 and 11 the operation

$$[f_1, f_2, g_1, \ldots, g_k] \to \Big(\mathcal{O}_\alpha \tilde{\mathcal{M}} f_2, \prod_{\substack{[\varkappa,\lambda]\in M \\ \delta(\varkappa,\lambda)=0}} \tilde{\mathcal{M}}\mathcal{E}_\lambda(f_1, g_\varkappa)\Big)$$

belongs to $\Omega^*(N_-^{k+2})$. (An empty product equals 1). By Lemmata 5 and 11 the operation

$$[f_1, f_2, g_1, \ldots, g_k] \to \frac{\mathcal{O}_\alpha \tilde{\mathcal{M}} f_2}{\Big(\mathcal{O}_\alpha \tilde{\mathcal{M}} f_2, \prod\limits_{\substack{[\varkappa,\lambda]\in M \\ \delta(\varkappa,\lambda)=0}} \tilde{\mathcal{M}}\mathcal{E}_\lambda(f_1, g_\varkappa)\Big)}$$

belongs to $\Omega^*(N_-^{k+2})$. Further, by Lemmata 8, 11 and 12 for every fixed $\delta, \varkappa, \lambda$ the operation

$$[f_1, g_\varkappa] \to \mathcal{O}_{\delta(\varkappa,\lambda)} \tilde{\mathcal{M}}\mathcal{E}_\lambda(f_1, g_\varkappa)$$

belongs to $\Omega^*(N_-^2)$. Hence by Lemmata 7 and 11 for every $\delta \in \Delta$ the operation

$$[f_1, f_2, g_1, \ldots, g_k] \to \left(\frac{\mathcal{O}_\alpha \tilde{\mathcal{M}} f_2}{\Big(\mathcal{O}_\alpha \tilde{\mathcal{M}} f_2, \prod\limits_{\substack{[\varkappa,\lambda]\in M \\ \delta(\varkappa,\lambda)=0}} \tilde{\mathcal{M}}\mathcal{E}_\lambda(f_1, g_\varkappa)\Big)}, \text{g.c.d. } \mathcal{O}_{\delta(\varkappa,\lambda)} \tilde{\mathcal{M}}\mathcal{E}_\lambda(f_1, g_\varkappa)\right)$$

belongs to $\Omega^*(N_-^{k+2})$. On the other hand, by Definition 6 the operation $\mathcal{M}$ is a specialization of the operation $\tilde{\mathcal{M}}$ to the case of

$$\tilde{K} = K, \quad \tilde{v} = v, \quad \tilde{p} = p.$$

This implies by the definition of $\Omega^*(N_-^{k+2})$ that for every $\delta \in \Delta$ there exist polynomials $F_\mu\langle\delta\rangle$ $(\mu \leqslant \mu_\delta)$, $G_\varrho\langle\delta\rangle$, $H_\varrho\langle\delta\rangle$ $(\varrho \leqslant \varrho_\delta)$ and a decomposition

$$N_+^{\mu_\delta} = \bigcup_{\varrho=1}^{\varrho_\delta} T_\varrho\langle\delta\rangle$$

independent of K, v and p and with the following properties:

(51) $F_\mu\langle\delta\rangle, G_\varrho\langle\delta\rangle \in C_0(m_1, m_2, n_1, \ldots, n_k)$, $\quad H_\varrho\langle\delta\rangle \in C_1(m_1, m_2, n_1, \ldots, n_k)$;

if $f_1, f_2, g_1, \ldots, g_k \in I[x]$, $\deg f_\sigma \leqslant m_\sigma$ $(\sigma = 1, 2)$, $\deg g_\varkappa \leqslant n_\varkappa$ $(\varkappa \leqslant k)$ and

$$[v(F_1\langle\delta\rangle(f_1, f_2, g_1, \ldots, g_k)), \ldots, v(F_{\mu_\delta}\langle\delta\rangle(f_1, f_2, g_1, \ldots, g_k))] \in T_\varrho\langle\delta\rangle$$

then

$$G_\varrho\langle\delta\rangle(f_1, f_2, g_1, \ldots, g_k) \neq 0,$$

$$I_\varrho\langle\delta\rangle := \frac{H_\varrho\langle\delta\rangle(f_1, f_2, g_1, \ldots, g_k, x)}{G_\varrho\langle\delta\rangle(f_1, f_2, g_1, \ldots, g_k)} \in I[x] \tag{52}$$

and

$$\left(\frac{\mathcal{O}_\alpha \mathcal{M} f_2}{\Big(\mathcal{O}_\alpha \mathcal{M} f_2, \prod\limits_{\substack{[\varkappa,\lambda]\in M \\ \delta(\varkappa,\lambda)=0}} \mathcal{M}\mathcal{E}_\delta(f_1, g_\varkappa)\Big)}, \operatorname*{g.c.d.}_{\substack{[\varkappa,\lambda]\in M \\ \delta(\varkappa,\lambda)>0}} \mathcal{O}_{\delta(\varkappa,\lambda)} \mathcal{M}\mathcal{E}_\lambda(f_1, g_\varkappa)\right) = \mathcal{L} I_\varrho\langle\delta\rangle. \tag{53}$$

Now let us fix $\delta \in \Delta$ and $\varkappa \leqslant k$ and consider the following operations:

$$\mathcal{G}_1\colon [f, g] \to \prod_{\substack{\lambda=0 \\ \delta(\varkappa,\lambda)>0}}^{n_\varkappa+1} (-1)^{\delta(\varkappa,\lambda)\delta(\varkappa,\lambda+1)} \left(\frac{\mathcal{E}_{\lambda+1}^{(\delta(\varkappa,\lambda+1))}(f, g)}{\delta(\varkappa, \lambda+1)!}\right)^{\delta(\varkappa,\lambda)},$$

$$\mathcal{G}_2\colon [f, g] \to \prod_{\substack{\lambda=1 \\ \delta(\varkappa,\lambda)>0}}^{n_\varkappa+1} \left(\frac{\mathcal{E}_{\lambda-1}^{(\delta(\varkappa,\lambda-1))}(f, g)}{\delta(\varkappa, \lambda-1)!}\right)^{\delta(\varkappa,\lambda)}.$$

The operation $f \to f^{(n)}/n!$ properly understood $\left(\sum a_\mu x^\mu \to \sum \binom{\mu}{n} x^{\mu-n}\right)$ belongs to $\Omega(N; -n, 1)$; hence in virtue of Lemmata 2, 4 and 6 we have

$$\mathcal{G}_1 \in \Omega\Big(N_-^2, \sum_{\lambda=0}^{n_\varkappa+1} \delta(\varkappa, \lambda)\delta(\varkappa, \lambda+1), \sum_{\substack{\lambda=1 \\ \lambda\equiv 1 \bmod 2}}^{n_\varkappa+1} \delta(\varkappa, \lambda), \sum_{\substack{\lambda=0 \\ \lambda\equiv 0 \bmod 2}}^{n_\varkappa+1} \delta(\varkappa, \lambda)\Big),$$

(54)

$$\mathcal{G}_2 \in \Omega\Big(N_-^2, -\sum_{\lambda=1}^{n_\varkappa+2} \delta(\varkappa, \lambda-1)\delta(\varkappa, \lambda), \sum_{\substack{\lambda=1 \\ \lambda\equiv 1 \bmod 2}}^{n_\varkappa+2} \delta(\varkappa, \lambda), \sum_{\substack{\lambda=2 \\ \lambda\equiv 0 \bmod 2}}^{n_\varkappa+2} \delta(\varkappa, \lambda)\Big).$$

Therefore for $\sigma = 1, 2$ there exist polynomials $A_\mu\langle\delta, \varkappa, \sigma\rangle$ $(\mu \leqslant \mu_{\delta\varkappa\sigma})$, $B_\varrho\langle\delta, \varkappa, \sigma\rangle$, $C_\varrho\langle\delta, \varkappa, \sigma\rangle$ $(\varrho \leqslant \varrho_{\delta\varkappa\sigma})$ and a decomposition

$$\{0, 1\}^{\mu_{\delta\varkappa\sigma}} = \bigcup_{\varrho=1}^{\mu_{\delta\varkappa\sigma}} S_\varrho\langle\delta, \varkappa, \sigma\rangle$$

independent of K, v and p and with the following properties:

(55) $A_\mu\langle\delta, \varkappa, \sigma\rangle, B_\varrho\langle\delta, \varkappa, \sigma\rangle \in C_0(m, n_\varkappa)$, $\quad C_\varrho\langle\delta, \varkappa, \sigma\rangle \in C_1(m, n_\varkappa)$;

if $f, g \in K[x]$, $\deg f \leqslant m$, $\deg g \leqslant n_\varkappa$ and $[\operatorname{sg}_K A_1\langle\delta, \varkappa, \sigma\rangle(f, g), \ldots, \operatorname{sg}_K A_{\mu_{\delta\varkappa\sigma}}(f, g)] \in S_\varrho\langle\delta, \varkappa, \sigma\rangle$ then

$$B_\varrho\langle\delta, k, \sigma\rangle(f, g) \neq 0 \tag{56}$$

and

$$\prod_{\substack{\lambda=0 \\ \delta(\varkappa,\lambda)>0}}^{n_\varkappa+1} (-1)^{\delta(\varkappa,\lambda)\delta(\varkappa,\lambda+1)} \left(\frac{E_{\lambda+1}^{(\delta(\varkappa,\lambda+1))}(f, g)}{\delta(\varkappa, \lambda+1)!}\right)^{\delta(\varkappa,\lambda)} = \frac{C_\varrho\langle\delta, \varkappa, 1\rangle(f, g, x)}{B_\varrho\langle\delta, \varkappa, 1\rangle(f, g)},$$

(57)

$$\prod_{\substack{\lambda=1 \\ \delta(\varkappa,\lambda)>0}}^{n_\varkappa+2} \left(\frac{E_{\lambda-1}^{(\delta(\varkappa,\lambda-1))}(f, g)}{\delta(\varkappa, \lambda-1)!}\right)^{\delta(\varkappa,\lambda)} = \frac{C_\varrho\langle\delta, \varkappa, 2\rangle(f, g, x)}{B_\varrho\langle\delta, \varkappa, 2\rangle(f, g)}.$$

Moreover by (54)

$$\deg^1 C_\varrho\langle\delta,\varkappa,\sigma\rangle-\deg^1 B_\varrho\langle\delta,\varkappa,\sigma\rangle = \sum_{\substack{\lambda=1\\ \lambda\equiv 1 \bmod 2}}^{n_\varkappa+\sigma} \delta(\varkappa,\lambda),$$

(58)

$$\deg^2 C_\varrho\langle\delta,\varkappa,\sigma\rangle-\deg^2 B_\varrho\langle\delta,\varkappa,\sigma\rangle = \sum_{\substack{\lambda=2\\ \lambda\equiv 0 \bmod 2}}^{n_\varkappa+\sigma} \delta(\varkappa,\lambda)+2-\sigma,$$

$$w(C_\varrho\langle\delta,\varkappa,\sigma\rangle)-w(B_\varrho\langle\delta,\varkappa,\sigma\rangle)$$
$$= -\sum_{\lambda=0}^{n_\varkappa+1}\delta(\varkappa,\lambda)\delta(\varkappa,\lambda+1)+m\sum_{\substack{\lambda=1\\ \lambda\equiv 1 \bmod 2}}^{n_\varkappa+\sigma}\delta(\varkappa,\lambda)+n_\varkappa\Big(\sum_{\substack{\lambda=2\\ \lambda\equiv 0 \bmod 2}}^{n_\varkappa+\sigma}(\varkappa,\lambda)+2-\sigma\Big).$$

We order all functions $\delta\in\Delta$ in a sequence $\delta_1,\delta_2,\ldots,\delta_{\nu_0}$. Next we order lexicographically all polynomials $F_\mu\langle\delta_\nu\rangle$, the order of letters being ν,μ, and then order lexicographically all polynomials $A_\mu\langle\delta_\nu,\varkappa,\sigma\rangle$, the order of letters being $\nu,\varkappa,\sigma,\mu$. The ith term of the sequence of polynomials so obtained will be called F_i ($i\leqslant i_0=\sum_{\nu=1}^{\nu_0}\mu_{\delta_\nu}+\sum_{\nu=1}^{\nu_0}\sum_{\varkappa=1}^{k}\sum_{\sigma=1}^{2}\mu_{\delta_\nu\varkappa\sigma}$). Now we order all elements of the multiple Cartesian product (the order of letters being $\nu,\varkappa,\sigma$)

$$\mathop{\boldsymbol{P}}_{\nu=1}^{n}(\{1,\ldots,\varrho_{\delta_\nu}\}\times\mathop{\boldsymbol{P}}_{\varkappa=1}^{k}\mathop{\boldsymbol{P}}_{\sigma=1}^{2}\{1,\ldots,\varrho_{\delta_\nu\varkappa\sigma}\}$$

in a sequence, denote the jth term of this sequence by

$$[\varrho_{j1},\varrho_{j111},\varrho_{j112},\varrho_{j121},\ldots,\varrho_{jnk2}]\qquad \Big(1\leqslant j\leqslant j_0=\prod_{\nu=1}^{n}\varrho_{\delta_\nu}\prod_{\varkappa=1}^{k}\prod_{\sigma=1}^{2}\varrho_{\delta_\nu\varkappa\sigma}\Big),$$

define a transformation $\tau\colon N_+^\mu\to\{0,1\}^\mu$ by the formula

$$\tau(v_1,\ldots,v_\mu):=[\mathrm{sg}_\Omega v_1^{-1},\ldots,\mathrm{sg}_\Omega v_\mu^{-1}]$$

and put

(59) $$T_j:=\mathop{\boldsymbol{P}}_{\nu=1}^{n}\big(T_{\varrho_{j\nu}}\langle\delta_\nu\rangle\times\mathop{\boldsymbol{P}}_{\varkappa=1}^{k}\mathop{\boldsymbol{P}}_{\sigma=1}^{2}\tau^{-1}S_{\varrho_{j\nu\varkappa\sigma}}\langle\delta_\nu,\varkappa,\sigma\rangle\big),$$

(60) $$G_{j\nu}:=G_{\varrho_{j\nu}}\langle\delta_\nu\rangle,\qquad H_{j\nu}:=H_{\varrho_{j\nu}}\langle\delta_\nu\rangle,$$

(61)

$$K_{j\nu\varkappa}:=\begin{cases}B_{\varrho_{j\nu\varkappa1}}\langle\delta_{\nu,\varkappa,1}\rangle C_{\varrho_{j\nu\varkappa2}}\langle\delta_{\nu,\varkappa,2}\rangle & \text{if } \delta_\nu(\varkappa,n_\varkappa+1)\delta_\nu(\varkappa,n_\varkappa+2)=0,\\ 1 & \text{otherwise},\end{cases}$$

$$L_{j\nu\varkappa}:=\begin{cases}B_{\varrho_{j\nu\varkappa2}}\langle\delta_{\nu\varkappa2}\rangle C_{\varrho_{j\nu\varkappa2}}\langle\delta_\nu,\varkappa,2\rangle & \text{if } \delta_\nu(\varkappa,n_\varkappa+1)=\delta_\nu(\varkappa,n_\varkappa+2)=0,\\ 0 & \text{otherwise}.\end{cases}$$

We proceed to prove that the sets and the polynomials so defined have the properties asserted in the lemma.

The sets T_j are disjoint and we have

$$\bigcup_{j=1}^{j_0}T_j=\mathop{\boldsymbol{P}}_{\nu=1}^{n}\Big(\bigcup_{\varrho=1}^{\varrho_{\delta\nu}}T_\varrho\langle\delta_\nu\rangle\times\mathop{\boldsymbol{P}}_{\varkappa=1}^{k}\mathop{\boldsymbol{P}}_{\sigma=1}^{2}\tau^{-1}\bigcup_{\varrho=1}^{\varrho_{\delta_\nu\varkappa\sigma}}S_\varrho\langle\delta_\nu,\varkappa,\sigma\rangle\Big)$$
$$=\mathop{\boldsymbol{P}}_{\nu=1}^{n}\big(N_+^{\mu_{\delta_\nu}}\times\mathop{\boldsymbol{P}}_{\varkappa=1}^{k}\mathop{\boldsymbol{P}}_{\sigma=1}^{2}\tau^{-1}\{0,1\}^{\mu_{\delta_\nu\varkappa\sigma}}\big)=\mathop{\boldsymbol{P}}_{\nu=1}^{n}\big(N_+^{\mu_{\delta_\nu}}\times\mathop{\boldsymbol{P}}_{\varkappa=1}^{k}\mathop{\boldsymbol{P}}_{\sigma=1}^{2}N_+^{\mu_{\delta_\nu\varkappa\sigma}}\big)=N_+^{i_0}.$$

Formula (44) follows from (51) and (59), formula (45) follows from (55), (58) and (61). Indeed, if $\delta_\nu(\varkappa,n_\varkappa+1)+\delta_\nu(\varkappa,n_\varkappa+2)>0$ we have $L_{j\nu\varkappa}=0$, if $\delta_\nu(\varkappa,n_\varkappa+1)=\delta_\nu(\varkappa,n_\varkappa+2)=0$ we have

$$\deg^\sigma L_{j\nu\varkappa}-\deg^\sigma K_{j\nu\varkappa}$$
$$=\deg^\sigma C_{\varrho_{j\nu\varkappa1}}\langle\delta_\nu,\varkappa,1\rangle-\deg^\sigma B_{\varrho_{j\nu k1}}\langle\delta_\nu,\varkappa,1\rangle-\deg^\sigma C_{\varrho_{j\nu\varkappa2}}\langle\delta_\nu,\varkappa,2\rangle+$$
$$+\deg^\sigma B_{\varrho_{j\nu\varkappa2}}\langle\delta_\nu,\varkappa,2\rangle$$
$$=\sum_{\substack{\lambda=1\\ \lambda\equiv 1 \bmod 2}}^{n_\varkappa+1}\delta_\nu(\varkappa,\lambda)-\sum_{\substack{\lambda=1\\ \lambda\equiv 1 \bmod 2}}^{n_\varkappa+2}\delta_\nu(\varkappa,\lambda)=\frac{(-1)^{n_\varkappa}-1}{2}\delta_\nu(\varkappa,n_\varkappa,+2)=0\ \text{if}\ \sigma=1,$$
$$=1+\sum_{\substack{\lambda=2\\ \lambda\equiv 0 \bmod 2}}^{n_\varkappa+1}\delta_\nu(\varkappa,\lambda)-\sum_{\substack{\lambda=2\\ \lambda\equiv 0 \bmod 2}}^{n_\varkappa+2}\delta_\nu(\varkappa,\lambda)=1-\frac{1+(-1)^{n_\varkappa}}{2}\delta_\nu(\varkappa,n_\varkappa+2)=1\ \text{if}\ \sigma=2,$$

$$w(L_{j\nu\varkappa})-w(K_{j\nu\varkappa})=m\cdot 0+n_\varkappa\cdot 1=n_\varkappa.$$

Assume now that polynomials $f_1,f_2,g_1,\ldots,g_k$ satisfy (46) and

(62) $$[v(F_1(f_1,f_2,g_1,\ldots,g_k)),\ldots,v(F_{i_0}(f_1,f_2,g_1,\ldots,g_k))]\in T_j.$$

Then by (59) for each $\nu\leqslant n$

$$[v(F_1\langle\delta_\nu\rangle(f_1,f_2,g_1,\ldots,g_k)),\ldots,v(F_{\mu_{\delta_\nu}}\langle\delta_\nu\rangle(f_1,f_2,g_1,\ldots,g_k))]\in T_{\varrho_{j\nu}}\langle\delta_\nu\rangle.$$

Hence, by (52) and (60)

(63) $$G_{j\nu}(f_1,f_2,g_1,\ldots,g_k)\neq 0,$$

(64) $$I_{j\nu}:=\frac{H_{j\nu}(f_1,f_2,g_1,\ldots,g_k)}{G_{j\nu}(f_1,f_2,g_1,\ldots,g_k)}=I_{\varrho_{j\nu}}\langle\delta_\nu\rangle\in I[x].$$

Now for every $\delta\in\Delta$ and every pair $[\varkappa,\lambda]\in M$ with $\delta(\varkappa,\lambda)>0$ we have by Lemma 9

$$\mathcal{O}_{\delta(\varkappa,\lambda)}\mathcal{M}\mathcal{E}_\lambda(f_1,g_\varkappa)=\prod_{\substack{\zeta\in\hat{\boldsymbol{R}}\\ (x-\zeta)^{\delta(\varkappa,\lambda)}\|\mathcal{M}\mathcal{E}_\lambda(f_1,g_\varkappa)}}(x-\zeta).$$

Hence we get

$$\mathop{\mathrm{g.c.d.}}_{\substack{[\varkappa,\lambda]\in M\\ \delta(\varkappa,\lambda)>0}}\mathcal{O}_{\delta(\varkappa,\lambda)}\mathcal{M}\mathcal{E}_\lambda(f_1,g_\varkappa)=\prod_{\substack{\zeta\in\hat{\boldsymbol{R}}\\ \delta(\varkappa,\lambda)\neq 0\to(x-\zeta)^{\delta(\varkappa,\lambda)}\|\mathcal{M}\mathcal{E}_\lambda(f_1,g_\varkappa)}}(x-\zeta).$$

On the other hand, since by Lemma 9 $\mathcal{O}_\alpha \mathcal{M} f_2$ is square-free

$$\frac{\mathcal{O}_\alpha \mathcal{M} f_2}{\Big(\mathcal{O}_\alpha \mathcal{M} f_2, \prod\limits_{\substack{[\varkappa,\lambda]\in M\\ \delta(\varkappa,\lambda)=0}} \mathcal{M}\mathcal{E}_\lambda(f_1, g_\varkappa)\Big)} = \text{const} \prod_{\substack{\zeta\in\hat{R},\ \mathcal{O}_\alpha\mathcal{M}f_2(\zeta)=0\\ \delta(\varkappa,\lambda)=0\to x-\zeta\nmid \mathcal{M}\mathcal{E}_\lambda(f_1,g_\varkappa)}} (x-\zeta).$$

Hence

$$(65)\quad \left(\frac{\mathcal{O}_\alpha \mathcal{M} f_2}{\Big(\mathcal{O}_\alpha \mathcal{M} f_2, \prod\limits_{\substack{[\varkappa,\lambda]\in M\\ \delta(\varkappa,\lambda)=0}} \mathcal{M}\mathcal{E}_\lambda(f_1, g_\varkappa)\Big)},\ \underset{\substack{[\varkappa,\lambda]\in M\\ \delta(\varkappa,\lambda)\neq 0}}{\text{g.c.d.}}\ \mathcal{O}_{\delta(k,\lambda)}\mathcal{M}\mathcal{E}_\lambda(f_1, g_\varkappa)\right) = \prod_{\substack{\zeta\in\hat{R},\ \mathcal{O}_\alpha\mathcal{M}f_2(\zeta)=0\\ \operatorname{ord}_{x-\zeta}\mathcal{M}\mathcal{E}_\lambda(f_1,g_\varkappa)=\delta(\varkappa,\lambda)}} (x-\zeta).$$

For every $\zeta\in\hat{R}$ satisfying $\mathcal{O}_\alpha\mathcal{M}f_2(\zeta)=0$ we have by (46) $\operatorname{ord}_{x-\zeta}\mathcal{M}f_1(x)=1$; hence $\operatorname{ord}_{x-\zeta}\mathcal{M}\mathcal{E}_1(f_1, g_\varkappa)=1$. Moreover for every positive λ and every $\zeta\in\hat{R}$ we have $\operatorname{ord}_{x-\zeta}\mathcal{M}\mathcal{E}_\lambda(f_1, g_\varkappa)\leqslant \deg\mathcal{E}_\lambda(f_1, g_\varkappa)\leqslant n_\varkappa$. Hence the function $\operatorname{ord}_{x-\zeta}\mathcal{M}\mathcal{E}_\lambda(f_1, g_\varkappa)$ defined on M belongs to Δ and we have by (53) and (64)

$$\mathcal{O}_\alpha\mathcal{M}f_2 = \text{const}\prod_{\nu=1}^{n}\mathcal{L}I_{j\nu}.$$

This together with (63) and (64) proves (47) and (48). Moreover it follows from (64) that

$$v(H_{j\nu}(f_1, f_2, g_1, \ldots, g_k, x)) = v(G_{j\nu}(f_1, f_2, g_1, \ldots, g_k)).$$

In order to prove the remaining part of the lemma let us assume that

$$\xi\in I,\quad f_1(\xi)=0 \quad\text{and}\quad \overline{\mathcal{K}H_{j\nu}(f_1, f_2, g_1, \ldots, g_k, x)|_{x=\xi}} = 0.$$

Then $\overline{\dfrac{H_{j\nu}(f_1, f_2, g_1, \ldots, g_k, \xi)}{G_{j\nu}(f_1, f_2, g_1, \ldots, g_k)}} = 0$ and by (53), (64) and (65)

$$\operatorname{ord}_{x-\bar{\xi}}\mathcal{M}\mathcal{E}_\lambda(f_1, g_\varkappa) = \delta_\nu(\varkappa,\lambda) \quad \text{for all } [\varkappa,\lambda]\in M.$$

Let $l_\varkappa$ be the least nonnegative integer such that $\mathcal{E}_{l_\varkappa+1}(f_1, g_\varkappa)=0$. We have $l_\varkappa \leqslant n_\varkappa+1$ and for $\lambda\geqslant l_\varkappa$

$$\mathcal{E}_\lambda(f_1, g_\varkappa) = \begin{cases} 0 & \text{if } \lambda\not\equiv l_\varkappa \bmod 2,\\ \mathcal{E}_{l_\varkappa}(f_1, g_\varkappa) & \text{if } \lambda\equiv l_\varkappa \bmod 2,\end{cases}$$

Hence by the definition of $\mathcal{M}$ and $\mathcal{E}_\lambda$ we have for $\lambda\geqslant l_\varkappa$

$$\mathcal{M}\mathcal{E}_\lambda(f_1, g_\varkappa) = \begin{cases} 1 & \text{if } \lambda\not\equiv l_\varkappa \bmod 2,\\ \mathcal{M}\mathcal{E}_{l_\varkappa}(f_1, g_\varkappa) & \text{if } \lambda\equiv l_\varkappa \bmod 2,\end{cases}$$

and

$$(66)\quad \delta_\nu(\varkappa,\lambda) = \begin{cases} 0 & \text{if } \lambda\not\equiv l_\varkappa \bmod 2,\\ \delta_\nu(\varkappa, l_\varkappa) & \text{if } \lambda\equiv l_\varkappa \bmod 2.\end{cases}$$

Therefore, if $\delta_\nu(\varkappa, n_\varkappa+1)=\delta_\nu(\varkappa, n_\varkappa+2)=0$ we have $\delta_\nu(\varkappa,\lambda)=0$ for all $\lambda\geqslant l_\varkappa$ and from Lemma 14 with $f=f_1$, $g=g_\varkappa$, $E_\lambda=\mathcal{E}_\lambda(f_1, g_\varkappa)$ we get $g_\varkappa(\xi)\neq 0$

$$g_\varkappa(\xi) \equiv \prod_{\lambda=1}^{l_\varkappa}(-1)^{d(\varkappa,\lambda-1)d(\varkappa,\lambda)}\left(\frac{E_\lambda^{d(\varkappa,\lambda)}(\xi)}{d(k,\lambda)!}\right)^{d(\varkappa,\lambda-1)}\left(\frac{E_{\lambda-1}^{d(\varkappa,\lambda-1)}(\xi)}{d(\varkappa,\lambda-1)!}\right)^{-d(\varkappa,\lambda)}$$

$$\equiv \prod_{\substack{\lambda=0\\ \delta(\varkappa,\lambda)>0}}^{n_\varkappa+1}(-1)^{d(\varkappa,\lambda)d(\varkappa,\lambda+1)}\left(\frac{E_{\lambda+1}^{d(\varkappa,\lambda+1)}(\xi)}{d(\varkappa,\lambda+1)!}\right)^{d(\varkappa,\lambda)} \Bigg/ \prod_{\substack{\lambda=1\\ \delta(\varkappa,\lambda)>0}}^{n_\varkappa+2}\left(\frac{E_{\lambda-1}^{d(\varkappa,\lambda-1)}(\xi)}{\delta(\varkappa,\lambda-1)!}\right)^{\delta(\varkappa,\lambda)} \bmod P^{v(g_\varkappa(\xi))+1}.$$

However, by (59) and (62), for $\sigma=1,2$,

$$[\operatorname{sg}_K A_1\langle\delta_\nu, \varkappa, \sigma\rangle(f_1, g_\varkappa), \ldots, \operatorname{sg}_K A_{\mu_{\delta_\nu\varkappa\sigma}}\langle\delta_\nu, \varkappa, \sigma\rangle(f_1, g_\varkappa)] \in S_{\varrho_{j\nu\varkappa\sigma}}\langle\delta_\nu, \varkappa, \sigma\rangle;$$

hence (56) and (57) hold with $\varrho=\varrho_{j\nu\varkappa\sigma}$, $\delta=\delta_\nu$, $f=f_1$, $g=g_\varkappa$ and in particular

$$(67)\quad \frac{C_{\varrho j\nu\varkappa 2}\langle\delta_\nu, \varkappa, 2\rangle(f_1, g_\varkappa, \xi)}{B_{\varrho j\nu\varkappa 2}\langle\delta_\nu, \varkappa, 2\rangle(f_1, g_\varkappa)} = \prod_{\substack{\lambda=1\\ \delta(\varkappa,\lambda)>0}}^{n_\varkappa+2}\left(\frac{E_{\lambda-1}^{d(\varkappa,\lambda-1)}(\xi)}{\delta(\varkappa,\lambda-1)!}\right)^{\delta(\varkappa,\lambda)} \neq 0.$$

Thus we have

$$g_\varkappa(\xi) \equiv \frac{C_{\varrho j\nu\varkappa 1}\langle\delta_\nu, \varkappa, 1\rangle(f_1, g_\varkappa, \xi)B_{\varrho j\nu\varkappa 2}\langle\delta_\nu, \varkappa, 2\rangle(f_1, g_\varkappa)}{B_{\varrho j\nu\varkappa 1}\langle\delta_\nu, \varkappa, 1\rangle(f_1, g_\varkappa)C_{\varrho j\nu\varkappa 2}\langle\delta_\nu, \varkappa, 2\rangle(f_1, g_\varkappa, \xi)} \bmod P^{v(g_\varkappa(\xi))+1}.$$

In virtue of (61) this gives (50) while (49) follows from (56) and (67).

Assume now that $\delta_\nu(\varkappa, n_\varkappa+1)+\delta_\nu(\varkappa, n_\varkappa+2)>0$. Then in virtue of (61)

$$\frac{L_{j\nu\varkappa}(f_1, g_\varkappa, \xi)}{K_{j\nu\varkappa}(f_1, g_\varkappa, \xi)} = \frac{0}{1} = 0$$

and (49), (60) follow. The last statement of the lemma follows from (57) and the last statement of Lemma 14.

LEMMA 16. *Let $m\in N$, $u=[u_0, \ldots, u_m]$ and for every pair $[\alpha,\beta]$ where $0\leqslant\alpha, \beta\leqslant m$, $\alpha\neq\beta$ let*

$$q(\alpha,\beta,u) := \frac{u_\beta-u_\alpha}{\beta-\alpha} \quad (u_\alpha+u_\beta<\infty).$$

Furthermore, let $S(\alpha,\beta)$ be the set of all vectors $u\in N_0^{m+1}$ satisfying

$$u_\alpha+u_\beta<\infty,\quad \varrho(\alpha,\beta,u)\in N$$

and

$$(68)\quad \text{for all } \gamma\leqslant m \text{ either } u_\gamma=\infty \text{ or } (q(\alpha,\gamma,u)-q(\alpha,\beta,u))(\gamma-\alpha)\geqslant 0.$$

If $\alpha'<\beta'$, $\alpha''<\beta''$ then for every $u\in S(\alpha',\beta')\cap S(\alpha'',\beta'')$ we have the implications

$$(69)\quad \alpha''<\beta' \to q(\alpha',\beta',u) = q(\alpha'',\beta'',u) \to u\in S(\alpha',\beta'').$$

Proof. Take a $u\in S(\alpha',\beta')\cap S(\alpha',\beta'')$. Using (68) with $\alpha=\alpha'$, $\beta=\beta'$, $\gamma=\alpha''$, we get

(70) $$\big(q(\alpha',\alpha'',u)-(\alpha',\beta',u)\big)(\alpha''-\alpha')\geqslant 0;$$

hence $q(\alpha',\alpha'',u)\geqslant q(\alpha',\beta',u)$.

Using (68) with $\alpha=\alpha''$, $\beta=\beta''$, $\gamma=\alpha'$ we get similarly

$$q(\alpha'',\alpha',u)\leqslant q(\alpha'',\beta'',u),$$

(71) $$q(\alpha',\beta',u)\leqslant q(\alpha',\alpha'',u)\leqslant q(\alpha'',\beta'',u).$$

Using (68) with $\alpha=\alpha''$, $\beta=\beta''$, $\gamma=\beta'$, we get

(72) $$\big(q(\alpha'',\beta',u)-q(\alpha'',\beta'',u)\big)(\beta'-\alpha'')\geqslant 0.$$

Addition of (71) and (72) gives

$$\big(q(\alpha',\beta',u)-q(\alpha'',\beta'',u)\big)(\beta'-\alpha'')\geqslant 0;$$

hence by the assumption

$$q(\alpha',\beta',u)\geqslant q(\alpha'',\beta'',u),$$

which together with (71) gives

$$q(\alpha',\beta',u)=q(\alpha'',\beta'',u).$$

Assume now the last equality. By (71) we have

$$q(\alpha',\alpha'',u)=q(\alpha',\beta',u)=q(\alpha'',\beta'',u)=q, \text{ say}.$$

Hence

$$u_{\alpha''}-u_{\alpha'}=q(\alpha''-\alpha'),\quad u_{\beta''}-u_{\alpha''}=q(\beta''-\alpha''),$$
$$u_{\beta''}-u_{\alpha'}=q(\beta''-\alpha'),\quad q(\alpha',\beta'',u)=q$$

and since $u\in S(\alpha',\beta')$, condition (68) is satisfied with $\alpha=\alpha'$, $\beta=\beta''$.

Lemma 17. *For every $m\in N$ there exists a decomposition*

(73) $$N_+^{m+1}=\bigcup_{r=1}^{r_m}U_r$$

and for each $r\leqslant r_m$ there are finitely many (possibly zero) N-valued functions $\pi\langle r,1\rangle,\ldots,\pi\langle r,s_r\rangle$ defined on U_r such that if

$$f(x)=\sum_{\mu=0}^{m}a_\mu x^{m-\mu}\in I[x],\quad f\neq 0,$$

(74) $$u:=[v(a_0),\ldots,v(a_m)]\in U_r,$$

(75) $$h_{rs}(y):=f(p^{\pi\langle r,s\rangle(u)}y)$$

then

(76) $$\operatorname{card}\{\xi\in P\setminus\{0\}:\ f(\xi)=0\}=\sum_{s=1}^{s_r}\operatorname{card}\{\eta\in I\setminus P:\ h_{rs}(\eta)=0\}.$$

The sets U_r and the functions $\pi\langle r,s\rangle$ are independent of K, v or p.

Proof. We shall use the notation of Lemma 16. Let us order all subsets of the set $\{S(\alpha,\beta):\ 0\leqslant\alpha<\beta\leqslant m\}$ in a sequence $T_1,\ldots,T_{r_m}$ and put for each $r\leqslant r_m$

(77) $$U_r=\bigcap_{S\in T_r}S\cap\bigcap_{S\notin T_r}(N_+^{m+1}\setminus S).$$

It is clear that the sets U_r are disjoint and (73) holds.

For every r with $T_r\neq\emptyset$ we consider the set $\bigcup_{S(\alpha,\beta)\in T_r}\{x \text{ real}:\ \alpha\leqslant x\leqslant\beta\}$ and represent it as the sum of disjoint open intervals:

(78) $$\bigcup_{S(\alpha,\beta)\in T_r}\{x\text{real}:\ \alpha\leqslant x\leqslant\beta\}=\bigcup_{s=1}^{s_r}\{x\text{real}:\ \alpha_{rs}<x<\beta_{rs}\}$$

where

(79) $$\beta_{rs}\leqslant\alpha_{rs+1}\quad(s=1,\ldots,s_r-1).$$

We shall show that for every $s\leqslant s_r$

(80) $$U_r\subset S(\alpha_{rs},\beta_{rs}).$$

Indeed, by (78)–(79) there exists a sequence $[\alpha_i,\beta_i]$ $(i\leqslant n)$ such that

$$\alpha_1=\alpha_{r1},\quad \alpha_{i+1}<\beta_i,\quad \beta_n=\beta_{rs},\quad S(\alpha_i,\beta_i)\in T_r.$$

By Lemma 16 we have by induction on ν

$$\bigcap_{i=1}^{\nu}S(\alpha_i,\beta_i)\subset S(\alpha_1,\beta_\nu);$$

hence by (77)

$$U_r\subset\bigcap_{i=1}^{n}S(\alpha_1,\beta_i)\subset S(\alpha_1,\beta_n)=S(\alpha_{rs},\beta_{rs}).$$

If $T_r=\emptyset$ we take $s_r=0$, otherwise we put

(81) $$\pi\langle r,s\rangle(u):=q(\alpha_{rs},\beta_{rs},u)\quad(1\leqslant s\leqslant s_r).$$

Clearly, the sets U_r and the functions $\pi\langle r,s\rangle(u)$ are independent of K, v or p. Suppose now that for some $u_0\in U_r$ and $s<t$ we have

$$\pi\langle r,s\rangle(u_0)=\pi\langle r,t\rangle(u_0).$$

Since by (80) $u_0\in S(\alpha_{rs},\beta_{rs})\cap S(\alpha_{rt},\beta_{rt})$, we have by Lemma 16

$$u_0\in S(\alpha_{rs},\beta_{rt}).$$

Since $u_0 \in U_r$, we have by (77) $S(\alpha_{rs}, \beta_{rt}) \in T_r$ and it follows from (78) that

$$\alpha_t \in \bigcup_{s=1}^{s_r} \{x \text{ real}: \alpha_{rs} < x < \beta_{rs}\},$$

contrary to (79). Thus for all $u \in U_r$ and $s \neq t$ we have

(82) $$\pi\langle r, s\rangle(u) \neq \pi(r, t)(u).$$

Assume now that (74) holds. Then by (80) for every s

(83) $$u \in S(\alpha_{rs}, \beta_{rs}).$$

Suppose that $f(\xi) = 0$, $\xi \in P \setminus \{0\}$. Let α be the least integer $\leqslant m$ such that

$$\varrho := \min_{\mu} (\alpha_\mu \xi^{m-\mu}) = v(a_\alpha \xi^{m-\alpha}) = v(\alpha_\alpha) + (m-\alpha) v(\xi).$$

From the ultrametric property of v we infer the existence of a $\beta > \alpha$ such that

$$\varrho = v(a_\beta \xi^{m-\beta}) = v(a_\beta) + (m-\beta) v(\xi).$$

It easily follows that

(84) $$v(\xi) = q(\alpha, \beta, u) > 0 \quad \text{and} \quad u \in S(\alpha, \beta).$$

Since $u \in U_r$, we have by (77) $S(\alpha, \beta) \in T_r$ and by (78)

$$\{x \text{ real}: \alpha < x < \beta\} \subset \bigcup_{s=1}^{s_r} \{x \text{ real}: \alpha_{rs} < x < \beta_{rs}\}.$$

Since by (79) the intervals on the right-hand side are disjoint, there exists an $s \leqslant s_r$ such that

$$\alpha_{rs} \leqslant \alpha < \beta \leqslant \beta_{rs}.$$

By (83) we have $u \in S(\alpha, \beta) \cap S(\alpha_{rs}, \beta_{rs})$ and by Lemma 16

$$q(\alpha, \beta, u) = q(\alpha_{rs}, \beta_{rs}, u).$$

By (81) and (84)

$$v(\xi) = \pi\langle r, s\rangle(u).$$

Putting $\eta = p^{-\pi\langle r,s\rangle(u)}\xi$, we get $\eta \in I \setminus P$ and by (75) $h_{rs}(\eta) = 0$. Conversely, if for some s: $\xi := p^{\pi\langle r,s\rangle(u)}$, $\eta \in I \setminus P$ and $h_{rs}(\eta) = 0$, we get by (75) $f(\xi) = 0$, and since $\pi\langle r, s\rangle(u) > 0$ also $\xi \in P \setminus \{0\}$. By (82) to different s correspond different values of $\pi\langle r, s\rangle(u)$ and consequently different ξ. This implies (76) and completes the proof of the lemma.

Definition 7. For a polynomial $f(x) \neq 0$

$$\mathscr{J} f := x^{-\operatorname{ord}_x f} f.$$

Lemma 18. *For an $f \in I[x]$ and a $\varrho \in R$ let*

(85) $$0 \leqslant \deg f \leqslant m, \quad \alpha := \operatorname{ord}_{x-\varrho} \mathscr{M} f \geqslant 1, \quad \Big(f, \prod_{i=1}^{\deg f - 1} f^{(i)}\Big) = 1,$$

let ξ_ϱ be a unique element of I satisfying

(86) $$f^{(\alpha-1)}(\xi_\varrho) = 0, \quad \bar{\xi}_\varrho = \varrho$$

and, in the notation of Lemma 17, let

$$u_\varrho := \left[v\left(\frac{f^{(m)}(\xi_\varrho)}{m!}\right), \ldots, v(f(\xi_\varrho))\right] \in U_r,$$

(87) $$h_{\varrho rs} := f(\xi_\varrho + p^{\pi\langle r,s\rangle(u_\varrho)} x) := f_\varrho(p^{\pi\langle r,s\rangle(u_\varrho)} x) \quad (1 \leqslant s \leqslant s_r).$$

Then

(88) $$\operatorname{card}\{\xi \in I \setminus P: f(\xi) = 0\}$$
$$= \operatorname{card}\{\varrho \in R: \mathscr{J}\mathscr{O}_1 \mathscr{M} f(x)|_{x=\varrho} = 0\} + \sum_{\substack{\varrho \in R\setminus\{0\} \\ \operatorname{ord}_{x-\varrho}\mathscr{M} f \geqslant 2}} \sum_{s=1}^{s_r} \operatorname{card}\{\xi \in I \setminus P: h_{\varrho rs}(\xi) = 0\}.$$

Proof. The existence and the uniqueness of ξ_ϱ follow from Hensel's lemma. Further, we have

(89) $$\operatorname{card}\{\xi \in I \setminus P: f(\xi) = 0\}$$
$$= \sum_{\varrho \in R\setminus\{0\}} \operatorname{card}\{\xi \in I: f(\xi) = 0,\ \bar{\xi} = \varrho\}$$
$$= \sum_{\substack{\varrho \in R\setminus\{0\} \\ \operatorname{ord}_{x-\varrho}\mathscr{M} f = 1}} \operatorname{card}\{\xi \in I: f(\xi) = 0,\ \bar{\xi} = \varrho\} +$$
$$+ \sum_{\substack{\varrho \in R\setminus\{0\} \\ \operatorname{ord}_{x-\varrho}\mathscr{M} f \geqslant 2}} \operatorname{card}\{\xi \in I: f(\xi) = 0,\ \bar{\xi} = \varrho\}.$$

By Lemma 9 and Definition 7 the condition $\varrho \in R \setminus \{0\}$, $\operatorname{ord}_{x-\varrho}\mathscr{M} f = 1$ is equivalent to

$$\mathscr{J}\mathscr{O}_1 \mathscr{M} f(x)|_{x=\varrho} = 0.$$

On the other hand, the condition

$$\xi \in I: f(\xi) = 0, \quad \bar{\xi} = \varrho$$

is in the case $\alpha = 1$ equivalent to $\xi = \xi_\varrho$. Thus the first sum on the right-hand side of (89) is equal to the first sum on the right-hand side of (88). On the other hand,

$$f_\varrho(x) = f(\xi_\varrho + x).$$

Thus $f(x) = f_\varrho(x - \xi_\varrho)$ and $f_\varrho(0) = f(\xi_\varrho)$. Since by (86) $f^{(\alpha-1)}(\xi_\varrho) = 0$ and $\alpha \geqslant 2$, it follows by (85) that $f_\varrho(0) \neq 0$. Thus

$$\operatorname{card}\{\xi \in I: f(\xi) = 0,\ \bar{\xi} = \varrho\} = \operatorname{card}\{\eta \in P \setminus \{0\}: f_\varrho(\eta) = 0\}.$$

However, by (87) and Lemma 17

$$\operatorname{card}\{\eta \in P \setminus \{0\}: f_\varrho(\eta) = 0\} = \sum_{s=1}^{s_r} \operatorname{card}\{\xi \in I \setminus P: h_{\varrho rs}(\xi) = 0\}.$$

Thus the second sums on the right-hand of (88) and of (89) coincide and the lemma follows.

LEMMA 19. *If* $A\in C_0(m)$, $f(x)=\sum_{\mu=0}^{m}a_\mu x^{m-\mu}$ *then*

$$\hat{A}(f,x):=A\left(\frac{f^{(m)}(x)}{m!},\ldots,f(x)\right)\in C_1(m).$$

Proof. For a typical term $a\prod_{\mu=0}^{m}x_\mu^{\alpha_\mu}$ of A we have

$$\sum_{\mu=0}^{m}\alpha_\mu=\deg A,\quad \sum_{\mu=0}^{m}\alpha_\mu\mu=w(A).$$

Hence

$$\deg_{a}\prod_{\mu=0}^{m}\left(\frac{f^{(m-\mu)}(x)}{(m-\mu)!}\right)^{\alpha_\mu}=\sum_{\mu=0}^{m}\alpha_\mu=\deg A,$$

$$w\left(\prod_{\mu=0}^{m}\left(\frac{f^{(m-\mu)}(x)}{(m-\mu)!}\right)^{\alpha_\mu}\right)=\sum_{\mu=0}^{m}\alpha_\mu\mu=w(A),$$

where $\deg_{a}$ denotes the degree with respect to variables $a_0,\ldots,a_m$. Thus $\hat{A}(f,x)$ is homogeneous in $a_0,\ldots,a_m$ and isobaric with respect to all the variables.

LEMMA 20. *If* $A\in C_l(m)$, $h(x)=g(cx)$, $g\in K[x]$, $\deg g\leqslant m$, $c\in K$ *then*

$$A(h,y_1,\ldots,y_l)=c^{m\deg^1A-w(A)}A(g,cy_1,\ldots,cy_l).$$

Proof. For a typical term $a\prod_{\mu=0}^{m}x_\mu^{\alpha_\mu}\prod_{\lambda=1}^{l}y_\lambda^{\beta_\lambda}$ of A we have

$$\sum_{\mu=\sigma}^{m}\alpha_\mu=\deg^1A,\quad \sum_{\mu=0}^{m}\alpha_\mu\mu+\sum_{\lambda=1}^{l}\beta_\lambda=w(A).$$

If $g(x)=\sum_{\mu=0}^{m}b_\mu x^{m-\mu}$ we get $h(x)=\sum_{\mu=0}^{m}(b_\mu c^{m-\mu})x^{m-\mu}$

$$a\Big(\prod_{\mu=0}^{m}b_\mu c^{m-\mu}\Big)^{\alpha_\mu}\prod_{\lambda=1}^{l}y_\lambda^{\beta_\lambda}=a\Big(\prod_{\mu=0}^{m}b_\mu^{\alpha_\mu}\Big)\prod_{\lambda=1}^{l}(cy_\lambda)^{\beta_\lambda}c^{\sum_{\mu=0}^{m}\alpha_\mu(m-\mu)-\sum_{\lambda=1}^{l}\beta_\lambda}.$$

Since the exponent of c equals $m\deg^1A-w(A)$ independently of the term, the lemma follows.

LEMMA 21. *Let* $a_\mu,b_\mu,\xi\in I$, $\bar{\xi}=\varrho$, $K_\mu,L_\mu\in I[y_1]$ $(1\leqslant\mu\leqslant\mu_l)$,

$$v(L_\mu(\xi))=v(L_\mu)\geqslant v(K_\mu)=v(K_\mu(\xi))<\infty,$$

$$a_\mu\equiv\frac{L_\mu(\xi)}{K_\mu(\xi)}\bmod P^{v(a_\mu)+1},\quad q_{\mu\lambda}\in N_0,$$

$$A(y_1,\ldots,y_{l+1}):=\mathscr{L}\mathscr{K}\sum_{\mu=1}^{\mu_l}\frac{L_\mu(y_1)}{K_\mu(y_1)}b_\mu\prod_{\lambda=1}^{l}y_{\lambda+1}^{q_{\mu\lambda}}\prod_{\mu=1}^{\mu_l}K_\mu(y_1).$$

Then

$$(90)\qquad \mathscr{L}\mathscr{K}\Big(\sum_{\mu=1}^{\mu_l}a_\mu b_\mu\prod_{\lambda=1}^{l}y_{\lambda+1}^{q_{\mu\lambda}}\Big)=A(\varrho,y_2,\ldots,y_{l+1})\prod_{\mu=1}^{\mu_l}\overline{\mathscr{K}K_\mu(\xi)}^{-1}$$

(*the operation* $\mathscr{K}$ *performed after the substitution of* $y_1=\xi$ *into* K_μ).

Proof. The assumptions imply for each $\mu\leqslant\mu_l$

$$(91)\qquad \overline{\mathscr{K}(b_\mu L_\mu(\xi))}=\mathscr{L}\mathscr{K}b_\mu L_\mu(y_1)|_{y_1=\varrho},\quad \overline{\mathscr{K}(K_\mu(\xi))}=\mathscr{L}\mathscr{K}K_\mu(y_1)|_{y_1=\varrho}.$$

Let $v(L_\mu)-v(K_\mu)+v(b_\mu)$ attains its minimum for $\mu\in S$ precisely. Then

$$A(y_1,\ldots,y_{l+1})=\sum_{\mu\in S}\mathscr{L}\mathscr{K}b_\mu L_\mu(y_1)\prod_{\lambda=1}^{l}y_{\lambda+1}^{q_{\mu\lambda}}\prod_{\nu\neq\mu}^{\mu_l}\mathscr{L}\mathscr{K}K_\nu(y_1).$$

On the other hand,

$$\mathscr{L}\mathscr{K}\Big(\sum_{\mu=1}^{\mu_l}a_\mu b_\mu\prod_{\lambda=1}^{l}y_{\lambda+1}^{q_{\mu\lambda}}\Big)\prod_{\mu=1}^{\mu_l}\overline{\mathscr{K}K_\mu(\xi)}=\sum_{\mu\in S}\overline{\mathscr{K}_\mu L_\mu(\xi)}\prod_{\lambda=1}^{l}y_{\lambda+1}^{q_{\mu\lambda}}\prod_{\nu\neq\mu}^{\mu_l}\overline{\mathscr{K}K_\nu(\xi)}$$

and (90) follows from (91).

LEMMA 22. *For every two nonnegative integers* m *and* $n\leqslant m$ *there exist finitely many forms* $M_i^{mn}(a)$ $(i\leqslant i_{mn})$ *and polynomials* $N_{jkl}^{mn}(a,y_1,\ldots,y_l)$ $(j\leqslant j_{mn},\ k\leqslant k_j^{mn},\ l\leqslant l_{jk}^{mn})$ *with integral coefficients, a decomposition*

$$(92)\qquad N_+^{i_{mn}}=\bigcup_{j=1}^{j_{mn}}V_j^{mn}$$

and N_0*-valued functions* $v^{mn}\langle j,k,l\rangle$ *defined on* V_j^{mn} *with the following properties (the superscripts are omitted)*:

$$(93)\qquad M_i\in C_0(m),\quad N_{jkl}\in C_l(m);$$

if $\operatorname{char}R=0$ *or* $\operatorname{char}R>m$,

$f(x)\in I[x]$, $0\leqslant\deg f\leqslant m$, *all zeros of* $\mathscr{L}\mathscr{K}f$ *except* 0 *have multiplicity* $\leqslant n$,

$$(94)\qquad \Big(f(x),\ \prod_{i=1}^{\deg f-1}f^{(i)}(x)\Big)=1,$$

$$(95)\qquad \boldsymbol{v}:=[v(M_1(f)),\ldots,v(M_{i_{mn}}(f))]\in V_j$$

and

$$(96)\qquad \tilde{N}_{jkl}(y_1,\ldots,y_l):=\mathscr{L}\mathscr{K}N_{jkl}(f,p^{v\langle j,k,1\rangle(v)}y_1,\ldots,p^{v\langle j,k,l\rangle(v)}y_l),$$

then

$$(97)\qquad \operatorname{card}\{\xi\in I\setminus P:f(\xi)=0\}=\sum_{k=1}^{k_j}\operatorname{card}\Big\{[\eta_1,\eta_2,\ldots]\in R^{l_{jk}}:\bigwedge_{l=1}^{l_{jk}}\tilde{N}_{jkl}(\eta_1,\ldots,\eta_l)=0\Big\}.$$

The polynomials M_i^{mn}, N_{jkl}^{mn}, *the sets* V_j^{mn} *and the functions* $v^{mn}\langle j,k,l\rangle$ *are independent of* K, v *and* p.

Proof by induction on n. Suppose first that $n=0$. Then we take $i_{m0}=j_{m0}=1$, $M_1^{m0}=a_0$, $V_1^{m0}=N_+$, $k_1^{m0}=0$. Both sides of (97) are equal to 0.

In the inductive step m is kept fixed; thus the polynomials M_i^{mn}, N_{jkl}^{mn}, the sets V_j^{mn} and the functions $v^{mn}\langle j,k,l\rangle$ will be distinguished only by the superscript n. Also, we shall write $i_n, j_n, k_j^n, l_{jk}^n$ instead of $i_{mn}, j_{mn}, k_j^{mn}, l_{jk}^{mn}$. We shall use formula (88) of Lemma 18 and consider first the term $\operatorname{card}\{\varrho \in \boldsymbol{R}\colon \mathscr{J}\mathcal{O}_1\mathscr{M}f(x)|_{x=\varrho} = 0\}$.

We have $\mathscr{J} \in \Omega(N_-)$ and thus, by Lemmata 8 and 11, $\mathscr{J}\mathcal{O}_1\mathscr{L} \in \Omega^*(N_-)$. Hence there exist polynomials $A_t, B_u \in C_0(m)$, $C_\mu \in C_1(m)$ $(t \leqslant t_0, u \leqslant u_0)$ and a decomposition

$$N_+^{t_0} = \bigcup_{u=1}^{u_0} \boldsymbol{T}_u \tag{98}$$

such that if

$$[v(A_1(f)), \ldots, v(A_{t_0}(f))] \in \boldsymbol{T}_u$$

then

$$B_u(f) \neq 0, \quad \frac{C_u(f,x)}{B_u(f)} \in \boldsymbol{I}[x]$$

and

$$\mathscr{J}\mathcal{O}_1\mathscr{M}f = \mathscr{L}\frac{C_u(f,x)}{B_u(f)} = \mathscr{L}\mathscr{K}\frac{C_u(f,x)}{B_u(f)}. \tag{99}$$

Consider now the term $\operatorname{card}\{\xi \in \boldsymbol{I}\setminus\boldsymbol{P}\colon h_{\varrho rs}(\xi) = 0\}$ in (88). As we shall show, all zeros of $\mathscr{M}h_{\varrho rs}$ except 0 have multiplicity less than n. Indeed, by (85) $(\mathscr{K}f)^{(\alpha)}(\xi_\varrho) \not\equiv 0 \bmod \boldsymbol{P}$; hence by (87)

$$\deg \mathscr{M}h_{\varrho rs} \leqslant \operatorname{ord}_x \mathscr{L}\mathscr{K}f(\xi_\varrho + x) \leqslant \alpha \leqslant n$$

and, if the multiplicity of a zero ζ of $\mathscr{M}h_{\varrho rs}$ were n, we should have

$$\mathscr{L}\mathscr{K}h_{\varrho rs}(x) = \frac{\overline{(\mathscr{K}f)^{(n)}(\xi_\varrho)}}{n!}(x-\zeta)^n.$$

However, also by (87)

$$\mathscr{L}\mathscr{K}h_{\varrho rs}(x) = \overline{\sum_{\mu=0}^{n}\frac{(\mathscr{K}f)^{(\mu)}(\xi_\varrho)}{\mu!}p^{\pi\langle r,s\rangle(v_\varrho)(\mu-n)}x^\mu}.$$

Comparing the coefficient of x^{n-1} in the two expressions for $\mathscr{L}\mathscr{K}h_{\varrho rs}$ and using (86), we find $\zeta = 0$. In view of (87) and (94) we have also

$$\Big(h_{\varrho rs}, \prod_{i=1}^{\deg h_{\varrho rs}-1} h_{\varrho rs}^{(i)}\Big) = 1;$$

thus the inductive assumption applies to $h_{\varrho rs}$. Accordingly, we shall apply Lemma 15 to the polynomials $f_1 = f^{(\alpha-1)}$, $f_2 = f$ and the following polynomials $g_\varkappa$:

$$f^{(i)}(x)\ (0 \leqslant i \leqslant n), \quad \widehat{M_i^{n-1}}(f,x)\ (i \leqslant i_{n-1}), \quad \widehat{N_{jkl\boldsymbol{q}}^{n-1}}(f,x)\ (j \leqslant j_{n-1}, k \leqslant k_j^{n-1}, l \leqslant l_{jk}^{n-1})$$

$$\boldsymbol{q} = [q_1, \ldots, q_l] \in N_0^l, \quad q_1 + \ldots + q_l \leqslant w(N_{jkl}^{n-1}),$$

where $N_{jkl\boldsymbol{q}}^{n-1}$ is the coefficient of $\prod_{\lambda=1}^{l} y_\lambda^{q_\lambda}$ in $N_{jkl}^{n-1}(\boldsymbol{a}, y_1, \ldots, y_l)$.

Therefore we order all vectors $[j,k,l,\boldsymbol{q}]$ in question lexicographically and let $[j,k,l,\boldsymbol{q}]$ (j,k,l, fixed, $\boldsymbol{q}$ variable) occupy the places

$$\lambda(j,k,l)+1 \quad \text{to} \quad \lambda(j,k,l)+\mu(j,k,l).$$

The relevant vectors $\boldsymbol{q}$ will be denoted by $\boldsymbol{q}(j,k,l,\mu)$ $(1 \leqslant \mu \leqslant \mu(j,k,l))$ and the λth component of $\boldsymbol{q}(j,k,l,\mu)$ by $q(j,k,l,\mu,\lambda)$ $(1 \leqslant \lambda \leqslant l)$.

If $[j,k,l,\boldsymbol{q}]$ occupies the νth place, we shall write

$$N_{jkl\boldsymbol{q}}^{n-1} = N_\nu^{n-1} \quad (\nu \leqslant \nu_{n-1}),$$

so that

$$N_{jkl}^{n-1}(\boldsymbol{a}, y_1, \ldots, y_l) = \sum_{\mu=1}^{\mu\langle j,k,l\rangle} N_{\lambda(j,k,l)+\mu}^{n-1}(\boldsymbol{a}) \prod_{\lambda=1}^{l} y_\lambda^{q(j,k,l,\mu,\lambda)}. \tag{100}$$

Now we set in Lemma 15

$$n_\varkappa = \begin{cases} m-\varkappa+1 & \text{if } 1 \leqslant \varkappa \leqslant m+1, \\ w(M_{\varkappa-m-1}^{n-1}) & \text{if } m+1 < \varkappa \leqslant m+1+i_{n-1}, \\ w(N_{\varkappa-m-1-i_{n-1}}^{n-1}) & \text{if } m+1+i_{n-1} < \varkappa \leqslant m+1+i_{n-1}+\nu_{n-1} = k_0 \end{cases}$$

and denote the corresponding parameters of that lemma by $i_0(\alpha), j_0(\alpha), \nu_0(\alpha)$, $F_{\alpha i}, G_{\alpha j\nu}, H_{\alpha j\nu}, K\langle\alpha,j,\nu,\varkappa\rangle, L\langle\alpha,j,\nu,\varkappa\rangle, \boldsymbol{T}_{\alpha j}$. Then we set

$$g_\varkappa(x) = \begin{cases} \Big(\sum_{\mu=0}^{m} a_\mu x^{m-\mu}\Big)^{(\varkappa-1)} & (1 \leqslant \varkappa \leqslant m+1), \\ \widehat{M_{\varkappa-m-1}^{n-1}}\Big(\sum_{\mu=0}^{m} a_\mu x^{m-\mu}, x\Big) & (m+1 < \varkappa \leqslant m+1+i_{n-1}), \\ \widehat{N_{\varkappa-m-1-i_{n-1}}^{n-1}}\Big(\sum_{\mu=0}^{m} a_\mu x^{m-\mu}, x\Big) & (m+1+i_{n-1} < \varkappa \leqslant m+1+i_{n-1}+\nu_{n-1}) \end{cases} \tag{101}$$

and observe that, in virtue of Lemma 19, $\deg g_\varkappa \leqslant n_\varkappa$.

Let

$$\boldsymbol{M} := \{[\alpha, \nu] \in N^2\colon 1 < \alpha \leqslant n,\ 1 < \nu \leqslant \nu_0(\alpha)\}$$

$$\boldsymbol{\Phi} := \{\varphi \in N^{\{1,2,\ldots,n\}}\colon \varphi(1) \leqslant u_0,\ \varphi(\alpha) \leqslant j_0(\alpha) \text{ for } \alpha > 1\},$$

$$\boldsymbol{X} := \{1, 2, \ldots, r_m\}^{\boldsymbol{M}},$$

and for $\chi \in X$ let

$$S_\chi := \{[\alpha, \nu, s]:\ [\alpha, \nu] \in M:\ 1 \leqslant s \leqslant s_{\chi(\alpha,\nu)}\},$$
$$\Psi_\chi := \{1, 2, \ldots, j_{n-1}\}^{S_\chi}.$$

Let us order in a sequence all triples $[\varphi, \chi, \psi]$ where $\varphi \in \Phi$, $\chi \in X$, $\psi \in \Psi_\chi$, and denote the jth term of this sequence by $[\varphi_j, \chi_j, \psi_j]$ $(j \leqslant j_n)$. For each $j \leqslant j_n$ we order all quadruples $[\alpha, \nu, s, \varkappa]$ where $1 < \alpha \leqslant n$, $\nu \leqslant \nu_0(\alpha)$, $s \leqslant s_{\chi_j(\alpha,\nu)}$, $\varkappa \leqslant k^{n-1}_{\psi_j(\alpha,\nu,s)}$ in a sequence and denote the kth term of this sequence by $[\alpha_{jk}, \nu_{jk}, s_{jk}, \varkappa_{jk}]$ $(k \leqslant k_j^n)$.

The sequence M_i^n is defined as consisting of blocks corresponding to $\alpha = 1, 2, \ldots, n$ in an increasing order. For $\alpha = 1$ we take polynomials $A_1, \ldots, A_{t_0}$, for an $\alpha > 1$ we take polynomials $F_{\alpha i}(\boldsymbol{g}_\alpha, \boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_{k_0})$ $(1 \leqslant i \leqslant i_0(\alpha))$, then all the coefficients of the polynomials $K\langle \alpha, \beta, \nu, \varkappa\rangle(\boldsymbol{g}_\alpha, \boldsymbol{g}_\varkappa, x)$ and $L\langle \alpha, \beta, \nu, \varkappa\rangle(\boldsymbol{g}_\alpha, \boldsymbol{g}_\varkappa, x)$ $(1 \leqslant \beta \leqslant j_0(\alpha),\ 1 \leqslant \nu \leqslant \nu_0(\alpha),\ 1 \leqslant \varkappa \leqslant k_0)$ ordered lexicographically (the order of letters being $\beta, \nu, \varkappa$).

Now we put for $\varkappa \leqslant k_0$

$$r_{\alpha\beta\nu\varkappa} := \sum_{\iota=1}^{\varkappa} (\deg_x K\langle \alpha, \beta, \nu, \iota\rangle + \deg_x L\langle \alpha, \beta, \nu, \iota\rangle + 2),$$

(102) $$r'_{\alpha\beta\nu\varkappa} := r_{\alpha\beta\nu\varkappa-1} + \deg_x K\langle \alpha, \beta, \nu, \varkappa\rangle + 1,$$

(103) $$a_{\alpha\beta} := \sum_{\lambda<\beta} \sum_{\nu=1}^{\nu_0(\alpha)} r_{\alpha\lambda\nu m+1+i_{n-1}}, \qquad b_{\alpha\beta} := \sum_{\lambda>\beta}^{j_0(\alpha)} \sum_{\nu=1}^{\nu_0(\alpha)} r_{\alpha\lambda\nu m+1+i_{n-1}},$$
$$d_\alpha := \sum_{\beta=1}^{j_0(\alpha)} \sum_{\nu=1}^{\nu_0(\alpha)} r_{\alpha\beta\nu m+1+i_{n-1}}, \qquad c_{\alpha\beta\nu} := r_{\alpha\beta\nu m+1+i_{n-1}} - r_{\alpha\beta\nu m+1},$$

so that

$$i_n = t_0 + \sum_{\alpha=2}^{n} (i_0(\alpha) + d_\alpha).$$

For a vector $\boldsymbol{u} = [u_1, u_2, \ldots] \in N_+^{r_{\alpha\beta\nu m+1}}$ define $\tau_{\alpha\beta\nu}(\boldsymbol{u})$ as a vector whose μth component $(1 \leqslant \mu \leqslant m+1)$ equals

(104) $$|\min\{u_{r'_{\alpha\beta\nu\mu}+1}, \ldots, u_{r_{\alpha\beta\nu\mu}}\} - \min\{u_{r_{\alpha\beta\nu\mu-1}+1}, \ldots, u_{r'_{\alpha\beta\nu\mu}}\}|,$$

where by convention $\infty - \infty = 0$, and for $\boldsymbol{u} \in N_+^{r_{\alpha\beta\nu m+1+i_{n-1}}}$, $\boldsymbol{u} = [u_1, \ldots, u_{r_{\alpha\beta\nu m+1}}]$ define $\omega_{\alpha\beta\nu s}(\boldsymbol{u})$ as follows. If $1 < \alpha \leqslant n$, $\beta \leqslant j_0(\alpha)$, $\nu \leqslant \nu_0(\alpha)$, $\tau_{\alpha\beta\nu}(\boldsymbol{u}) \in U_r$, $s \leqslant s_r$ (notation of Lemma 17), then for $\mu \leqslant m+1$ the μth component $\omega_{\alpha\beta\nu s\mu}(\boldsymbol{u})$ of $\omega_{\alpha\beta\nu s}(\boldsymbol{u})$ equals the μth component of $\tau_{\alpha\beta\nu}(\boldsymbol{u}')$ and for $\mu > m+1$ the μth component of $\omega_{\alpha\beta\nu s}(\boldsymbol{u})$ equals

(105) $$|\min\{u_{r'_{\alpha\beta\nu\mu}+1}, \ldots, u_{r_{\alpha\beta\nu\mu}}\} - \min\{u_{r_{\alpha\beta\nu\mu-1}+1}, \ldots, u_{r'_{\alpha\beta\nu\mu}}\}| + \pi\langle r, s\rangle\big(\tau_{\alpha\beta\nu}(\boldsymbol{u}')\big(m \deg M^{n-1}_{\mu-m-1} - w(M^{n-1}_{\mu-m-1})\big),$$

where again $\infty - \infty = 0$.

Further, let us put

(106) $$V\langle \alpha, \beta, \nu, \chi, \psi\rangle := \begin{cases} \tau^{-1}_{\alpha\beta\nu}(U_{\chi(\alpha,\nu)}) \times N_+^{c_{\alpha\beta\nu}} & \text{if } s_{\chi(\alpha,\nu)} = 0, \\ \bigcap\limits_{s \leqslant s_{\chi(\alpha,\nu)}} \omega^{-1}_{\alpha\beta\nu s}(U_{\chi(\alpha,\nu)} \times V_{\psi(\alpha,\nu,s)}) & \text{otherwise} \end{cases}$$

and for $j \leqslant j_n$

(107) $$V_j^n = T_{1\varphi_j(1)} \times \mathop{\boldsymbol{P}}_{\alpha=2}^{n} \Big(T_{\alpha\varphi_j(\alpha)} \times N_+^{a_{\alpha\varphi_j(\alpha)}} \times \Big(\mathop{\boldsymbol{P}}_{\nu=1}^{\nu_0(\alpha)} V\langle \alpha, \varphi_j(\alpha), \nu, \chi_j, \psi_j\rangle\Big) \times N_+^{b_{\alpha\varphi_j(\alpha)}}\Big).$$

Furthermore, for $j \leqslant j_n$, $k < k_j^n$ put

(108) $$\lambda\big(\psi_j(\alpha_{jk}, \nu_{jk}, s_{jk}), \varkappa_{jk}, l\big) = \lambda_{jkl},$$
$$\mu\big(\psi_j(\alpha_{jk}, \nu_{jk}, s_{jk}), \varkappa_{jk}, l\big) = \mu_{jkl},$$
$$q\big(\psi_j(\alpha_{jk}, \nu_{jk}, s_{jk}), \varkappa_{jk}, l, \mu, \lambda\big) = q_{jkl\mu\lambda},$$

(109) $$K\langle \alpha_{jk}, \varphi_j(\alpha_{jk}), \nu_{jk}, m+1+i_{n-1}+\lambda_{jkl}+\mu\rangle\big(\boldsymbol{a}^{(\alpha_{jk}-1)}, \widehat{N^{n-1}_{\lambda_{jkl}+\mu}}(\boldsymbol{a}, y_1), y_1\big) = K_{jkl\mu}(\boldsymbol{a}, y_1),$$

(110) $$L\langle \alpha_{jk}, \varphi_j(\alpha_{jk}), \nu_{jk}, m+1+i_{n-1}+\lambda_{jkl}+\mu\rangle\big(\boldsymbol{a}^{(\alpha_{jk}-1)}, \widehat{N^{n-1}_{\lambda_{jkl}+\mu}}(\boldsymbol{a}, y_1), y_1\big) = L_{jkl\mu}(\boldsymbol{a}, y_1),$$

(111) $$N^n_{jk1}(\boldsymbol{a}, y_1) = H\langle \alpha_{jk}, \varphi_j(\alpha_{jk}), \nu_{jk}\rangle \big(\boldsymbol{a}^{(\alpha_{jk}-1)}, \boldsymbol{a};\ \boldsymbol{a}, \boldsymbol{a}', \ldots, \boldsymbol{a}^{(m)}, \ldots, *\widehat{M^{n-1}_1}(\boldsymbol{a}, y_1), \ldots, *\widehat{M^{n-1}_{i_{n-1}}}(\boldsymbol{a}, y_1), *\widehat{N^{n-1}_1}(\boldsymbol{a}, y_1), \ldots, *\widehat{N^{n-1}_{\nu_{n-1}}}(\boldsymbol{a}, y_1), y_1\big),$$

(112) $$\nu^n\langle j, k, 1\rangle(\boldsymbol{v}) = 0,$$

(113) $$N^n_{jkl+1}(\boldsymbol{a}, y_1, \ldots, y_{l+1}) = \sum_{\mu=1}^{\mu_{jkl}} \frac{L_{jkl\mu}(\boldsymbol{a}, y_1)}{K_{jkl\mu}(\boldsymbol{a}, y_1)} \prod_{\lambda=1}^{l} y_{\lambda+1}^{q_{jkl\lambda}} \prod_{\mu=1}^{\mu_{jkl}} K_{jkl\mu}(\boldsymbol{a}, y_1)$$

and for $1 \leqslant l \leqslant l^{n-1}_{\psi_j(\alpha_{jk}, \nu_{jk}, s_{jk})\varkappa_{jk}} := l^n_{jk} - 1$

(114) $$\nu^n\langle j, k, l+1\rangle(\boldsymbol{v})$$
$$= \pi\langle \chi_j(\alpha_{jk}, \nu_{jk}), s_{jk}\rangle\big(\omega_{\alpha_{jk}\varphi_j(\alpha_{jk})\nu_{jk}s_{jk}1}(\boldsymbol{v}_{jk}), \ldots, \omega_{\alpha_{jk}\varphi_j(\alpha_{jk})\nu_{jk}s_{jk}m+1}(\boldsymbol{v}_{jk})\big) +$$
$$+ \nu^{n-1}\langle \psi_j(\alpha_{jk}, \nu_{jk}, s_{jk}), \varkappa_{jk}, l\rangle\big(\omega_{\alpha_{jk}\varphi_j(\alpha_{jk})\nu_{jk}s_{jk}m+2}(\boldsymbol{v}_{jk}), \ldots, \omega_{\alpha_{jk}\varphi_j(\alpha_{jk})\nu_{jk}s_{jk}m+1+i_{n-1}}(\boldsymbol{v}_{jk})\big),$$

where, assuming $\boldsymbol{v} = [v_1, v_2, \ldots]$, we set

$$\boldsymbol{v}_{jk} := [v_{a+1}, \ldots, v_{a+b}],$$
$$a := t_0 + a_{\alpha_{jk}\varphi_j(\alpha_{jk})} + \sum_{\lambda<\nu_{jk}} r_{\alpha_{jk}\varphi_j(\alpha_{jk})\lambda m+1+i_{n-1}}, \qquad b := r_{\alpha_{jk}\varphi_j(\alpha_{jk})\nu_{jk}m+1+i_{n-1}}.$$

Finally, for $j \leqslant j_n$: $l^n_{jk_j^n} = 1$,

(115) $$N^n_{jk_j^n1}(\boldsymbol{a}, y_1) = C_{\varphi_j(1)}(\boldsymbol{a}, y_1),$$

(116) $$\nu^n\langle j, k_j^n, 1\rangle(\boldsymbol{v}) = 0.$$

Clearly the sets, the polynomials and the functions defined above are independent of $\boldsymbol{K}$, v and p. Now we proceed to prove that they have all the properties asserted in the lemma.

The polynomials M_i^n of the first block ($\alpha = 1$) belong to $C_0(m)$ since $A_t \in C_0(m)$, ($t \leqslant t_0$). Consider now the block α for $\alpha > 1$. By Lemma 15, $F_{\alpha i} \in C_0(m, m, n_1, \ldots, n_{k_0})$ while the coefficients of polynomials $K\langle\alpha, \beta, \nu, \varkappa\rangle$, $L\langle\alpha, \beta, \nu, \varkappa\rangle$ belong to $C_0(m, n_\varkappa)$. Since by the inductive assumption $M_i^{n-1} \in C_0(m)$ and $N_i^{n-1} \in C_1(m)$, we have, by Lemma 19 and (101), $g_\varkappa \in C_1(m)$ $(1 \leqslant \varkappa \leqslant k_0)$.

It follows now from Lemma 1 that

$$F_{\alpha i}(g_{\alpha-1}, g_1;\ g_1, \ldots, g_{k_0}) \in C_0(m),$$

and the same applies to the coefficients of polynomials $K\langle\alpha, \beta, \nu, \varkappa\rangle(g_\alpha, g_\varkappa, x)$ and $L\langle\alpha, \beta, \nu, \varkappa\rangle(g_\alpha, g_\varkappa, x)$. Hence all polynomials M_i^n belong to $C_0(m)$ and the first part of (93) is proved. The proof of the second part of (93) for $l = 1$ is similar in view of (111). For $l > 1$ we need in view of (108) and (113) to show that, for fixed j, k and each μ,

$$R_\mu := \frac{L_{jkl\mu}(\boldsymbol{a}, y_1)}{K_{jkl\mu}(\boldsymbol{a}, y_1)} \prod_{\mu=1}^{\mu_{jkl}} K_{jkl\mu}(\boldsymbol{a}, y_1) \in C_1(m), \tag{117}$$

the degree of all non-zero polynomials R_μ with respect to $\boldsymbol{a}$ is the same and their weight differs from $w(N^{n-1}_{\lambda_{jkl}+\mu})$ by a constant summand. Now (117) follows from Lemma 19 and (109)-(110). Moreover, by (109), (110), Lemma 15 and Lemma 19, either $R_\mu = 0$ or

$$\begin{aligned}
\deg_{\boldsymbol{a}} R_\mu - \sum_{\mu=1}^{\mu_{jkl}} \deg_{\boldsymbol{a}} K_{jkl\mu} &= \deg_{\boldsymbol{a}} L_{jkl\mu} - \deg_{\boldsymbol{a}} K_{jkl\mu} \\
&= \deg^1 L\langle\alpha_{jk}, \varphi_j(\alpha_{jk}), m+1+i_{n-1}+\lambda_{jkl}+\mu\rangle + \\
&\quad + \deg^2 L\langle\alpha_{jk}, \varphi_j(\alpha_{jk}), m+1+i_{n-1}+\lambda_{jkl}+\mu\rangle \deg_{\boldsymbol{a}} \widehat{M}^{n-1}_{\lambda_{jkl}+\mu} - \\
&\quad - \deg^1 K\langle\alpha_{jk}, \varphi_j(\alpha_{jk}), m+1+i_{n-1}+\lambda_{jkl}+\mu\rangle - \\
&\quad - \deg^2 K\langle\alpha_{jk}, \varphi_j(\alpha_{jk}), m+1+i_{n-1}+\lambda_{jkl}+\mu\rangle \deg_{\boldsymbol{a}} \widehat{N}^{n-1}_{\lambda_{jkl}+\mu} \\
&= \deg_{\boldsymbol{a}} \widehat{N}^{n-1}_{\lambda_{jkl}+\mu} = \deg_{\boldsymbol{a}} N^{n-1}_{\lambda_{jkl}+\mu},
\end{aligned}$$

but, by (100) and (108), $N^{n-1}_{\lambda_{jkl}+\mu}$ are the coefficients of $N^{n-1}_{\psi_j(\alpha_{jk}, \nu_{jk}, s_{jk})\varkappa_{jk}l}$ and thus they all have the same degree with respect to $\boldsymbol{a}$.

Furthermore, by Lemma 1, Lemma 15 and (101), either $R_\mu = 0$ or

$$\begin{aligned}
w(R_\mu) - \sum_{\mu=1}^{\mu_{jkl}} w(K_{jkl\mu}) &= w(L_{jkl\mu}) - w(K_{jkl\mu}) \\
&= w\Big(L\langle\alpha_{jk}, \varphi_j(\alpha_{jk}), m+1+i_{n-1}+\lambda_{jkl}+\mu\rangle\big(*\big(\sum_{\mu=0}^{m} a_\mu x^{m-\mu}\big)^{(\alpha_{jk}-1)}, *\widehat{N}^{n-1}_{\alpha_{jkl}+\mu}, x\big)\Big) \\
&= w\Big(K\langle\alpha_{jk}, \varphi_j(\alpha_{jk}), m+1+i_{n-1}+\lambda_{jkl}+\mu\rangle\big(*\big(\sum_{\mu=0}^{m} a_\mu x^{m-\mu}\big)^{(\alpha_{jk}-1)}, *\widehat{N}^{n-1}_{\alpha_{jkl}+\mu}, x\big)\Big)
\end{aligned}$$

$$\begin{aligned}
&= w(L\langle\alpha_{jk}, \varphi_j(\alpha_{jk}), m+1+i_{n-1}+\lambda_{jkl}+\mu\rangle) - (\alpha_{jk}-1)\deg^1 L\langle\alpha_{jk}, \varphi_j(\alpha_{jk}), m+1+ \\
&\quad + i_{n-1}+\lambda_{jkl}+\mu\rangle - w(K\langle\alpha_{jk}, \varphi_j, (\alpha_{jk}), m+1+i_{n-1}+\lambda_{jkl}+\mu\rangle) + \\
&\quad + (\alpha_{jk}-1)\deg^1 K\langle\alpha_{jk}, \varphi_j(\alpha_{jk}), m+1+i_{n-1}+\lambda_{jkl}+\mu\rangle \\
&= n_{m+1+i_{n-1}+\lambda_{jkl}+\mu} = w(N^{n-1}_{\alpha_{jkl}+\mu}).
\end{aligned}$$

This completes the proof of (93).

Now we shall show that sets V_j^n are disjoint and (92) holds. If $j \neq j'$ we have one of the following cases:

$$\varphi_j \neq \varphi_{j'}, \tag{118}$$

$$\varphi_j = \varphi_{j'}, \quad \chi_j \neq \chi_{j'}, \tag{119}$$

$$\varphi_j = \varphi_{j'}, \quad \chi_j = \chi_{j'}, \quad \psi_j \neq \psi_{j'}. \tag{120}$$

In the case (118) there exists an $\alpha \leqslant n$ such that

$$\varphi_j(\alpha) \neq \varphi_{j'}(\alpha), \quad \text{thus} \quad T_{\alpha\varphi_j(\alpha)} \cap T_{\alpha\varphi_{j'}(\alpha)} = \emptyset;$$

hence by (107)

$$V_j^n \cap V_{j'}^n = \emptyset.$$

In the case (119) there exists a pair $[\alpha, \nu]$ such that $1 \leqslant \alpha \leqslant n$, $1 \leqslant \nu \leqslant \nu_0(\alpha)$ and $\chi_j(\alpha, \nu) \neq \chi_{j'}(\alpha, \nu)$, whence $U_{\chi_j}(\alpha, \nu) \cap U_{\chi_{j'}}(\alpha, \nu) = \emptyset$. Let us observe that by (103) and (106) for all α, β, ν, χ

$$\bigcup_{\psi \in \Psi_\chi} V\langle\alpha, \beta, \nu, \chi, \psi\rangle = \tau^{-1}_{\alpha\beta\nu}(U_{\chi(\alpha,\nu)}) \times N_+^{c_{\alpha\beta\nu}}. \tag{121}$$

If $\varphi_j(\alpha) = \varphi_{j'}(\alpha) = \beta$ we have

$$\tau^{-1}_{\alpha\beta\nu}(U_{\chi_j(\alpha,\nu)}) \cap \tau^{-1}_{\alpha\beta\nu}(U_{\chi_{j'}(\alpha,\nu)}) = \emptyset$$

and by (121)

$$V\langle\alpha, \varphi_j(\alpha), \nu, \chi_j, \psi_j\rangle \cap V\langle\alpha, \varphi_{j'}(\alpha), \nu, \chi_{j'}, \psi_{j'}\rangle = \emptyset;$$

thus by (107) $V_j^n \cap V_{j'}^n = \emptyset$.

In the case (120) there exists a triple $[\alpha, \nu, s]$ such that $1 < \alpha \leqslant n$, $1 \leqslant \nu \leqslant \nu_0(\alpha)$, $1 \leqslant s \leqslant \chi_j(\alpha, \nu) = \chi_{j'}(\alpha, \nu)$ and $\psi_j(\alpha, \nu, s) \neq \psi_{j'}(\alpha, \nu, s)$. Therefore

$$V^{n-1}_{\psi_j(\alpha,\nu,s)} \cap V^{n-1}_{\psi_{j'}(\alpha,\nu,s)} \neq \emptyset$$

and if $\beta = \varphi_j(\alpha) = \varphi_{j'}(\alpha)$, $\gamma = \chi_j(\alpha, \nu) = \chi_{j'}(\alpha, \nu)$ we have

$$\omega^{-1}_{\alpha\beta\nu s}(U_\gamma \times V^{n-1}_{\psi_j(\alpha,\nu,s)}) \cap \omega^{-1}_{\alpha\beta\nu s}(U_\gamma \times V_{\psi_{j'}(\alpha,\nu,s)}) = \emptyset.$$

Hence by (106)

$$V\langle\alpha, \varphi_j(\alpha), \nu, \chi_j, \psi_j\rangle \cap V\langle\alpha, \varphi_{j'}(\alpha), \nu, \chi_{j'}, \psi_{j'}\rangle = \emptyset$$

and by (107) $V_j^n \cap V_{j'}^n = \emptyset$.

In order to show that $\bigcup_{j=1}^{j_n} V_j^n = N_+^{i_n}$ we proceed as follows. By (121) we have for every pair $[\alpha, \nu] \in \boldsymbol{M}$ and every number $\beta \leqslant j_0(\alpha)$

$$\bigcup_{\chi \in X} \bigcup_{\psi \in \Psi_\chi} V\langle \alpha, \beta, \nu, \chi, \psi \rangle = \tau_{\alpha\beta\nu}^{-1}\Big(\bigcup_{\chi \in X} U_{\chi(\alpha,\nu)}\Big) \times N_+^{c_{\alpha\beta\nu}} = \tau_{\alpha\beta\nu}^{-1}(N_+^{m-1}) \times N_+^{c_{\alpha\beta\nu}}$$
$$= N_+^{r_{\alpha\beta\nu m+1}+c_{\alpha\beta\nu}} = N_+^{r_{\alpha\beta\nu m+1}+i_{n-1}};$$

hence by Lemma 15, (98) and (103)

$$\bigcup_{j=1}^{j_n} V_j^n = \bigcup_{\varphi \in \Phi} \bigcup_{\chi \in X} \bigcup_{\psi \in \Psi_\chi} \Big(\boldsymbol{T}_{1\varphi(1)} \times \mathop{\boldsymbol{P}}_{\alpha=2}^{n} \Big(\boldsymbol{T}_{\alpha\varphi(\alpha)} \times N_+^{a_{\alpha\varphi(\alpha)}} \times \mathop{\boldsymbol{P}}_{\nu=1}^{\nu_0(\alpha)} V\langle \alpha, \varphi(\alpha), \nu, \chi, \psi \rangle \times N_+^{b_{\alpha\varphi(\alpha)}}\Big)\Big)$$
$$= \bigcup_{\varphi \in \Phi} \Big(\boldsymbol{T}_{1\varphi(1)} \times \mathop{\boldsymbol{P}}_{\alpha=2}^{n} \Big(\boldsymbol{T}_{\alpha\varphi(\alpha)} \times N_+^{a_{\alpha\varphi(\alpha)}} \times \mathop{\boldsymbol{P}}_{\nu=1}^{\nu_0(\alpha)} \bigcup_{\chi \in X} \bigcup_{\psi \in \Psi_\chi} V\langle \alpha, \varphi(\alpha), \nu, \chi, \psi \rangle \times N_+^{b_{\alpha\varphi(\alpha)}}\Big)\Big)$$
$$= \bigcup_{\varphi \in \Phi} \Big(\boldsymbol{T}_{1\varphi(1)} \times \mathop{\boldsymbol{P}}_{\alpha=2}^{n} \Big(\boldsymbol{T}_{\alpha\varphi(\alpha)} \times N_+^{a_{\alpha\varphi(\alpha)}} \times \mathop{\boldsymbol{P}}_{\nu=1}^{\nu_0(\alpha)} N_+^{r_{\alpha\varphi(\alpha)\nu m+1}+i_{n-1}} \times N_+^{b_{\alpha\varphi(\alpha)}}\Big)\Big)$$
$$= \bigcup_{\varphi \in \Phi} \Big(\boldsymbol{T}_{1\varphi(1)} \times \mathop{\boldsymbol{P}}_{\alpha=2}^{n} (\boldsymbol{T}_{\alpha\varphi(\alpha)} \times N_+^{d_\alpha})\Big) = \bigcup_{u=1}^{u_0} \Big(\boldsymbol{T}_{1u} \times \mathop{\boldsymbol{P}}_{\alpha=2}^{n} \Big(\bigcup_{\beta=1}^{j_0(\alpha)} \boldsymbol{T}_{\alpha\beta} \times N_+^{d_\alpha}\Big)\Big)$$
$$= N_+^{t_0} \times \mathop{\boldsymbol{P}}_{\alpha=2}^{n} N_+^{i_0(\alpha)+d_\alpha} = N_+^{i_n}.$$

The claim that the functions $v^n\langle j, k, l\rangle(\boldsymbol{v})$ are N_0-valued is obvious from (112), (114) and (116).

Now we assume (94)-(96) and proceed to prove (97), using the formula (88), Lemma 18. Let $f(x) = \sum_{\mu=0}^{m} a_\mu x^{m-\mu}$. By (95) and (107) we have

$$[v(A_1(f)), \ldots, v(A_{t_0}(f))] \in \boldsymbol{T}_{1\varphi_j(1)};$$

hence by (99) and (115)

$$\mathscr{J}\mathscr{O}_1\mathscr{M}f = \mathscr{L}\frac{C_{\varphi_j(1)}(f, x)}{B_{\varphi_j(1)}(f)} = c_j\mathscr{L}\mathscr{K}N_{jk_j^n1}^n(f, x), \quad \text{where} \quad c_j \neq 0.$$

Therefore, by (96) and (116)

$$(122) \qquad \operatorname{card}\{\varrho \in R\colon \mathscr{J}\mathscr{O}_1\mathscr{M}f(x)|_{x=\varrho} = 0\} = \operatorname{card}\{\eta_1 \in \boldsymbol{R}\colon N_{jk_j^n1}^n(\eta_1) = 0\}.$$

Let

$$\boldsymbol{g} = [\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_{k_0}].$$

By (95) and (107) for each $\alpha > 1$, $\alpha \leqslant k_0$

$$(123) \qquad [v(F_{\alpha 1}(\boldsymbol{g}_\alpha, \boldsymbol{g})), \ldots, v(F_{\alpha i_0(\alpha)}(\boldsymbol{g}_\alpha, \boldsymbol{g}))] \in \boldsymbol{T}_{\alpha\varphi_j(\alpha)}.$$

Hence by Lemma 15

$$\mathscr{J}\mathscr{O}_\alpha\mathscr{M}f(x) = \prod_{\nu=1}^{\nu_0(\alpha)} \mathscr{L}\frac{H\langle \alpha, \varphi_j(\alpha), \nu\rangle(\boldsymbol{g}_\alpha, \boldsymbol{g}, x)}{G\langle \alpha, \varphi_j(\alpha), \nu\rangle(\boldsymbol{g}_\alpha, \boldsymbol{g})}.$$

Therefore, if (86) holds and r and $h_{\varrho rs}$ are defined by (87), we have

$$(124) \qquad \sum_{\substack{\varrho \in \boldsymbol{R}\setminus\{0\} \\ \operatorname{ord}_{x-\varrho}\mathscr{M}f(x)=\alpha}} \sum_{s=1}^{s_r} \operatorname{card}\{\xi \in \boldsymbol{I}\setminus\boldsymbol{P}\colon h_{\varrho rs}(\xi) = 0\}$$
$$= \sum_{\nu=1}^{\nu_0(\alpha)} \sum_{\substack{\varrho \in \boldsymbol{R} \\ \mathscr{L}\mathscr{K}H\langle \alpha, \varphi_j(\alpha), \nu\rangle(\boldsymbol{g}_\alpha, \boldsymbol{g}, \varrho) = 0}} \sum_{s=1}^{s_r} \operatorname{card}\{\xi \in \boldsymbol{I}\setminus\boldsymbol{P}\colon h_{\varrho rs}(\xi) = 0\},$$

where

$$\mathscr{L}\mathscr{K}H\langle \alpha, \varphi_j(\alpha), \nu)\rangle(\boldsymbol{g}_\alpha, \boldsymbol{g}, \varrho) = \mathscr{L}\mathscr{K}H\langle \alpha, \varphi_j(\alpha), \nu)\rangle(\boldsymbol{g}_\alpha, \boldsymbol{g}, x)|_{x=\varrho}.$$

On the other hand, if $\mathscr{L}\mathscr{K}H\langle \alpha, \varphi_j(\alpha), \nu\rangle(\boldsymbol{g}_\alpha, \boldsymbol{g}, \varrho) = 0$, we have

$$\overline{\mathscr{K}H\langle \alpha, \varphi_j(\alpha), \nu\rangle(\boldsymbol{g}_\alpha, \boldsymbol{g}, x)}|_{x=\xi_\varrho} = 0;$$

thus by (86), (123) and Lemma 15 for $\mu = 0, 1, \ldots, m$

$$f^{(\mu)}(\xi_\varrho) \equiv \frac{L\langle \alpha, \varphi_j(\alpha), \nu, \mu\rangle(f^{(\alpha-1)}, f^{(\alpha)}, \xi_\varrho)}{K\langle \alpha, \varphi_j(\alpha), \nu, \mu\rangle(f^{(\alpha-1)}, f^{(\alpha)}, \xi_\varrho)} \bmod \boldsymbol{P}^{v(f^{(\mu)}(\xi_\varrho))+1}$$

and

$$v\big(L\langle \alpha, \varphi_j(\alpha), \nu, \mu+1\rangle(f^{(\alpha-1)}, f^{(\mu)}, \xi_\varrho)\big)$$
$$= v\big(L\langle \alpha, \varphi_j(\alpha), \nu, \mu+1\rangle(f^{(\alpha-1)}, f^{(\mu)}, x)\big)$$
$$= v\big(K\langle \alpha, \varphi_j(\alpha), \nu, \mu+1\rangle(f^{(\alpha-1)}, f^{(\mu)}, \xi_\varrho)\big)$$
$$= v\big(K\langle \alpha, \varphi_j(\alpha), \nu, \mu+1\rangle(f^{(\alpha-1)}, f^{(\mu)}, x)\big).$$

Hence

$$v\big(f^{(\mu)}(\xi_\varrho)\big) = \big|v\big(L\langle \alpha, \varphi_j(\alpha), \nu, \mu+1\rangle(f^{(\mu-1)}, f^{(\mu)}, x)\big) - v\big(K\langle \alpha, \varphi_j(\alpha), \nu, \mu+1\rangle(f^{(\alpha-1)}, f^{(\alpha)}, x)\big)\big|$$

and by (102)-(104)

$$(125) \quad \boldsymbol{u}_\varrho := [v(f^{(m)}(\xi_\varrho)), \ldots, v(f(\xi_\varrho))] = \tau_{\alpha\varphi_j(\alpha)\nu}[v(M_{z+1}^n(f)), \ldots, v(M_{z+z_1}^n(f))],$$

where

$$z := t_0 + a_{\alpha\varphi_j(\alpha)} + \sum_{\lambda<\nu} r_{\alpha\varphi_j(\alpha)\lambda m+1+i_{n-1}}, \qquad z_1 := \nu_{\alpha\varphi_j(\alpha)\nu m+1}.$$

Since by (95), (106) and (107)

$$[v(M_{z+1}^n(f)), \ldots, v(M_{z+z_1}^n(f))] \in \tau_{\alpha\varphi_j(\alpha)\nu}^{-1}(U_{\chi_j(\alpha,\nu)}),$$

we get

$$\boldsymbol{u}_\varrho \in \boldsymbol{U}_{\chi_j(\alpha,\nu)};$$

thus by (87)

$$(126) \qquad r = \chi_j(\alpha, \nu),$$

$$(127) \qquad h_{\varrho rs}(y) = f_\varrho(p^{\pi\langle r,s\rangle(\boldsymbol{u}_\varrho)}y).$$

Since $M_i^{n-1} \in C_0(m)$, we have by Lemma 20 for all $i \leqslant i_{n-1}$

$$M_i^{n-1}(\boldsymbol{h}_{\varrho rs}) = p^{(m\deg M_i^{n-1} - w(M_i^{n-1}))\pi\langle r,s\rangle(\boldsymbol{u}_\varrho)}\,\widehat{M_i^{n-1}}(f, \xi_\varrho).$$

Now by Lemma 15

$$v(\widehat{M_i^{n-1}}(f, \xi_\varrho)) = |v(L\langle \alpha, \varphi_j(\alpha), v, m+1+i\rangle(f^{(\alpha-1)}, *\widehat{M_i^{n-1}}(f, x), x)) - $$
$$ - v(K\langle \alpha, \varphi_j(\alpha), v, m+1+i\rangle(f^{(\alpha-1)}, *\widehat{M_i^{n-1}}(f, x), x))|;$$

hence by (105)

$$\text{(128)}\qquad \boldsymbol{u}_{\varrho s} := [v(f(\xi_\varrho)), \ldots, v(f^{(m)}(\xi_\varrho)), v(M_1^{n-1}(\boldsymbol{h}_{\varrho rs})), \ldots, v(M_{i_{n-1}}^{n-1}(\boldsymbol{h}_{\varrho rs}))]$$
$$= \omega_{\alpha\varphi_j(\alpha)vs}(v(M_{z+1}^n(f)), \ldots, v(M_{z+z_2}^n(f))).$$

where

$$z_2 := r_{\alpha\varphi_j(\alpha)vm+1+i_{n-1}}.$$

Since by (95), (106) and (107)

$$[v(M_{z+1}^n(f)), \ldots, v(M_{z+z_2}^n(f))] \in \omega_{\alpha\varphi_j(\alpha)vs}^{-1}(U_{\chi_j(\alpha,v)} \times V_{\psi_j(\alpha,v,s)}^{n-1}),$$

we get

$$\boldsymbol{u}_{\varrho s} \in U_{\chi_j(\alpha,v)} \times V_{\psi_j(\alpha,v,s)}^{n-1}$$

and

$$\text{(129)}\qquad w_{\varrho s} := [v(M_1^{n-1}(\boldsymbol{h}_{\varrho rs})), \ldots, v(M_{i_{n-1}}^{n-1}(\boldsymbol{h}_{\varrho rs}))] \in V_{\psi_j(\alpha,v,s)}^{n-1}.$$

It follows from the inductive assumption that

$$\text{(130)}\qquad \operatorname{card}\{\xi \in \boldsymbol{I}\setminus\boldsymbol{P}\colon h_{\varrho rs}(\xi) = 0\}$$
$$= \sum_{\varkappa=1}^{k_{\psi_j(\alpha,v,s)}^{n-1}} \operatorname{card}\{[\eta_2, \eta_3, \ldots] \in \boldsymbol{R}^{l_{\psi_j(\alpha,v,s)\varkappa}^{n-1}}\colon \bigwedge_{l=1}^{l_{\psi_j(\alpha,v,s)\varkappa}^{n-1}} N_{\alpha vs\varkappa l\varrho}(\eta_2, \ldots, \eta_{l+1}) = 0\},$$

where

$$N_{\alpha vs\varkappa l\varrho}(y_2, \ldots, y_{l+1}) := \mathscr{L}\mathscr{K} N_{\psi_j(\alpha,v,s)}^{n-1}(\boldsymbol{h}_{\varrho rs}, p^{\pi_{\alpha vs\varkappa l\varrho}} y_2, \ldots, p^{\pi_{\alpha vs\varkappa l\varrho}} y_{l+1}),$$
$$\text{(131)}\qquad \pi_{\alpha vs\varkappa\lambda\varrho} := v^{n-1}\langle \psi_j(\alpha, v, s), \varkappa, \lambda\rangle(w_{\varrho s}).$$

Since by the inductive assumption

$$N_{\psi_j(\alpha,v,s)\varkappa l}^{n-1} \in C_l(m),$$

we have by Lemma 20, (126) and (127)

$$\text{(132)}\qquad N_{\alpha vs\varkappa l\varrho}(y_2, \ldots, y_{l+1})$$
$$= \mathscr{L}\mathscr{K} N_{\psi_j(\alpha,v,s)\varkappa l}^{n-1}(\boldsymbol{f}_\varrho, p^{\pi\langle\chi_j(\alpha,v),s\rangle(\boldsymbol{u}_\varrho)+\pi_{\alpha vs\varkappa l\varrho}} y_2, \ldots, p^{\pi\langle\chi_j(\alpha,v),s\rangle(\boldsymbol{u}_\varrho)+\pi_{\alpha vs\varkappa l\varrho}} y_{l+1}).$$

It follows from (124), (126) and (136) that

$$\text{(133)}\qquad \sum_{\substack{\varrho \in \boldsymbol{R}\setminus\{0\} \\ \operatorname{ord}_{x-\varrho}\mathscr{M}f \geqslant 2}} \sum_{s=1}^{s_r} \operatorname{card}\{\xi \in \boldsymbol{I}\setminus\boldsymbol{P}\colon h_{\varrho rs}(\xi) = 0\}$$
$$= \sum_{\alpha=2}^{n} \sum_{v=1}^{v_0(\alpha)} \sum_{\substack{\varrho \in \boldsymbol{R} \\ \mathscr{L}\mathscr{K}H\langle\alpha,\varphi_j(\alpha),v\rangle(\boldsymbol{g}_\alpha,\boldsymbol{g},\varrho)=0}} \sum_{s=1}^{s_{\chi_j(\alpha,v)}} \sum_{\varkappa=1}^{k_{\psi_j(\alpha,v,s)}^{n-1}} \operatorname{card}\{[\eta_2, \eta_3, \ldots] \in \boldsymbol{R}^{l_{\psi_j(\alpha,v,s)\varkappa}^{n-1}}\colon$$
$$\bigwedge_{l=1}^{l_{\psi_j(\alpha,v,s)\varkappa}^{n-1}} N_{\alpha vs\varkappa l\varrho}(\eta_2, \ldots, \eta_{l+1}) = 0\}$$
$$= \sum_{k=1}^{k_j^n} \sum_{\substack{\varrho \in \boldsymbol{R} \\ \mathscr{L}\mathscr{K}H\langle\alpha_{jk},\varphi_j(\alpha_{jk}),v_{jk}\rangle(\boldsymbol{g}_{\alpha_{jk}},\boldsymbol{g},\varrho)=0}} \operatorname{card}\{[\eta_2, \eta_3, \ldots] \in \boldsymbol{R}^{l_{jk}^n-1}\colon$$
$$\bigwedge_{l=1}^{l_{jk}^n-1} N_{\alpha_{jk}v_{jk}s_{jk}\varkappa_{jk}l\varrho}(\eta_2, \ldots, \eta_{l+1}) = 0\}.$$

Let us consider a typical summand in the last sum. By (96), (111) and (112)

$$\text{(134)}\qquad \mathscr{L}\mathscr{K}H\langle\alpha_{jk}, \varphi_{jk}(\alpha_{jk}), v_{jk}\rangle(\boldsymbol{g}_{\alpha_{jk}}, \boldsymbol{g}, y_1) = \tilde{N}_{jk1}^n(y_1).$$

Furthermore, by (132), (100) and (108)

$$\text{(135)}\qquad N_{\alpha_{jk}v_{jk}s_{jk}\varkappa_{jk}\varrho}(y_2, \ldots, y_{l+1}) = \mathscr{L}\mathscr{K} N_{\psi_j(\alpha_{jk},v_{jk},s_{jk})\varkappa_{jk}l}^{n-1}(\boldsymbol{f}_\varrho, p^{e_1}y_2, \ldots, p^{e_l}y_{l+1})$$
$$= \mathscr{L}\mathscr{K}\Big(\sum_{\mu=1}^{\mu_{jkl}} N_{\lambda_{jkl}+\mu}^{n-1}(\boldsymbol{f}_\varrho) \prod_{\lambda=1}^{l} y_{\lambda+1}^{q_{jkl\mu\lambda}} p^{e_\lambda q_{jkl\mu\lambda}}\Big),$$

where

$$e_\lambda := \pi\langle\chi_j(\alpha_{jk}, v_{jk}), s_{jk}\rangle(\boldsymbol{u}_\varrho) + \pi_{\alpha_{jk}v_{jk}s_{jk}\varkappa_{jk}\lambda\varrho}.$$

By the definition of f_ϱ (formula (87)) and of the operation $\wedge$ (Lemma 19) we have

$$\text{(136)}\qquad N_{\lambda_{jkl}+\mu}^{n-1}(\boldsymbol{f}_\varrho) = \widehat{N_{\lambda_{jkl}+\mu}^{n-1}}(f, \xi_\varrho).$$

Now by Lemma 15 and (109)-(110)

$$\widehat{N_{\lambda_{jkl}+\mu}^{n-1}}(f, \xi_\varrho) \equiv \frac{L_{jkl\mu}(f, \xi_\varrho)}{K_{jkl\mu}(f, \xi_\varrho)} \bmod \boldsymbol{P}^{v(\widehat{N_{\lambda_{jkl}+\mu}^{n-1}}(f,\xi_\varrho))+1},$$
$$v(K_{jkl}(f, \xi_\varrho)) = v(K_{jkl}(f, x)), \qquad v(L_{jkl}(f, \xi_\varrho)) = v(L_{jkl}(f, x)).$$

On the other hand, by (125), (128) and (129) the relations $\boldsymbol{v} \in V_j^n$, $\operatorname{ord}_{x-\varrho}\mathscr{M}f(x) = \alpha_{jk}$ imply

$$[\boldsymbol{u}_\varrho, \boldsymbol{w}_{\varrho s_{jk}}] = \boldsymbol{u}_{\varrho s_{jk}} = \omega_{\alpha_{jk}\varphi_j(\alpha_{jk})v_{jk}s_{jk}}(\boldsymbol{v}_{jk});$$

thus by (114) and (131)

$$e_\lambda = v^n\langle j, k, \lambda+1\rangle(\boldsymbol{v}) \qquad (1 \leqslant \lambda \leqslant l).$$

Put in Lemma 21

$$a_\mu = \widehat{N^{n-1}_{\lambda jkl+\mu}}(f,\xi_\varrho),\quad b_\mu = p^{\sum\limits_{\lambda=1}^{l} e_\lambda q_{jkl\mu\lambda}},$$
$$\xi=\xi_\varrho,\quad K_\mu = K_{jkl\mu}(f,y_1),\quad L_\mu = L_{jkl\mu}(f,y_1),$$
$$\mu_l = \mu_{jkl},\quad q_{\mu\lambda} = q_{jkl\mu\lambda}.$$

It follows by (113) and (96) that in the notation of that lemma

$$\begin{aligned} A(y_1,\dots,y_{l+1}) &= \mathscr{L}\mathscr{K}\sum_{\mu=1}^{\mu_{jkl}} \frac{L_{jkl\mu}(f,y_1)}{K_{jkl\mu}(f,y_1)}\prod_{\lambda=1}^{l} y_{\lambda+1}^{q_{jkl\mu\lambda}} p^{v^n\langle j,k,\lambda+1\rangle(v) q_{jkl\mu\lambda}} \prod_{\mu=1}^{\mu_{jkl}} K_{jkl\mu}(f,y_1)\\ &= \tilde N^n_{jkl+1}(y_1,\dots,y_{l+1})\end{aligned}$$

and by Lemma 21, (135) and (136) that

$$N_{\alpha_{jk}v_{jk}s_{jk}k_{jk}l_\varrho}(y_2,\dots,y_{l+1}) = \tilde N^n_{jkl}(\varrho,y_2,\dots,y_{l+1})\prod_{\mu=1}^{\mu_{jkl}}\overline{\mathscr{K}K_{jkl\mu}(f,\xi_\varrho)}^{-1}.$$

Hence by (133) and (134)

$$\sum_{\substack{\varrho\in R\setminus\{0\}\\ \operatorname{ord}_{x-\varrho}\mathscr{M}f\geqslant 2}}\ \sum_{s=1}^{s_r}\operatorname{card}\{\xi\in I\setminus P:\ h_{\varrho rs}(\xi)=0\}$$
$$= \sum_{k=1}^{k_j^n}\operatorname{card}\{[\eta_1,\eta_2,\dots]\in R^{l^n_{jk}}:\ \bigwedge_{l=1}^{l^n_{jk}}\tilde N^n_{jkl}(\eta_1,\eta_2,\dots,\eta_l)=0\},$$

and (97) follows from (88) and (122).

LEMMA 23. *Lemma 22 holds with $n=m$ and without any restriction on a polynomial $f\in I[x]$ except $0\leqslant\deg f\leqslant m$. Polynomials M_i^{mm}, N_{jkl}^{mm} are to be replaced by P_i^m, $Q^m_{jkl}(i\leqslant i_m,\ j\leqslant j_m,\ k\leqslant k_j^m,\ l\leqslant l^m_{jk})$, sets V_j^{mm} by W_j^m and functions $v^{mm}\langle j,k,l\rangle$ by $\varrho^m\langle j,k,l\rangle$.*

Proof. We proceed by induction on m. For $m=0$ we take $i_0=0$, $j_0=k_1^0=l^0_{11}=1$, $Q^0_{111}(y_1)=a_0$, $\varrho^0\langle 1,1,1\rangle(t)=0$. Assume now that the lemma is true for polynomials f of degree less than $m\geqslant 1$. By Lemmata 6.8 and 3 the operations (f,g) and $f/(f,g)$ belong to $\Omega(N_0\times N_-)$. Hence there exist polynomials $A_i, B_j, D_j\in C_0(m,n)$ and $C_j, E_j\in C_1(m,m)$ $(i\leqslant i^0,\ j\leqslant j^0)$ and a decomposition

$$N_+^{i^0} = \bigcup_{j=1}^{j^0} S_j \tag{137}$$

with the following property. If $0\leqslant\deg f\leqslant m$, $\deg g\leqslant m$ and

$$[v(A_1(f,g)),\dots,v(A_{i^0}(f,g))]\in S_j$$

then

$$B_j(f,g)\neq 0,\quad D_j(f,g)\neq 0 \tag{138}$$

and

$$\frac{f}{(f,g)} = \frac{C_j(f,g,x)}{B_j(f,g)},\quad (f,g) = \frac{E_j(f,g,x)}{D_j(f,g)}.$$

Let

$$C_j(f,g,x) = \sum_{\mu=0}^{m} C_{j\mu}(f,g)x^{m-\mu},\quad E_j(f,g,x) = \sum_{\mu=0}^{m} E_{j\mu}(f,g)x^{m-\mu}. \tag{139}$$

We take as P_i^m the following polynomials:

$$P_i^m = \left|\begin{matrix} a_0\,a_1\,\dots\,a_m & & \\ \dots\dots\dots\dots & & \\ & & a_0\,a_1\,\dots\,a_m \\ \binom{m}{i}a_0\binom{m-1}{i}a_1\dots a_{m-i} & & \\ \dots\dots\dots\dots & & \\ & & \binom{m}{i}a_0\binom{m-1}{i}a_1\dots a_{m-i}\end{matrix}\right| \begin{matrix}\left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\}m-i\\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\}m\end{matrix}\quad (1\leqslant i<m), \tag{140}$$

$$P_i^m = M^{mm}_{i-m+1}\quad (m\leqslant i<m+i_{mm});$$

then in the increasing order of μ $(=1,2,\dots,m-1)$ and v

$$A_v(a,a^{(\mu)})\quad (v\leqslant i^0),$$
$$P_v^{m-1}(C_{j1}(a,a^{(\mu)}),\dots,C_{jm}(a,a^{(\mu)}))\quad (v\leqslant i_{m-1},\ j\leqslant j^0),$$
$$P_v^{m-1}(E_{j1}(a,a^{(\mu)}),\dots,E_{jm}(a,a^{(\mu)}))\quad (v\leqslant i_{m-1},\ j\leqslant j^0),$$

so that

$$i_m = m-1+i_{mm}+(i^0+2j^0 i_{m-1})(m-1).$$

Further, let us order in a sequence all quadruples $[\alpha,\beta,\gamma,\delta]$, where $1\leqslant\alpha<m$, $1\leqslant\beta\leqslant j^0$, $1\leqslant\gamma\leqslant j_{m-1}$, $1\leqslant\delta\leqslant j_{m-1}$ and call the vth term of this sequence $[\alpha_v,\beta_v,\gamma_v,\delta_v]$ $(v\leqslant j'_m)$. Let $v=[v_1,\dots,v_{i_m}]$. Then put for $j\leqslant j'_m$

$$\begin{aligned} W_j^m = {} & N_0^{\alpha_j-1}\times\{\infty\}\times N_+^{m-1-\alpha_j}\times N_+^{i_{mm}+(i^0+2j^0 i_{m-1})(\alpha_j-1)}\times S_{\beta_j}\times\\ & \times N_+^{2i_{m-1}(\beta_j-1)}\times W^{m-1}_{\gamma_j}\times W^{m-1}_{\delta_j}\times N_+^{2i_{m-1}(j^0-\beta_j)+(i^0+2j^0 i_{m-1})(m-1-\alpha_j)}\end{aligned} \tag{141}$$

and for $k\leqslant k^{m-1}_{\gamma_j}$, $l\leqslant l^{m-1}_{\gamma_j k} := l^m_{jk}$

$$Q^m_{jkl} = Q^{m-1}_{\gamma_j kl}(C_{\beta_j 1}(a,a^{(\alpha_j)}),\dots,C_{\beta_j m}(a,a^{(\alpha_j)}),y_1,\dots,y_l), \tag{142}$$

$$\varrho^m\langle j,k,l\rangle(v) = \varrho^{m-1}\langle\gamma_j,k,l\rangle(v_j),\quad\text{where}\quad v_j := [v_{r_j+1},\dots,v_{r_j+i_{m-1}}], \tag{143}$$
$$r_j := m-1+i_{mm}+(i^0+2j^0 i_{m-1})(\alpha_j-1)+i^0+2i_{m-1}(\beta_j-1).$$

If $\alpha_j=1$ we take $k_j^m := k^{m-1}_{\gamma_j}$, but if $\alpha_j>1$ we put further, for $k>k^{m-1}_{\gamma_j}$, $k\leqslant k^{m-1}_{\gamma_j}+k^{m-1}_{\delta_j} := k_j^m$, $l\leqslant l^{m-1}_{\delta_j,k-k^{m-1}_{\gamma_j}} := l^m_{jk}$,

$$Q^m_{jkl} = Q^{m-1}_{\delta_j kl}(E_{\beta_j 1}(a,a^{(\alpha_j)}),\dots,E_{\beta_j m}(a,a^{(\alpha_j)}),y_1,\dots,y_l), \tag{144}$$

$$\varrho^m\langle j,k,l\rangle(v) = \varrho^{m-1}_{\delta_j kl}(v'_j),\quad\text{where}\quad v'_j := [v_{r_j+i_{m-1}+1},\dots,v_{r_j+2i_{m-1}}]. \tag{145}$$

If $j'_m < j \leqslant j'_m + j_{mm} = j_m$ we take

$$W_j^m = N_0^{m-1} \times N_+^{(m-1)(i^0+2j^0 i_{m-1})+i_{m-1}} \tag{146}$$

and for $k \leqslant k^{mm}_{j-j'_m} := k^m_j$, $l \leqslant l^{mm}_{j-j'_m,k} := l^m_{jk}$

$$Q^m_{jkl} = N^{mm}_{j-j_m kl}, \tag{147}$$

$$\varrho^m\langle j,k,l\rangle(v) = \nu^{mm}\langle j-j'_m,k,l\rangle(v_0), \quad \text{where} \quad v_0 := [v_m, \ldots, v_{m+i_{mm}-1}]. \tag{148}$$

Clearly the polynomials P^m_i, Q^m_{jkl}, the sets W^m_j and the functions $\varrho^m\langle j,k,l\rangle$ are independent of K, v, and p. We proceed to show that they have all the properties asserted in Lemma 22 for the polynomials M^{mm}_i, N^{mm}_{jkl}, the sets V^{mm}_j and the functions $\nu^{mm}\langle j,k,l\rangle$.

The relation $P^m_i \in C_0(m)$ follows for $i<m$ directly from (139) for all other $i<m+i_{mm}$ from $M^{mm}_\nu \in C_0(m)$, and for $i \geqslant m+i_{mm}$ from the properties of A_ν, C_ν, E_ν, the inductive assumption and Lemma 1. The relation $Q^{mm}_{jkl} \in C_l(m)$ follows for $j \leqslant j'_m$ from the properties of C_ν, E_ν, the inductive assumption and Lemma 1, and for $j > j'_m$ from $M^{mm}_{\nu kl} \in C_l(m)$.

If $j < j' \leqslant j_m$ we shall show that

$$W^m_j \cap W^m_{j'} = \varnothing. \tag{149}$$

Indeed, we have the following possibilities:

1) $j \leqslant j'_m$, 2) $j \leqslant j'_m < j$, 3) $j'_m < j$.

In case 1) we have four subcases: 1a) $\alpha_j \neq \alpha_{j'}$; 1b) $\alpha_j = \alpha_{j'}$, $\beta_j \neq \beta_{j'}$; 1c) $\alpha_j = \alpha_{j'}$, $\beta_j = \beta_{j'}$, $\gamma_j \neq \gamma_{j'}$; 1d) $\alpha_j = \alpha_{j'}$, $\beta_j = \beta_{j'}$, $\gamma_j = \gamma_{j'}$, $\delta_j \neq \delta_{j'}$.

In case 1a) the projections of W^m_j and $W^m_{j'}$ on the axis of $\min(\alpha_j, \alpha_{j'})$th coordinate are in some order N_0 and $\{\infty\}$, and hence disjoint.

In case 1b) the projections of W^m_j and $W^m_{j'}$ on a suitable linear space are S_{β_j} and $S_{\beta_{j'}}$ and hence disjoint.

In case 1c) the projections of W^m_j and $W^m_{j'}$ on a suitable linear space are $W^{m-1}_{\gamma_j}$ and $W^{m-1}_{\gamma_{j'}}$, and hence disjoint by the inductive assumption. A similar argument applies in case 1d).

In case 2) the projection of W^m_j and $W^m_{j'}$ on the axis of α_jth coordinate are $\{\infty\}$ and N_0 respectively, and hence disjoint.

In case 3) the projections of W^m_j and $W^m_{j'}$ on a suitable linear space are $V^{mm}_{j-j'_m}$ and $V^{mm}_{j'-j'_m}$, and hence disjoint by Lemma 22.

In every case (149) follows. On the other hand,

$$\bigcup_{j=1}^{j_m} W^m_j = \bigcup_{j=1}^{j'_m} W^m_j \cup \bigcup_{j=j'_m+1}^{j_m} W^m_j.$$

Now by the inductive assumption and (137)

$$\begin{aligned}
\bigcup_{j=1}^{j'_m} W^m_j &= \bigcup_{j=1}^{j'_m} \big(N_0^{\alpha_j-1} \times \{\infty\} \times N_0^{m-1-\alpha_j+i_{mm}+(i^0+2j^0 i_{m-1})(\alpha_j-1)} \times S_{\beta_j} \times \\
&\quad \times N_+^{2i_{m-1}(\beta_j-1)} \times W^{m-1}_{\gamma_j} \times W^{m-1}_{\delta_j} \times N_+^{2i_{m-1}(j^0-\beta_j)+(i^0+2j^0 i_{m-1})(m-\alpha_j-1)}\big) \\
&= \bigcup_{\alpha=1}^{m-1} \bigcup_{\beta=1}^{i^0} \bigcup_{\gamma=1}^{j^0} \bigcup_{\delta=1}^{j^0} \big(N_0^{\alpha-1} \times \{\infty\} \times N_+^{m-1-\alpha+i_{mm}+(i^0+2j^0 i_{m-1})(\alpha-1)} \times S_\beta \times \\
&\quad \times N_+^{2i_{m+1}(\beta-1)} \times W^{m-1}_\gamma \times W^{m-1}_\delta \times N_+^{2i_{m-1}(j^0-\beta)+(i^0+2j^0 i_{m-1})(m-\alpha-1)}\big) \\
&= \bigcup_{\alpha=1}^{m-1} \bigcup_{\beta=1}^{i^0} \big(N_0^{\alpha-1} \times \{\infty\} \times N_+^{m-1-\alpha+i_{mm}+(i^0+2j^0 i_{m+1})(\alpha-1)} \times S_\beta \times N_+^{2i_{m-1}(\beta-1)} \times \\
&\quad \times N_+^{2i_{m-1}+2i_{m-1}(j^0-\beta)+(i^0+2j^0 i_{m-1})(m-\alpha-1)}\big) \\
&= \bigcup_{\alpha=1}^{m-1} \big(N_0^{\alpha-1} \times \{\infty\} \times N_+^{m-1-\alpha+i_{mm}+(i^0+2j^0 i_{m-1})(\alpha-1)+i+2j^0 i_{m-1}+(i^0+2j^0 i_{m-1})(m-\alpha-1)}\big) \\
&= \big(\bigcup_{\alpha=1}^{m-1} N_0^{\alpha-1} \times \{\infty\} \times N_+^{m-1-\alpha}\big) \times N_+^{i_{mm}+(i^0+2j^0 i_{m-1})(m-1)};
\end{aligned}$$

$$\bigcup_{j=j'_m+1}^{j_m} W^m_j = N_0^{m-1} \times N_+^{i_{mm}+(i^0+2j^0 i_{m-1})(m-1)}.$$

Hence

$$\bigcup_{j=1}^{j_m} W^m_j = \big(\bigcup_{\alpha=1}^{m-1} N_0^{\alpha-1} \times \{\infty\} \times N_+^{m-1-\alpha} \cup N_0^{m-1}\big) \times N_+^{i_{mm}+(i^0+2j^0 i_{m-1})(m-1)}$$

and in order to show that

$$\bigcup_{j=1}^{j_m} W^m_j = N_+^{i_m}$$

if suffices to notice that

$$\bigcup_{\alpha=1}^{m-1} (N_0^\alpha \times \{\infty\} \times N_+^{m-1-\alpha}) \cup N_0^{m-1} = N_+^{m-1}.$$

The claim that the functions $\varrho^m\langle j,k,l\rangle$ are N_0-valued of W^m_j follows by (143) for $j \leqslant j'_m$ from the fact that $\varrho^{m-1}\langle \gamma_j,k,l\rangle$ and $\varrho^{m-1}\langle \delta_j,k,l\rangle$ are by the inductive assumption N_0-valued on $W^{m-1}_{\gamma_j}$ and $W^{m-1}_{\delta_j}$ respectively.

If $j'_m < j \leqslant j_m$ then by (117) $\varrho^m\langle j,k,l\rangle$ is N_0-valued on W^m_j since, by Lemma 22, $\nu^{mm}\langle j-j'_m,k,l\rangle$ is such on $V^{mm}_{j-j'_m}$. Assume now that

$$f(x) = \sum_{\mu=0}^{m} a_\mu x^{m-\mu} \in I[x], \quad f \neq 0$$

and

$$v = [v(P^m_1(f)), \ldots, v(P^m_{i_m}(f))] \in W^m_j. \tag{150}$$

We distinguish three cases:

(i) $j' \leqslant j'_m$, $\alpha_j = 1$;

(ii) $j \leqslant j'_m$, $\alpha_j > 1$;

(iii) $j > j'_m$.

In case (i) we have by (140), (141) and (150)

$$a_0 = 0 \quad \text{or} \quad \operatorname{res}(f, f') = 0,$$
$$[v(A_1(f, f')), \ldots, v(A_{i_0}(f, f'))] \in S_{\beta_j},$$

(151) $$[v(P_1^{m-1}(C_{\beta_j 1}(f, f'), \ldots, C_{\beta_j m}(f, f'))), \ldots \\ \ldots, v(P_{i_{m-1}}^{m-1}(C_{\beta_j 1}(f, f'), \ldots, C_{\beta_j m}(f, f')))] \in W_{\gamma_j}^{m-1}.$$

Thus

(152) $$\deg f/(f, f') < m$$

and by (138)

$$B_j(f, f') \neq 0 \quad \text{and} \quad \frac{f}{(f, f')} = \frac{C_j(f, f', x)}{B_j(f, f')} \neq 0.$$

By (139) and (152)

(153) $$C_{\beta_j}(f, f', x) = \sum_{\mu=1}^{m} C_{\beta_j \mu}(f, f') x^{m-\mu}.$$

Since $\operatorname{char} K = 0$ or $\operatorname{char} K = \operatorname{char} R > m \geqslant \deg f$, each zero of f is a simple zero of $f/((f, f')$ and we have

$$\operatorname{card}\{\xi \in I \setminus P \colon f(\xi) = 0\} = \operatorname{card}\Big\{\xi \in I \setminus P \colon \frac{f}{(f, f')}(\xi) = 0\Big\} \\ = \operatorname{card}\{\xi \in I \setminus P \colon C_{\beta_j}(f, f', \xi) = 0\}.$$

By the inductive assumptions and (142), (143), (151) and (153),

$$\operatorname{card}\{\xi \in I \setminus P \colon f(\xi) = 0\} \\ = \sum_{k=1}^{k_{\gamma_j}^{m-1}} \operatorname{card}\Big\{[\eta_1, \eta_2, \ldots] \in R^{l_{\gamma_j k}^{m-1}} \colon \bigwedge_{l=1}^{l_{\gamma_j k}^{m-1}} \mathscr{L}\mathscr{K} Q_{\gamma_j kl}^{m-1}(*C_{\beta_j}(f, f', x), p^{\varrho^{m-1}\langle\gamma_j, k, 1\rangle(v_j)} y_1, \ldots \\ \ldots, p^{\varrho^{m-1}\langle\gamma_j, k, l\rangle(v_j)} y_l)|_{y_\lambda = \eta_\lambda} = 0\Big\} \\ = \sum_{k=1}^{k_j^m} \operatorname{card}\Big\{[\eta_1, \eta_2, \ldots] \in R^{l_{jk}^m} \colon \bigwedge_{l=1}^{l_{jk}^m} (\tilde{Q}_{jkl}^m(f, \eta_1, \ldots, \eta_l) = 0)\Big\}.$$

In case (ii) we have by (140) and (150)

$$a_0 \neq 0, \quad \operatorname{res}(f, f') \neq 0 \quad \operatorname{res}(f, f^{(\alpha_j)}) = 0,$$
$$[v(A_1(f, f^{(\alpha_j)})), \ldots, v(A_{i_0}(f, f^{(\alpha_j)}))] \in S_{\beta_j},$$

(154) $$[v(P_1^{m-1}(C_{\beta_j 1}(f, f^{(\alpha_j)}), \ldots, C_{\beta_j m}(f, f^{(\alpha_j)}))), \ldots \\ \ldots, v(P_{i_{m-1}}^{m-1}(C_{\beta_j 1}(f, f^{(\alpha_j)}), \ldots, C_{\beta_j m}(f, f^{(\alpha_j)})))] \in W_{\gamma_j}^{m-1},$$

(155) $$[v(P_1^{m-1}(E_{\beta_j 1}(f, f^{(\alpha_j)}), \ldots, E_{\beta_j m}(f, f^{(\alpha_j)}))), \ldots \\ \ldots, v(P_{i_{m-1}}^{m-1}(E_{\beta_j}(f, f^{(\alpha_j)}), \ldots, E_{\beta_j m}(f, f^{(\alpha_j)})))] \in W_{\delta_j}^{m-1};$$

hence

(156) $$(f, f') = 1, \quad (f, f^{(\alpha_j)}) \neq 1, \quad \deg f/(f, f^{(\alpha_j)}) < m$$

and by (136), (137)

(157) $$B_{\beta_j}(f, f^{(\alpha_j)}) \neq 0, \quad D_{\beta_j}(f, f^{(\alpha_j)}) \neq 0;$$

(158) $$\frac{f}{(f, f^{(\alpha_j)})} = \frac{C_{\beta_j}(f, f^{(\alpha_j)}, x)}{B_{\beta_j}(f, f^{(\alpha_j)})} \neq 0, \quad (f, f^{(\alpha_j)}) = \frac{E_{\beta_j}(f, f^{(\alpha_j)}, x)}{D_{\beta_j}(f, f^{(\alpha_j)})} \neq 0.$$

By (139) and (156)

(159) $$C_{\beta_j}(f, f^{(\alpha_j)}, x) = \sum_{\mu=1}^{m} C_{\beta_j \mu}(f, f^{(\alpha_j)}) x^{m-\mu},$$

(160) $$E_{\beta_j}(f, f^{(\alpha_j)}, x) = \sum_{\mu=1}^{m} E_{\beta_j}(f, f^{(\alpha_j)}) x^{m-\mu}.$$

Since $(f, f') = 1$ f has no multiple zeros in K, thus

$$\operatorname{card}\{\xi \in I \setminus P \colon f(\xi) = 0\} \\ = \operatorname{card}\Big\{\xi \in I \setminus P \colon \frac{f}{(f, f^{(\alpha_j)})}(\xi) = 0\Big\} + \operatorname{card}\{\xi \in I \setminus P \colon (f, f^{(\alpha_j)})(\xi) = 0\} \\ = \operatorname{card}\{\xi \in I \setminus P \colon C_{\beta_j}(f, f^{(\alpha_j)}, \xi) = 0\} + \operatorname{card}\{\xi \in I \setminus P \colon E_{\beta_j}(f, f^{(\alpha_j)}, \xi) = 0\}.$$

By the inductive assumption and by (142)-(144), (154), (155), (159) and (160) we have

$$\operatorname{card}\{\xi \in I \setminus P \colon f(\xi) = 0\} \\ = \sum_{k=1}^{k_{\gamma_j}^{m-1}} \operatorname{card}\Big\{[\eta_1, \eta_2, \ldots] \in R^{l_{\gamma_j k}^{m-1}} \colon \bigwedge_{l=1}^{l_{\gamma_j k}^{m-1}} \mathscr{L}\mathscr{K} Q_{\gamma_j kl}^{m-1}(*C_{\beta_j}(f, f^{(\alpha_j)}, x), p^{\varrho^{m-1}\langle\gamma_j, k, 1\rangle(v_j)} y_1, \ldots \\ \ldots, p^{\varrho^{m+1}\langle\gamma_j, k, l\rangle(v_j)} y_l)|_{y_\lambda = \eta_\lambda} = 0\Big\} + \\ + \sum_{k=1}^{k_{\delta_j}^{m-1}} \operatorname{card}\Big\{[\eta_1, \eta_2, \ldots] \in R^{l_{\delta_j k}^{m-1}} \colon \bigwedge_{l=1}^{l_{\delta_j k}^{m-1}} \mathscr{L}\mathscr{K} Q_{\delta_j kl}^{m-1} \\ (*E_{\beta_j}(f, f^{(\alpha_j)}, x), p^{\varrho^{m-1}\langle\sigma_j, k, 1\rangle(v_j)} y_1, \ldots, p^{\varrho^{m-1}\langle\sigma_j, k, l\rangle(v_j)} y_l)|_{y_\lambda = \eta_\lambda} = 0\Big\} \\ = \sum_{k=1}^{k_j} \operatorname{card}\Big\{[\eta_1, \eta_2, \ldots] \in R^{l_{jk}^m} \colon \bigwedge_{l=1}^{l_{jk}^m} \tilde{Q}_{jkl}^m(\eta_1, \ldots, \eta_l) = 0\Big\}.$$

In case (iii) we have by (140), (146) and (150)

$$a_0 \neq 0, \quad \operatorname{res}(f, f^{(\alpha)}) \neq 0 \quad \text{for all positive } \alpha < m,$$
$$\boldsymbol{v}_0 = \left[v\left(M_1^{mm}(f)\right), \ldots, v\left(M_{i_{mm}}^{mm}(f)\right)\right] \in \boldsymbol{V}_{j-j'_m}^{mm};$$

thus

$$\Big(f, \prod_{\alpha=1}^{\deg f-1} f^{(\alpha)}\Big) = 1.$$

Applying Lemma 22, we get

$$\operatorname{card}\{\xi \in \boldsymbol{I} \setminus \boldsymbol{P} : f(\xi) = 0\}$$
$$= \sum_{k=1}^{k_{j-j'_m}} \operatorname{card}\{[\eta_1, \eta_2, \ldots] \in \boldsymbol{R}^{l^{mm}_{j-j'_m k}} : \tilde{N}^{mm}_{j-j'_m kl}(\eta_1, \ldots, \eta_l) = 0\},$$

where

$$\tilde{N}^{mm}_{j-j'_m kl}(y_1, \ldots, y_l) = \mathscr{L}\mathscr{K} N^{mm}_{j-j'_m kl}(f, p^{\nu^{mm}\langle j-j'_m, k, l\rangle(\boldsymbol{v}_0)} y_1, \ldots, p^{\nu^{mm}\langle j-j'_m, k, l\rangle(\boldsymbol{v}_0)} y_l).$$

By (147) and (148)

$$\tilde{N}^{mm}_{j-j'_m kl}(y_1, \ldots, y_l) = \tilde{Q}^m_{jkl}(y_1, \ldots, y_l),$$

and the proof is complete.

LEMMA 24. *Let $f \in \boldsymbol{Z}[x]$, $h(f)$ be the height of f. Denoting by p a rational prime, by a bar the residue map $\boldsymbol{Z}_p \to \boldsymbol{F}_p$ and by a double bar the residue map $\boldsymbol{Z}[[t]] \to \boldsymbol{F}_p \bmod t, p$, we have*

$$\overline{f(p)} = \overline{\overline{f(t)}}$$

and if $p > h(f)$

$$\operatorname{ord}_p f(p) = \operatorname{ord}_t f(t).$$

Proof. We have

$$\overline{f(p)} = \overline{f(0)} = \overline{\overline{f(0)}} = \overline{\overline{f(t)}}.$$

If $h(f) = 0$ then $\operatorname{ord}_p f(p) = \operatorname{ord}_t f(t) = \infty$. If $p > h(f) > 0$ then

$$f(t) = t^\alpha g(t), \quad g(0) \neq 0, \quad p > h(g) \geqslant |g(0)|$$

and

$$\operatorname{ord}_p g(p) = \operatorname{ord}_p g(0) = 0 = \operatorname{ord}_t g(t),$$
$$\operatorname{ord}_p f(p) = \alpha = \operatorname{ord}_t f(t).$$

§ 3. Proofs of the theorems.

Proof of Theorem 1. Let r_m, s_r, have the meaning of Lemma 17 and j_n, k_j^m of Lemma 23, and for positive integers $r \leqslant r_m$ let

$$\Delta_r := \{1, 2, \ldots, j_m\}^{\{1, \ldots, s_r\}}.$$

Let us order all quadruples $[\beta, \gamma, \delta, \varepsilon]$, where $\beta \leqslant r_m$, $\gamma \leqslant j_m$, $\delta \in \Delta_\beta$, $\varepsilon \in \{0, 1\}$, in a sequence and call the jth term of this sequence $[\beta_j, \gamma_j, \delta_j, \varepsilon_j]$ $(j \leqslant j^*)$. For each $j \leqslant j^*$ let us order all pairs $[s, t]$, where $1 \leqslant s \leqslant s_{rj}$, $1 \leqslant t \leqslant k^m_{\delta_j(s)}$, and call the kth term of the sequence thus obtained $[s_{jk}, t_{jk}]$ $(k \leqslant k'_j - \varepsilon_j)$. Then, using the notation of Lemma 23, set $i^* = i_m + m + 1$ and let $\tau_{rs}: \boldsymbol{N}_+^{i^*} \to \boldsymbol{N}_+^{i_m}$ be a function defined by the formula

$$(161) \qquad \tau_{rs}(v_1, \ldots, v_{i^*}) = [v_{m+2}, \ldots, v_{i^*}] +$$
$$+ [m \deg P_1^m - w(P_1^m), \ldots, m \deg P_{i_m}^m - w(P_{i_m}^m)]\, \pi\langle r, s\rangle(\boldsymbol{u}), \qquad \boldsymbol{u} := [v_1, \ldots, v_{m+1}].$$

Moreover, put

$$(162) \qquad R_i = \begin{cases} a_{i-1} & (1 \leqslant i \leqslant m+1), \\ P^m_{i-m-1} & (m+1 \leqslant i \leqslant i^*), \end{cases}$$

for the sake of expediency $N_1 := \{\infty\}$ and for $j \leqslant j^*$

$$(163) \qquad \boldsymbol{X}_j = \big((U_{\beta_j} \cap (\boldsymbol{N}_+^m \times \boldsymbol{N}_{\varepsilon_j})) \times \boldsymbol{N}_+^{i_m}\big) \cap (\boldsymbol{N}_+^{m+1} \times \boldsymbol{W}^m_{\gamma_j}) \cap \bigcap_{s=1}^{s_{\beta_j}} \tau^{-1}_{\beta_j s}(\boldsymbol{W}^m_{\delta_j(s)}).$$

Further, for $k \leqslant k'_j - \varepsilon_j$, $l \leqslant l_{jk} := l^m_{\delta_j(s_{jk}) t_{jk}}$, put

$$(164) \qquad S_{jkl} = Q^m_{\delta_j(s_{jk}) t_{jk} l},$$

$$(165) \qquad \sigma_{jkl}(\boldsymbol{v}) = \pi\langle \beta_j, s_{jk}\rangle(\boldsymbol{u}) + \varrho^m\langle \delta_j(s_{jk}), t_{jk}, l\rangle(\tau_{\beta_j s_{jk}}(\boldsymbol{v})),$$

if $\varepsilon_j = 1$ then put $l_{jk'_j} = 1$;

$$(166) \qquad S_{jk'_j 1}(y_1) = y_1, \quad \sigma_{jk'_j 1}(\boldsymbol{v}) = 0$$

and if $k'_j < k \leqslant k'_j + k^m_{\gamma_j} := k_j$, $l \leqslant l^m_{\gamma_j k - k_j} := l_{jk}$ then

$$(167) \qquad S_{jkl} = Q^m_{\gamma_j k - k'_j l},$$

$$(168) \qquad \sigma_{jkl} = \varrho^m\langle \gamma_j, k - k'_j, l\rangle.$$

Clearly the polynomials R_i, Q_{jkl}, the sets $\boldsymbol{X}_j$ and the functions σ_{jkl} defined above are independent of $\boldsymbol{K}$, v and p. We proceed to show that they have all the properties asserted in the theorem. The claim that R_j are forms and S_{jkl} polynomials with integral coefficients follows from (162), (164), (166), (167) and Lemma 23. In order to prove that the sets $\boldsymbol{X}_j$ are disjoint let $j < j' \leqslant j^*$ and distinguish four cases:

1) $\beta_j \neq \beta_{j'}$, 2) $\beta_j = \beta_{j'}$, $\gamma_j \neq \gamma_{j'}$, 3) $\beta_j = \beta_{j'}$, $\gamma_j = \gamma_{j'}$, $\delta_j \neq \delta_{j'}$, 4) $\beta_j = \beta_{j'}$, $\gamma_j = \gamma_{j'}$, $\delta_j = \delta_{j'}$, $\varepsilon_j \neq \varepsilon_{j'}$.

In case 1), by Lemma 17, $U_{\beta_j} \cap U_{\beta_{j'}} = \varnothing$, and hence by (163)

$$(169) \qquad \boldsymbol{X}_j \cap \boldsymbol{X}_{j'} = \varnothing.$$

In case 2), by Lemma 23, $\boldsymbol{W}^m_{\gamma_j} \cap \boldsymbol{W}^m_{\gamma_{j'}} = \varnothing$, and hence again (169).

In case 3) there exists an $s \leqslant s_{\beta_j}$ such that $\delta_j(s) \neq \delta_{j'}(s)$. Hence by Lemma 23

$$\boldsymbol{W}^m_{\delta_j(s)} \cap \boldsymbol{W}^m_{\delta_{j'}(s)} = \varnothing$$

and, since $\beta_j = \beta_{j'}$ we have also

$$\tau_{\beta_j s}^{-1}(W^m_{\delta_j(s)}) \cap \tau_{\beta_{j'} s}^{-1}(W^m_{\delta_{j'}(s)}) = \emptyset,$$

which by (163) implies (169).

In case 4) $N_{\varepsilon_j} \cap N_{\varepsilon_{j'}} = \emptyset$, which by (163) again implies (169). On the other hand, by (163), Lemma 23 and Lemma 17

$$\bigcup_{j=1}^{j^*} X_j = \bigcup_{r\le r_m} \bigcup_{\gamma\le j_m} \bigcup_{\delta\in\Delta_r} \bigcup_{\varepsilon\in\{0,1\}} \Big(\big((U_r \cap (N_+^m N_\varepsilon)) \times N_+^{i_m}\big) \cap (N_+^{m+1} \times W_\gamma^m) \cap \bigcap_{s=1}^{s_r} \tau_{rs}^{-1}(W^m_{\delta(s)})\Big)$$

$$= \bigcup_{r\le r_m} \Big(\big((U_r \cap (N_+^m \times \bigcup_{\varepsilon\in\{0,1\}} N_\varepsilon)) \times N_+^{i_m}\big) \cap (N_+^{m+1} \times \bigcup_{\gamma\le j_m} W_\gamma^m \cap \bigcap_{s=1}^{s_r} \tau_{rs}^{-1}(\bigcup_{\delta\le j_m} W_\delta^m)\Big)$$

$$= \bigcup_{r\le r_m} \big((U_r \times N_+^{i_m}) \cap N_+^{i^*} \cap \bigcap_{s=1}^{s_r} N_+^{i^*}\big) = (\bigcup_{r\le r_m} U_r) \times N_+^{i_m} = N_+^{i^*}.$$

The claim that the functions σ_{jkl} are N_0-valued on X_j follows directly from (165), (166), (168) and Lemma 23. Assume now that

$$f = \sum_{\mu=0}^{m} a_\mu x^{m-\mu} \in I[x], \quad f \ne 0,$$

and

(170) $$v := [v(R_1(f)), \ldots, v(R_{i^*}(f))] \in X_j.$$

By (161) and (163) we have

$$u = [v(a_0), \ldots, v(a_m)] \in U_{\beta_j}$$

and hence by Lemma 17

(171) $$\operatorname{card}\{\xi \in P\setminus\{0\}: f(\xi) = 0\} = \sum_{s=1}^{s_{\beta_j}} \operatorname{card}\{\xi \in I\setminus P: f(p^{\pi\langle\beta_j,s\rangle(u)}\xi) = 0\}.$$

To compute the right-hand side we apply Lemma 23 with f replaced by $f^s := f(p^{\pi\langle\beta_j,s\rangle(u)}x)$. Since $P_i^m \in C_0(m)$, $Q_{jkl}^m \in C_l(m)$, we have by Lemma 20

$$P_i^m(f^s) = P_i^m(f)p^{\pi\langle\beta_j,s\rangle(u)(m\deg P_i^m - w(P_i^m))},$$

(172) $$Q_{jkl}^m(f^s, y_1, \ldots, y_l) = Q_{jkl}^m(f, p^{\pi\langle\beta_j,s\rangle(u)}y_1, \ldots, p^{\pi\langle\beta_j,s\rangle(u)}y_l)p^{\pi\langle\beta_j,s\rangle(u)(m\deg^1 Q_{jkl}^m - w(Q_{jkl}^m))};$$

hence by (161), (162) and (170)

$$[v(P_1^m(f^s)), \ldots, v(P_{i_m}^m(f^s))] = \tau_{\beta_j s}(v).$$

Thus by (163)

$$[v(P_1^m(f^s)), \ldots, v(P_{i_m}^m(f^s))] \in W^m_{\delta_j(s)} \quad (1 \le s \le s_{\beta_j}).$$

This implies by Lemma 23 that

(173) $$\operatorname{card}\{\xi \in I\setminus P: f^s(\xi) = 0\} = \sum_{t=1}^{k_{\delta_j(s)}} \operatorname{card}\Big\{[\eta_1, \eta_2, \ldots] \in R^{l^m_{\delta_j(s)t}}: \bigwedge_{l=1}^{l^m_{\delta_j(s)t}} \mathscr{L}\mathscr{K} Q^m_{\delta_j(s)tl}(f^s, p^{\varrho\langle\delta_j(s),t,1\rangle(\tau_{\beta_j s}(v))}y_1, \ldots, p^{\varrho\langle\delta_j(s),t,l\rangle(\tau_{\beta_j s}(v))}y_l)|_{y_\lambda=\eta_\lambda} = 0\Big\}.$$

Using (171), (173) and (164), (165), we get

$$\operatorname{card}\{\xi \in P\setminus\{0\}: f(\xi) = 0\}$$

$$= \sum_{s=1}^{s_{\beta_j}} \sum_{t=1}^{k^m_{\delta_j(s)}} \operatorname{card}\Big\{[\eta_1, \eta_2, \ldots] \in R^{l^m_{\delta_j(s)t}}: \bigwedge_{l=1}^{l^m_{\delta_j(s)t}} \mathscr{L}\mathscr{K} Q_{\delta_j(s)tl}(f, p^{\pi\langle\beta_j,s\rangle(u)+\varrho\langle\delta_j(s),t,1\rangle(\tau_{\beta_j s}(v))}y_1, \ldots, p^{\pi\langle\beta_j,s\rangle(u)+\varrho\langle\delta_j(s),t,l\rangle(\tau_{\beta_j s}(v))}y_l)|_{y_\lambda=\eta_\lambda} = 0\Big\}$$

$$= \sum_{k=1}^{k'_j-\varepsilon_j} \operatorname{card}\Big\{[\eta_1, \eta_2, \ldots] \in R^{l_{jk}}: \bigwedge_{l=1}^{l_{jk}} \tilde{S}_{jkl}(f, \eta_1, \ldots, \eta_l) = 0\Big\},$$

where for all j, k, l

(174) $$\tilde{S}_{jkl}(f, y_1, \ldots, y_l) := \mathscr{L}\mathscr{K} S_{jkl}(f, p^{\sigma_{jk1}(v)}y_1, \ldots, p^{\sigma_{jkl}(v)}y_l).$$

If $\varepsilon_j = 0$ we have by (162) and (163) $v(a_m) \in N_0$; thus $f(0) = a_m \ne 0$ and

$$\operatorname{card}\{\xi \in P: f(\xi) = 0\} = \operatorname{card}\{\xi \in P\setminus\{0\}: f(\xi) = 0\}.$$

If $\varepsilon_j = 1$ we have by (162) and (163) $v(a_m) \in N_1 = \{\infty\}$; thus $f(0) = a_m = 0$ and by (166)

$$\operatorname{card}\{\xi \in P: f(\xi) = 0\} - \operatorname{card}\{\xi \in P\setminus\{0\}: f(\xi) = 0\} = 1 = \operatorname{card}\{\eta \in R: \eta = 0\} = \operatorname{card}\{\eta \in R: \tilde{S}_{jk'_j1}(f, \eta) = 0\}.$$

Since in the latter case $l_{jk'_j} = 1$, we have in both cases

(175) $$\operatorname{card}\{\xi \in P: f(\xi) = 0\} = \sum_{k=1}^{k'_j} \operatorname{card}\Big\{[\eta_1, \eta_2, \ldots] \in R^{l_{jk}}: \bigwedge_{l=1}^{l_{jk}} \tilde{S}_{jkl}(f, \eta_1, \ldots, \eta_l) = 0\Big\}.$$

On the other hand, by (163) and (170)

$$v \in N_+^{m+1} \times W^m_{\gamma_j};$$

hence by (162)

$$[v(P_1^m(f)), \ldots, v(P_{i_m}^m(f))] \in W^m_{\gamma_j},$$

and by Lemma 23

(176) $$\operatorname{card}\{\xi \in I\setminus P: f(\xi) = 0\} = \sum_{k=1}^{k^m_{\gamma_j}} \operatorname{card}\Big\{[\eta_1, \eta_2, \ldots] \in R^{l^m_{\gamma_j k}}: \bigwedge_{l=1}^{l^m_{\gamma_j k}} \tilde{Q}^m_{\gamma_j kl}(f, \eta_1, \eta_2, \ldots, \eta_l) = 0\Big\}.$$

Since $l_{\gamma jk}^{m} = l_{jk'_j+k}$ and by (167), (168) and (174)

$$\tilde{Q}^{m}_{\gamma_j kl}(f, \eta_1, \ldots, \eta_l) = \tilde{S}^{m'}_{jk'_j+kl}(f, \eta_1, \ldots, \eta_l),$$

it follows from (175) and (176) that

$$\operatorname{card}\{\xi \in \boldsymbol{I}\colon f(\xi) = 0\} = \sum_{k=1}^{k_j} \operatorname{card}\{[\eta_1, \eta_2, \ldots] \in \boldsymbol{R}^{l_{jk}}\colon \bigwedge_{l=1}^{l_{jk}} \tilde{S}_{jkl}(f, \eta_1, \ldots, \eta_l) = 0,$$

and the proof of the theorem is complete.

Proof of Theorem 2. For a polynomial $G \in \boldsymbol{Z}[x]$, let $l(G)$ be the sum of the absolute values of the coefficients of G. Take

$$c_1(m) = \max\{l(R_i), l(S_{jkl})\} + m,$$
$$c_2(m) = \max\{\deg R_i, \deg_a S_{jkl}\},$$

where the maximum is taken over all polynomials occurring in Theorem 1 for a given m. Let

$$F(x, t) = \sum_{\mu=0}^{m} a_\mu(t) t^{m-\mu}, \quad \boldsymbol{a}(t) = [a_0(t), \ldots, a_m(t)].$$

If $p > c_1(m) l(F)^{c_2(m)}$ we have

$$h(R_i(\boldsymbol{a}(t))) \leqslant l(R_i(\boldsymbol{a}(t))) \leqslant l(R_i) \max_{0 \leqslant \mu \leqslant m} l(a_\mu)^{\deg R_i} \leqslant c_1(m) l(F)^{c_2(m)} < p;$$

thus by Lemma 24

(177) $$\operatorname{ord}_p R_i(\boldsymbol{a}(p)) = \operatorname{ord}_t R_i(\boldsymbol{a}(t)).$$

Similarly

(178) $$\operatorname{ord}_p S_{jkl\boldsymbol{q}}(\boldsymbol{a}(p)) = \operatorname{ord}_t S_{jkl\boldsymbol{q}}(\boldsymbol{a}(t)),$$

where for $\boldsymbol{q} = [q_1, \ldots, q_l]$ $S_{jkl\boldsymbol{q}}$ is the coefficient of $\prod_{\lambda=1}^{l} y_\lambda^{q_\lambda}$ in S_{jkl}.

We apply Theorem 1 twice, namely for $\boldsymbol{K} = \boldsymbol{Q}_p$, $v = \operatorname{ord}_p$, $\boldsymbol{P} = (p)$, and for $\boldsymbol{K} = \boldsymbol{F}_p((t))$, $v = \operatorname{ord}_t$, $\boldsymbol{P} = (t)$. In both cases $\boldsymbol{R} = \boldsymbol{F}_p$, but the operations $\mathscr{K}$ and $\mathscr{L}$ in the first case and in the second case are different and we shall denote them by $\mathscr{K}_p$, $\mathscr{L}_p$ and $\mathscr{K}_t$, $\mathscr{L}_t$ respectively. It follows from (177) that for $p > c_1(m) l(F)^{c_2(m)} \geqslant m$

$$[\operatorname{ord}_p R_1(\boldsymbol{a}(p)), \ldots, \operatorname{ord}_p R_{i^*}(\boldsymbol{a}(p))] = [\operatorname{ord}_t R_1(\boldsymbol{a}(t)), \ldots, \operatorname{ord}_t R_{i^*}(\boldsymbol{a}(t))] = \boldsymbol{v}.$$

Put

(179) $$S^*_{jkl}(t, y_1, \ldots, y_l) = S_{jkl}(\boldsymbol{a}(t), t^{\sigma_{jk1}(\boldsymbol{v})} y_1, \ldots, t^{\sigma_{jkl}(\boldsymbol{v})} y_l).$$

If $\boldsymbol{v} \in \boldsymbol{X}_j$ we have by Theorem 1

(180) $$\operatorname{card}\{\xi \in \boldsymbol{Z}_p\colon F(\xi, p) = 0\} = \sum_{k=1}^{k_j} \operatorname{card}\{[\eta_1, \eta_2, \ldots] \in \boldsymbol{F}_p^{l_{jk}}\colon \bigwedge_{l=1}^{l_{jk}} \mathscr{L}_p \mathscr{K}_p S^*_{jkl}(p, y_1, \ldots, y_l)|_{y_\lambda = \eta_\lambda} = 0\},$$

(181) $$\operatorname{card}\{\xi \in \boldsymbol{F}_p[[t]]\colon F(\xi, t) = 0\} = \sum_{k=1}^{k_j} \operatorname{card}\{[\eta_1, \eta_2, \ldots] \in \boldsymbol{F}_p^{l_{jk}}\colon \bigwedge_{l=1}^{l_{jk}} \mathscr{L}_t \mathscr{K}_t S^*_{jkl}(t, y_1, \ldots, y_l)|_{y_\lambda = \eta_\lambda} = 0\}.$$

Now by (178) and (179)

$$\operatorname{ord}_p S^*_{jkl}(p, y_1, \ldots, y_l) = \operatorname{ord}_t S^*_{jkl}(t, y_1, \ldots, y_l);$$

hence by the definition of the operation $\mathscr{K}$

$$\mathscr{K}_p S^*_{jkl}(p, y_1, \ldots, y_l) = \mathscr{K}_t S^*_{jkl}(t, y_1, \ldots, y_l)|_{t=p}.$$

Taking in Lemma 24 for f all the coefficients of S^*_{jkl} viewed as a polynomial in $y_1, \ldots, y_l$, we get

$$\mathscr{L}_p \mathscr{K}_p S^*_{jkl}(p, y_1, \ldots, y_l) = \mathscr{L}_t \mathscr{K}_t S^*_{jkl}(t, y_1, \ldots, y_l),$$

and the theorem follows from (180) and (181).

§ 4. Examples and comments. We shall give explicitly, for $m = 1, 2, 3$, polynomials, sets and functions whose existence is asserted in Theorem 1. By convention $\boldsymbol{v} = [v_1, v_2, \ldots, v_{i^*}]$, $\infty \equiv 0 \bmod 6$.

$m = 1$: $i^* = 0$, $j^* = 1$, $k_1 = l_{11} = 1$, $S_{111} = a_0 y_1 + a_1$, $\sigma_{111}(\boldsymbol{v}) = 0$.

$m = 2$: $i^* = 4$, $R_i = a_{i-1}$ $(i = 1, 2, 3)$, $R_4 = a_1^2 - 4a_0 a_2$;
$j^* = 6$;
$\boldsymbol{X}_1 = \{\infty\}^4$, $k_1 = 0$;
$\boldsymbol{X}_2 = \{\boldsymbol{v} \in \boldsymbol{N}_+^4\colon v_3 < \min\{v_1, v_2\}\}$, $k_2 = 0$;
$\boldsymbol{X}_3 = \{\boldsymbol{v} \in \boldsymbol{N}_+^4\colon v_3 \geqslant v_2 < v_1\}$, $k_3 = 1$, $l_{31} = 1$, $S_{311} = a_1 x + a_2$, $\sigma_{311}(\boldsymbol{v}) = v_3 - v_2$;
$\boldsymbol{X}_4 = \{\boldsymbol{v} \in \boldsymbol{N}_+^4\colon v_1 \leqslant \min\{v_2, v_3\},\ v_4 \equiv 1 \bmod 2\}$, $k_4 = 0$;
$\boldsymbol{X}_5 = \{\boldsymbol{v} \in \boldsymbol{N}_+^4\colon v_1 \leqslant \min\{v_2, v_3, \infty > v_4 \equiv 0 \bmod 2\}$, $k_5 = 1$, $l_{51} = 1$, $S_{511} = y_1^2 - R_4$, $\sigma_{511}(\boldsymbol{v}) = \frac{1}{2} v_4$;
$\boldsymbol{X}_6 = \{\boldsymbol{v} \in \boldsymbol{N}_+^4 \setminus \boldsymbol{X}_1\colon v_1 \leqslant \min\{v_2, v_3\},\ v_4 = \infty\}$, $k_6 = 1$, $l_{61} = 1$, $S_{611} = y_1^2 - R_1$, $\sigma_{611}(\boldsymbol{v}) = 0$.

$m = 3$: $i^* = 7$, $R_i = a_{i-1}$ $(i = 1, 2, 3, 4)$,
$R_5 = 3(3a_0 a_2 - a_1^2)$, $R_6 = 2a_1^3 - 9a_0 a_1 a_2 + 27 a_0^2 a_3$,
$R_7 = a_1^2 a_2^2 - 4a_1^3 a_3 - 4a_0 a_2^3 + 18 a_0 a_1 a_2 a_3 - 27 a_0^2 a_3^2$;
$j^* = 13$;
$\boldsymbol{X}_1 = \{\infty\}^7$, $k_1 = 0$;
$\boldsymbol{X}_2 = \{\boldsymbol{v} \in \boldsymbol{N}_+^7\colon v_4 < \min\{v_1, v_2, v_3\}\}$, $k_2 = 0$;
$\boldsymbol{X}_3 = \{\boldsymbol{v} \in \boldsymbol{N}_+^7\colon v_4 \geqslant v_3 \leqslant \min\{v_1, v_2\}\}$, $k_3 = 1$, $l_{31} = 1$, $S_{311} = a_2 y_1 + a_3$, $\sigma_{311}(\boldsymbol{v}) = v_4 - v_3$;
$\boldsymbol{X}_4 = \{\boldsymbol{v} \in \boldsymbol{N}_+^7\colon \min\{v_3, v_4\} \geqslant v_2 < v_1,\ v_7 \equiv 1 \bmod 2\}$, $k_4 = 0$;

$X_5 = \{v \in N_+^7 \colon \min\{v_3, v_4\} \geqslant v_2 < v_1, v_7 = \infty\}$, $k_5 = 1$, $l_{51} = 1$, $S_{511} = 2a_1 y_1 + a_2$, $\sigma_{511}(v) = v_3 - v_2$;

$X_6 = \{v \in N_+^7 \colon \min\{v_3, v_4\} \geqslant v_2 \langle v_1, \infty \rangle v_7 \equiv 0 \bmod 2\}$, $k_6 = 1$, $l_{61} = 1$, $S_{611} = y_1^2 - R_7$; $\sigma_{611}(v) = \frac{1}{2} v_7$;

$X_7 = \{v \in N_+^7 \colon v_1 \leqslant \min\{v_2, v_3, v_4\},\ \infty > 2v_6 > 3v_5;\ v_5 \equiv 1 \bmod 2\}$, $k_7 = 1$, $l_{71} = 1$, $S_{711} = R_5 y_1 + R_6$, $\sigma_{711}(v) = v_6 - v_5$;

$X_8 = \{v \in N_+^7 \colon v_1 \leqslant \min\{v_2, v_3, v_4\},\ \infty > 2v_6 > 3v_5,\ v_5 \equiv 0 \bmod 2\}$, $k_8 = 2$, $l_{81} = 1$, $l_{82} = 1$, $S_{811} = R_5 y_1 + R_6$, $\sigma_{811}(v) = v_6 - v_5$, $S_{821} = y_1^2 + R_5$, $\sigma_{821}(v) = \frac{1}{2} v_5$;

$X_9 = \{v \in N_+^7 \colon v_1 \leqslant \min\{v_2, v_3, v_4\},\ 2v_6 \leqslant 3v_5,\ v_6 \not\equiv 0 \bmod 3\}$, $k_9 = 0$;

$X_{10} = \{v \in N_+^7 \colon v_1 \leqslant \min\{v_2, v_3, v_4\},\ 2v_6 \leqslant 3v_5,\ \infty > v_6 \equiv 0 \bmod 3\}$, $k_{10} = 1$, $l_{10,1} = 1$, $S_{10,1,1} = y_1^3 + R_5 y_1 + R_6$, $\sigma_{10,1,1}(v) = \frac{1}{3} v_6$;

$X_{11} = \{v \in N_+^7 \colon v_1 \leqslant \min\{v_2, v_3, v_4\},\ v_6 = \infty > v_5 \equiv 1 \bmod 2\}$, $k_{11} = 1$, $l_{11,1} = 1$, $S_{11,1,1} = y_1$, $\sigma_{11,1,1}(v) = 0$;

$X_{12} = \{v \in N_+^7 \colon v_1 \leqslant \min\{v_2, v_3, v_4\},\ v_6 = \infty > v_5 \equiv 0 \bmod 2\}$, $k_{12} = 1$, $l_{12,1} = l_{12,2} = 1$, $S_{12,1,1} = y_1$, $\sigma_{12,1,1}(v) = 0$, $S_{12,2,1} = y_1^2 + R_5$, $\sigma_{12,2,1}(v) = \frac{1}{2} v_5$;

$X_{13} = \{v \in N_+^7 \setminus X_1 \colon v_1 \leqslant \min\{v_2, v_3, v_4\},\ v_5 = v_6 = \infty\}$, $k_{13} = 1$, $l_{13,1} = 1$, $S_{13,1,1} = y_1$, $\sigma_{13,1,1}(v) = 0$.

Let us observe that for $m = 3$ R_7 is the discriminant of the cubic form $F(x, y) = \sum_{i=0}^{3} a_i x^{3-i} y^i$ while R_5 and R_6 are constant multiples of the Cayley invariants of the quartic form $yF(x, y)$. The inspection of the data given above shows that in every case where $k_j \geqslant 1$ we have $l_{jk} = 1$. Therefore it is of some interest to exhibit for $m = 4$ the case where $l_{jk} = 2$:

$$v(a_0) = 0,\ v(a_1) = \infty,\quad \tfrac{1}{3} v(a_3) > \tfrac{1}{4} v(\alpha_4) = \tfrac{1}{2} v(a_2) \in N_0,$$
$$v(a_3) - \tfrac{3}{2} v(a_2) > v(a^2 - 4a_0 a_4) - 2v(a_2) \in 2N.$$

Here $k_j = 1$, $l_j = 2$,

$$S_{j11} = 2a_0 y_1 + a_2, \qquad \sigma_{j11}(v) = \tfrac{1}{2} v(a_2),$$
$$S_{j12} = y^2 + (4a_0 a_4 - a_2^2), \qquad \sigma_{j12}(v) = \tfrac{1}{2} v(a_2^2 - 4a_0 a_4).$$

Again in this case there is no variable occurring simultaneously in S_{j11} and S_{j12}. The first case encountered by the writer in which $\bigwedge_{l=1}^{l_{jk}} \bar{S}_{jkl}(\eta_1, \ldots, \eta_l) = 0$ is a system of interrelated equations occurs for $m = 6$ and f of the type $g(x)^2 + p^2 h(x)$, $g, h \in \boldsymbol{I}[x]$.

Finally we remark that the method of proof of Theorem 1 leads to a similar theorem about congruences modulo powers of $\boldsymbol{P}$. Namely, we have

THEOREM 3. *For every $m \in N$ there exists a system of forms $R_i^*(\boldsymbol{a})$ $(i \leqslant i^{**})$ and polynomials $S_{jkl}(\boldsymbol{a}, y_1, \ldots, y_l)$ $(j \leqslant j^{**},\ k \leqslant k_j^*,\ l \leqslant l_{jk}^*)$ with integral coefficients, a decomposition*

$$N \times N_+^{i^{**}} = \bigcup_{j=j}^{j^{**}} X_j^*$$

and N_0-valued functions $\sigma_{jkl}^*(v)$ defined on X_j^* with the following property.

If $\operatorname{char} \boldsymbol{R} = 0$ *or* $\operatorname{char} \boldsymbol{R} > m$,

$$f(x) = \sum_{\mu=0}^{m} a x^{m-\mu} \in \boldsymbol{I}[x], \quad f \neq 0, \quad \boldsymbol{a} = [a_0, \ldots, a_m],$$
$$v = [n, v(R_1^*(\boldsymbol{a})), \ldots, v(R_{i^{**}}^*(\boldsymbol{a}))] \in X_j$$

and

$$\bar{S}_{jkl}^*(y_1, \ldots, y_l) = \mathscr{L}\mathscr{K} S_{jkl}^*(\boldsymbol{a}, p^{\sigma_{jk1}^*(v)} y_1, \ldots, p^{\sigma_{jkl}^*(v)} y_l),$$

then the congruence

$$f(x) \equiv 0 \bmod \boldsymbol{P}^n$$

is solvable in $\boldsymbol{I}$ if only if for some $k \leqslant k_j^*$ the system of equations

$$\bar{S}_{jkl}^*(\eta_1, \ldots, \eta_l) = 0 \quad (1 \leqslant l \leqslant l_{jk}^*)$$

is solvable in $\boldsymbol{R}$.

The polynomials R_i^*, S_{jkl}^*, the sets X_j^* and the functions σ_{jkl}^* are independent of $\boldsymbol{K}$, v and p.

References

[1] P. Cohen, *Decision procedures for real and p-adic fields*, Comm. Pure Appl. Math. 22 (1969), pp. 131–151.

[2] T. Nagel, *Généralisation d'un théorème de Tchebycheff*, J. Math. Pur. Appl. (8) 4 (1921), pp. 343–356.

[3] T. Nagell, *Introduction to Number Theory*, New York 1964.

[4] O. Perron *Über den grössten gemeinsamen Teiler vor zwei Polynomen*, S. B. Bayer. Akad. Wiss. 1928, pp. 105–112.

Received 21 June 1982