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Borel sets in compact spaces: some Hurewicz type theorems

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Abstract. Let X be a compact metric space, and let A be a Borel subset of X. We identify two subspaces S and T of the Cantor set, and prove that:

(1) A is not the union of a complete and a countable subset if and only if X contains a Cantor set K such that $K \setminus A \approx \boldsymbol{P}$ and $K \cap A \approx \boldsymbol{Q} \times \boldsymbol{C}$.

(2) A is not strongly σ-complete if and only if X contains a Cantor set K such that $K \setminus A \approx \boldsymbol{Q} \times \boldsymbol{P}$ and $K \cap A \approx T$.

(3) A is not the union of a strongly σ-complete and a countable subset if and only if X contains a Cantor set K such that $K \setminus A \approx \boldsymbol{Q} \times \boldsymbol{P}$ and $K \cap A \approx S$.

As an application, we give topological characterizations of $\boldsymbol{Q} \times S$ and $\boldsymbol{Q} \times T$.

1. Introduction.

All spaces under discussion are separable metric.

In his 1928 paper [6], Hurewicz proved that a Borel subset A of a compact space X is not a G_δ in X (i.e. A is not topologically complete) if and only if there exists a compact subset K of X such that $K \cap A \approx \boldsymbol{Q}$ (the rationals) and $K \setminus A \approx \boldsymbol{P}$ (the irrationals). A theorem of the same type was proved in 1978 by Saint Raymond ([10]): he showed, among others, that a Borel subset A of a compact space X is not the union of an F_σ and a G_δ of X (i.e. A is not the union of a σ-compact and a topologically complete subspace) if and only if there exists a compact subspace K of X such that $K \cap A \approx \boldsymbol{Q} \times \boldsymbol{P}$. However, he did not prove anything concerning $K \setminus A$. In the light of Hurewicz's result, this suggests an obvious question; in this paper, we will answer this question, and prove some more "Hurewicz-type" theorems.

We identify a certain zero-dimensional space T, which can easily be visualized as the remainder of $\boldsymbol{Q} \times \boldsymbol{P}$ in some compactification of $\boldsymbol{Q} \times \boldsymbol{P}$, and we prove that a Borel subset A of a compact space X is not the union of a σ-compact and a topologically complete subspace if and only if there exists a Cantor set K in X such that $K \cap A \approx \boldsymbol{Q} \times \boldsymbol{P}$ and $K \setminus A \approx T$. This theorem can also be stated in a slightly different way. Call a subset Y of a space X *strongly σ-complete* if $Y = \bigcup \{Y_i \colon i \in N\}$, where each Y_i is topologically complete and closed in Y; it is easily seen that a subset Y of a compact space X is strongly σ-complete if and only if Y is the intersection of a σ-compact and a topologically complete subspace of X (Lemma 2.1). The above-mentioned theorem then states that a Borel subset A of a compact space X is not strongly σ-complete if and only if there exists a Cantor set K in X such that $K\cap A\approx T$ and $K\setminus A\approx \boldsymbol{Q}\times \boldsymbol{P}$. Thus, in a sense, T is minimal among the non-strongly σ-complete Borel sets. If a Borel set in a compact space does not come too close to being strongly σ-complete, then we can prove a somewhat stronger statement; for this purpose, we identify another zero-dimensional remainder of $\boldsymbol{Q}\times \boldsymbol{P}$, which we call S, and which is "larger" than T: S contains a closed copy of T, but not conversely. Then we show that a Borel subset A of a compact space X is not the union of a countable and a strongly σ-complete subspace if and only if there exists a Cantor set K in X such that $K\cap A\approx S$ and $K\setminus A\approx \boldsymbol{Q}\times \boldsymbol{P}$. In the proof, we will use another result à la Hurewicz: a Borel subset A of a compact space X is not the union of a countable and a topologically complete subset if and only if there exists a Cantor set K in X such that $K\cap A\approx \boldsymbol{Q}\times C$ and $K\setminus A\approx \boldsymbol{P}$ (here C denotes the Cantor set). As an application of the above results, we obtain topological characterizations of $\boldsymbol{Q}\times S$ and $\boldsymbol{Q}\times T$.

The space S has been topologically characterized by van Mill in [9]. For T, this was first done by van Douwen ([4]); since his proof has not yet been published, we include a new proof of this characterization in an appendix to this paper.

2. Known results and preliminary lemmas. Notation is standard, as e.g. in Engelking [5]; $A\approx B$ means that A and B are homeomorphic. The diameter of a set A is denoted by $\operatorname{diam}(A)$. All metrics in this paper are denoted by d and assumed to be bounded by 1; also, if the space in question is topologically complete, we always take d to be a complete metric. A subset of a space X is *clopen* if it is both closed and open in X. A space X is *strongly σ-complete* if it is the countable union of closed and complete subsets of X; we use "complete" as an abbreviation for "topologically complete", i.e. being an absolute G_δ.

2.1. LEMMA. *A subset A of a compact space X is strongly σ-complete if and only if it is the intersection of an F_σ and a G_δ of X.*

Proof. If A is strongly σ-complete, then $A=\bigcup_{i=1}^{\infty} A_i$, where A_i is closed in A and complete. Hence A_i is a G_δ in $\overline{A}_i$, so $\overline{A}_i\setminus A_i$ is an F_σ in $\overline{A}_i$ and hence also in X. So $X\setminus A=\bigcup_{i=1}^{\infty}(\overline{A}_i\setminus A_i)\cup\bigcap_{i=1}^{\infty}(X\setminus\overline{A}_i)$ is the union of an F_σ and a G_δ of X; equivalently, A is the intersection of an F_σ and a G_δ of X. Conversely, suppose $A=(\bigcup_{i=1}^{\infty}F_i)\cap G$, where each F_i is closed in X and G is complete. Then $F_i\cap G=F_i\cap A$ is closed in A, and in G, hence complete. So $A=\bigcup_{i=1}^{\infty}(F_i\cap G)$ is strongly σ-complete. ∎

We will now give topological characterizations of some Borel subsets of the Cantor set. Here, if $\mathscr{P}$ is a topological property, then a space is *nowhere* $\mathscr{P}$ if none of its non-empty open subsets has $\mathscr{P}$; note that if $\mathscr{P}$ is a closed-hereditary property of a zero-dimensional space, this is equivalent to none of the non-empty clopen subsets of the space having $\mathscr{P}$. Of course, "unique" is "unique up to homeomorphism".

2.2. THEOREM. (a) (Brouwer [3]). *The Cantor set C is the unique zero-dimensional compact space without isolated points.*

(b) (Alexandroff and Urysohn [1]). *The set of irrationals $\boldsymbol{P}$ is the unique zero-dimensional complete, nowhere locally compact space.*

(c) (Sierpiński [11]). *The set of rationals $\boldsymbol{Q}$ is the unique countable space without isolated points.*

(d) (Alexandroff and Urysohn [1]; van Mill [8]). *$\boldsymbol{Q}\times C$ is the unique zero-dimensional σ-compact, nowhere countable, nowhere locally compact space; equivalently, it is the unique zero-dimensional space X such that $X=\bigcup_{i=1}^{\infty}C_i$, where $C_i\approx C$, and each C_i is a nowhere dense subset of C_{i+1}.*

(e) (van Mill [8]). *$\boldsymbol{Q}\times\boldsymbol{P}$ is the unique zero-dimensional strongly σ-complete, nowhere σ-compact, nowhere complete space; equivalently, it is the unique zero-dimensional space X such that $X=\bigcup_{i=1}^{\infty}P_i$, where $P_i\approx\boldsymbol{P}$ is closed in X and a nowhere dense subset of P_{i+1}.*

Now let K, Q be dense subsets of C such that $K\approx\boldsymbol{Q}\times C$, $Q\approx\boldsymbol{Q}$. Let $P_1=C\setminus K$, $P_2=C\setminus Q$. Then $P_1\approx P_2\approx\boldsymbol{P}$. Define $S,T\subset C\times C$ by $S=(C\times C)\setminus(Q\times P_1)$, $T=(C\times C)\setminus(Q\times P_2)$; then $(C\times C)\setminus S\approx(C\times C)\setminus T\approx\boldsymbol{Q}\times\boldsymbol{P}$. The spaces S and T can be characterized as follows (a proof of (b) will be given in an appendix to this paper):

2.3. THEOREM. (a) (van Mill [9]). *S is the unique zero-dimensional space which is the union of a complete and a σ-compact subspace, and which is nowhere σ-compact and nowhere the union of a countable and a complete subspace.*

(b) (van Douwen [4]). *T is the unique zero-dimensional space which is the union of a complete and a countable subspace, and which is nowhere σ-compact and nowhere complete.*

Let us note that $\boldsymbol{Q}\times\boldsymbol{P}$, S and T are pairwise non-homeomorphic. S and T are Baire (they contain a dense copy of $\boldsymbol{P}$), whereas $\boldsymbol{Q}\times\boldsymbol{P}$ is not; and S is not the union of a complete and a countable subspace. Also observe that each of the above spaces is homeomorphic to any of its non-empty clopen subspaces (this follows easily from the characterizations). This implies that each of these spaces is homogeneous; in fact, any homeomorphism between closed and nowhere dense subsets can be extended to an autohomeomorphism of the whole space ([8], Theorem 3.1). That C, $\boldsymbol{P}$, $\boldsymbol{Q}$, $\boldsymbol{Q}\times C$ and $\boldsymbol{Q}\times\boldsymbol{P}$ are homogeneous is trivial of course; however, no easy proofs for the homogeneity of S or T are known.

2.4. LEMMA. *Let X be compact zero-dimensional, and let A be dense in X. Then $A\approx\boldsymbol{Q}\times\boldsymbol{P}$ if and only if $X\setminus A$ is a nowhere σ-compact, nowhere complete space which is the union of a complete and a σ-compact subspace.*

Proof. If $A \approx Q\times P$, then A is strongly σ-complete by Theorem 2.2(e); hence $X\setminus A$ is the union of a σ-compact and a complete subspace by Lemma 2.1. If U is open in X and $U\setminus A$ is σ-compact, then $U\cap A$ is a complete open subspace of A, contradicting nowhere completeness of $Q\times P$; thus, $X\setminus A$ is nowhere σ-compact. Similarly, nowhere σ-compactness of $Q\times P$ yields nowhere completeness of $X\setminus A$. The converse statement follows by exactly the same argumentation. ∎

2.5. Lemma. *Let X be any space, and let A, B and K be subspaces such that $\emptyset \neq K$ is compact, A is nowhere dense in B, and $A\cap K$ is dense in K. Then there exists a countable discrete subset D of $B\setminus\bar{A}$ such that $\bar{D} = D\cup K$; furthermore, if $(\varepsilon_n)_{n\in N}$ is a given sequence of positive numbers, we can choose $D = \{d_n: n\in N\}$ in such a way that $d(K, d_n) < \varepsilon_n$.*

Proof. For each $i\in N$, let $\mathscr{D}_i$ be a finite cover of K by open sets of X of diameter less than $1/i$, say $\mathscr{D}_i = \{D(i,j): j = 1, \dots, n_i\}$. Then for each $i\in N$ and $j\leqslant n_i$, we have $D(i,j)\cap A\cap K\neq\emptyset$, say $p(i,j)\in D(i,j)\cap A\cap K$. Let $n\in N$. Then there exists a unique $i\in N$ such that $n = (\sum_{k=1}^{i-1} n_k)+j$ for some $j\leqslant n_i$, and we choose $y(i,j) = d_n\in \big(B(p(i,j), \varepsilon_n)\cap D(i,j)\cap B\big)\setminus\bar{A}$. We claim that $D = \{d_n: n\in N\}$ is as required. Clearly $D\subset B\setminus\bar{A}$, and if $n = (\sum_{k=1}^{i-1} n_k)+j$ then $d(d_n, K)\leqslant d\big(d_n, p(i,j)\big)<\varepsilon_n$. Now take $x\in\bar{D}$ and let $x_i\in D$ be such that $\lim_{i\to\infty} x_i = x$; then either $(x_i)_i$ contains a constant subsequence, or we may assume that, for each $i\in N$, $x_i = y(i,j_i)$ for some $j_i\in\{1,\dots,n_i\}$. In the first case, $x\in D$; otherwise, $d(x,K)\leqslant d(x,x_k)+d(x_k,K)\leqslant d(x,x_k)+1/k$ which converges to 0. So $x\in K$. Hence D is discrete and $\bar{D}\subset D\cup K$. Conversely, suppose $x\in K$. Then for each $i\in N$, $x\in D(i,j_i)$ for some $j_i\in\{1,\dots,n_i\}$. Since $\operatorname{diam} D(i,j_i)<1/i$ for each i, $\lim_{i\to\infty} y(i,j_i) = x$, so $x\in\bar{D}$. ∎

3. Dense copies of $Q\times C$ in P.

In this section we will show that, essentially, $Q\times C$ can be densely embedded in P in only one way; more precisely, we well show that, given two dense copies A_1 and A_2 of $Q\times C$ in P, there exists an autohomeomorphism h of P which maps A_1 onto A_2 (i.e. P is "$Q\times C$ dense homogeneous").

3.1. Definition. Let $X\in\{C, P\}$. A *skeletoid in X* is a subset $A = \bigcup_{i=1}^{\infty} A_i$ of X, with $A_i\subset A_{i+1}$, such that each A_i is nowhere dense and compact, and such that for each $\varepsilon>0$, each $m\in N$, and each nowhere dense compact subset B of X, there exists an $n\in N$ and a homeomorphic embedding $f: A_m\cup B\to A_n$ such that $f|A_m = \mathrm{id}$ and $\sup\{d(f(x), x): x\in A_m\cup B\}<\varepsilon$.

Note that if $X = P$, then the "nowhere dense" can be deleted. Using [2] (Corollary IV. 3.1 and Proposition IV.4.1) and [8] (Theorem 3.1) it is easily seen that this is equivalent to the usual definition of a $\mathscr{K}$-skeletoid in P (resp. C), where $\mathscr{K}$ is the set of compacta in P (resp. nowhere dense compacta in C). Thus we have:

3.2. Theorem ([2], Theorem IV.2.1). *If A_1 and A_2 are two skeletoids in P, then there exists a homeomorphism h: $P\to P$ such that $h[A_1] = A_2$.*

Hence to show that P is $Q\times C$ dense homogeneous, it suffices to prove:

3.3. Theorem. *If A is a dense copy of $Q\times C$ in P, then A is a skeletoid in P.*

Proof. Fix $\varepsilon>0$, $m\in N$, and a compact subset B of P. Let $A = \bigcup_{i=1}^{\infty} A_i$, where $A_i\approx C$ and A_i is a nowhere dense subset of A_{i+1} (Theorem 2.2(d)). Embed P in C as a dense subset. Since P is nowhere locally compact, each A_i is a now here dense compact subset of C; by [9], A is a skeletoid in C. Also, B is nowhere dense in C, so by Definition 3.1 there is an $n\in N$ and a homeomorphic embedding f: $A_m\cup B\to A_n$ such that $f|A_m = \mathrm{id}$ and $\sup\{d(f(x), x): x\in A_m\cup B\}<\varepsilon$. Again by Definition 3.1, A is a skeletoid in P. ∎

3.4. Corollary. *If A_1 and A_2 are two dense copies of $Q\times C$ in P, then there exists a homeomorphism h: $P\to P$ such that $h[A_1] = A_2$.*

4. Borel sets which are not complete.

Recall that Hurewicz's original theorem states that if A is a Borel subset of a compact space X which is not complete, then there exists a compact subset K of X such that $K\cap A\approx Q$ and $K\setminus A\approx P$. We will first show that for K we can even choose a Cantor set.

4.1. Theorem. *Let X be compact, and let A be a Borel subset of X. Then A is not complete if and only if there exists a Cantor set K in X such that $K\cap A\approx Q$ and $K\setminus A\approx P$.*

Proof. Clearly, if A contains a closed copy of Q, then A cannot be complete. Conversely, if A is not complete, then by Hurewicz's theorem ([6]; for a different proof see [10] or [12]) there exists a compact subset K' of X such that $K'\cap A\approx Q$ and $K'\setminus A\approx P$. Let Q be a copy of Q in C, and let f: $Q\to K'\cap A$ be a homeomorphism. Since both C and K' are compact, we can apply Lavrentieff's theorem (see e.g. [5], Theorem 4.3.21) to obtain homeomorphic G_δ-subsets B and D of C and K', respectively, such that $B\supset Q$ and $D\supset K'\cap A$. In particular, D is zero-dimensional. Now if D contains an open (in D) compact subset U, then $U\approx C$ and $U\cap A\approx Q$ whence $U\setminus A\approx P$; so put $K = U$. If this is not the case, then D is nowhere locally compact, hence homeomorphic to P by Theorem 2.2(b). Now let P be any copy of P, and let K_0 be a copy of C contained in P; let Q_0 be countable and dense in K_0, and let Q_1 be countable and dense in $P\setminus K_0$. Then $Q = Q_0\cup Q_1$ is countable and dense in P. By [5] (Exercise 4.3.H) there exists a homeomorphism h: $P\to D$ such that $h[Q] = K'\cap A$. Then $K = h[K_0]$ is as required. ∎

We will now turn to the proof of our first Hurewicz-type theorem; in our arguments we will use techniques from [10]. In the remainder of this section, as well as in the next section, we will denote by M the set of all finite sequences of natural numbers (including the empty sequence $\emptyset$). If $s = (s_1, \dots, s_k)\in M$, then for each $n\in N$, "s, n" denotes the sequence $(s_1, \dots, s_k, n)\in M$; $|s| = k$ is the length of s and $\nu(s) = s_1+\dots+s_k$; we put $|\emptyset| = \nu(\emptyset) = 0$. If σ is an infinite sequence of natural numbers, then "$s<\sigma$" means that s is an initial segment of σ.

4.2. Theorem. *Let X be compact, and let A be a Borel subset of X. Then A is* *the union of a countable and a complete subset if and only if A does not contain a closed copy of $Q\times C$.*

Proof. If A is the union of a countable and a complete subspace, and if B is closed in A, then also $B = F\cup G$, where F is countable and G is complete. If $B\approx Q\times C$, then B contains uncountably many closed disjoint copies of Q, hence one of those is contained in G; but Q is not complete, a contradiction.

Conversely, suppose that A is not the union of a countable and a complete subset. Since A is Borel, there exists by [7] a continuous surjection $\varphi\colon \boldsymbol{P}\to X\setminus A$. Put $W = \{x\in\boldsymbol{P}$: there exists a neighborhood V_x of x in $\boldsymbol{P}$ and a σ-compact subset E_x of X such that $\varphi[V_x]\subset E_x$ and $E_x\cap A$ is countable$\}$. Then W is open in $\boldsymbol{P}$, and there exist countably many open V_i in $\boldsymbol{P}$ and σ-compact E_i in X such that $W = \bigcup_{i=1}^{\infty} V_i$, $\varphi[V_i]\subset E_i$, and $E_i\cap A$ is countable for each $i\in N$. Then $E = \bigcup_{i=1}^{\infty} E_i$ is σ-compact, $\varphi[W]\subset E$, and $E\cap A$ is countable. Put $F = \boldsymbol{P}\setminus\varphi^{-1}[E\setminus A]$. Then F is a G_δ in $\boldsymbol{P}$, hence complete. We claim that F is non-empty. Indeed, if $\varphi^{-1}[E\setminus A] = \boldsymbol{P}$, then $E\supset X\setminus A$ and $A = (X\setminus E)\cup(E\cap A)$. However, E is σ-compact so $X\setminus E$ is complete, and $E\cap A$ is countable, contradicting our hypothesis on A. So $F\neq\emptyset$. Also if $\emptyset\neq U$ is open in F, say $U = U'\cap F$ where U' is open in $\boldsymbol{P}$, then $\varphi[U'] = \varphi[U]\cup\varphi[U'\setminus U]\subset\overline{\varphi[U]}\cup E$ which is σ-compact; since $U'\not\subset W$, $(\overline{\varphi[U]}\cup E)\cap A = (\overline{\varphi[U]}\cap A)\cup(E\cap A)$ is uncountable, whence $\overline{\varphi[U]}\cap A$ is uncountable. Thus, $\overline{\varphi[U]}\cap A$ contains a copy of the Cantor set ([7]).

For each $s\in M$, we will now construct Cantor sets K_s in A and open subsets W_s of F, such that the following hold for each $s\in M$:

(1) $K_s\subset A\cap\overline{\varphi[W_s]}$;

(2) for each $n\in N$: $\overline{\varphi[W_{s,n}]}\cap K_s = \emptyset$;

(3) for each $n, m\in N$: $\overline{\varphi[W_{s,n}]}\cap\overline{\varphi[W_{s,m}]} = \emptyset$ if $n\neq m$;

(4) for each $n\in N$: $\overline{W}_{s,n}\subset W_s$ (here closure is taken in F);

(5) $\operatorname{diam}(W_s)\leqslant 2^{-|s|}$ (here the diameter is taken w.r.t. a complete metric for F);

(6) $\operatorname{diam}(\varphi[W_s])\leqslant 2^{-\nu(s)}$;

(7) for each $n\in N$: $d(K_s, K_{s,n})\leqslant 2^{1-\nu(s,n)}$.

We proceed by induction on $|s|$. First, put $W_\emptyset = F$; then $\overline{\varphi[W_\emptyset]}\cap A$ is uncountable, so it contains a Cantor set $K_\emptyset$. Then (1), (5) and (6) are satisfied since our metrics are bounded by 1. Next, suppose that W_s and K_s have been constructed for $|s|\leqslant k$, in accordance with conditions (1) through (7). Fix $s\in M$ with $|s| = k$; we will construct $W_{s,n}$ and $K_{s,n}$ for each $n\in N$. By (1), $K_s\subset A\cap\overline{\varphi[W_s]}$; since $\varphi[W_s]\subset X\setminus A$, K_s is nowhere dense in $K_s\cup\varphi[W_s]$ and we can apply Lemma 2.5 to obtain a countable discrete subset $D_s = \{y_{s,n}\colon n\in N\}$ of $\varphi[W_s]$ such that $\overline{D}_s = D_s\cup K_s$, and $d(y_{s,n}, K_s)\leqslant 2^{-\nu(s,n)}$ for each $n\in N$. Let $U_{s,n}$ be a neighborhood

of $y_{s,n}$ in $X\setminus A$ such that $\operatorname{diam}(U_{s,n})\leqslant 2^{-\nu(s,n)}$, $\overline{U}_{s,n}\cap\overline{U}_{s,m} = \emptyset$ if $n\neq m$, and $\overline{U}_{s,n}\cap K_s = \emptyset$, for each $n\in N$. Since $y_{s,n}\in\varphi[W_s]$, $y_{s,n} = \varphi(x_{s,n})$ for some $x_{s,n}\in W_s$; hence there is an open neighborhood $W_{s,n}$ of $x_{s,n}$ in F such that $\overline{W}_{s,n}\subset W_s$, $\operatorname{diam}(W_{s,n})\leqslant 2^{-|s|-1}$, and $\varphi[W_{s,n}]\subset U_{s,n}$. Then (2)-(6) are satisfied. Since $\overline{\varphi[W_{s,n}]}\cap A$ is uncountable, it contains a Cantor set $K_{s,n}$. To verify (7), note that

$$d(K_s, K_{s,n})\leqslant d(K_s, \varphi[W_{s,n}])+\operatorname{diam}(\varphi[W_{s,n}])\leqslant d(K_s, y_{s,n})+2^{-\nu(s,n)}.$$

This completes the induction. Now define $B_i = \bigcup\{K_s\colon |s|\leqslant i\}$. We claim that $\bigcup_{i=0}^{\infty} B_i$ is closed in A and homeomorphic to $Q\times C$. We first show that $B_i\approx C$ for each $i\in N$. Since $K_s\approx C$ for each $s\in M$, each B_i is zero-dimensional, being the countable union of closed zero-dimensional subspaces. Clearly, it contains no isolated points; hence it suffices to show that each B_i is closed in X. This is trivial for $i = 0$, so suppose it is true for $i = k$. For each $\varepsilon>0$, let $B_k^\varepsilon = \{x\in X\colon d(B_k, x)\leqslant\varepsilon\}$, and $M_k^\varepsilon = \{s\in M\colon |s| = k+1, K_s\not\subset B_k^\varepsilon\}$; then each B_k^ε is compact, and each M_k^ε is finite by (1), (6) and (7). Since B_k is compact, $B_k = \bigcap_{\varepsilon>0} B_k^\varepsilon$, and

$$B_{k+1} = \bigcap_{\varepsilon>0}(B_k^\varepsilon\cup\bigcup\{K_s\colon s\in M_k^\varepsilon\})$$

is compact being the intersection of compacta. From (2) and (7) it is obvious that B_i is nowhere dense in B_{i+1} for each i, and thus we may conclude from Theorem 2.2(d) that $B = \bigcup_{i=1}^{\infty} B_i\approx Q\times C$. To show that B is closed in A, take $x\in\overline{B}\setminus B$, and fix $i\in N$. Since $x\notin B_i$, $x\notin B_i^\varepsilon$ for some $\varepsilon>0$. From (1) and (4) it follows that $B\subset\bigcup_{|s|=i}\overline{\varphi[W_s]}\cup B_i$, and from (1) and (6) that $\overline{\varphi[W_s]}\subset B_i^\varepsilon$ for all but finitely many $s\in M$ with $|s| = i$. Hence for some finite $M_0\subset\{s\in M\colon |s| = i\}$, we have $\overline{B}\subset B_i^\varepsilon\cup\bigcup_{s\in M_0}\overline{\varphi[W_s]}$. Then $x\in\overline{\varphi[W_s]}$ for some $s\in M_0$, and this s is unique with $|s| = i$ by (3). So by (4), there exists an infinite sequence σ of natural numbers such that $x\in\bigcap_{s<\sigma}\overline{\varphi[W_s]}$ which is a one-point set by (6); also $\bigcap_{s<\sigma}\overline{W}_s = \bigcap_{s<\sigma} W_s$ is a one-point set by (5) and by completeness of F. Hence if $\{z\} = \bigcap_{s<\sigma} W_s$, then $\varphi(z)\in\bigcap_{s<\sigma}\overline{\varphi[W_s]} = \{x\}$, so $x\in\varphi[\boldsymbol{P}] = X\setminus A$; thus $\overline{B}\setminus B\subset X\setminus A$ and B is closed in A. ■

4.3. Theorem. *Let X be compact, and let A be a Borel subset of X which is not the union of a complete and a countable subset. Then X contains a Cantor set K such that $K\cap A\approx Q\times C$ and $K\setminus A\approx\boldsymbol{P}$.*

Proof. By Theorem 4.2, A contains a closed copy B of $Q\times C$. Also, we can embed $Q\times C$ as a subset D of the Cantor set. Let $f\colon B\to D$ be a homeomorphism. As in the proof of Theorem 4.1, we can apply Lavrentieff's theorem to obtain a zero-dimensional G_δ-subset G of $\overline{B}$ such that $G\supset B$. Now if G contains an open (in G) compact subset U, then $U\approx C$ and $U\cap A = U\cap B\approx Q\times C$ whence $U\setminus A = U\setminus B\approx\boldsymbol{P}$; so put $K = U$. If this is not the case, then G is nowhere locally compact, hence homeomorphic to $\boldsymbol{P}$ by Theorem 2.2(b). Now let P be any copy of $\boldsymbol{P}$, and let K_0 be a copy of C contained in P. Let $B_0 \approx \boldsymbol{Q} \times C$ be dense in K_0; since $P \setminus K_0 \approx \boldsymbol{P}$, there also exists a dense copy B_1 of $\boldsymbol{Q} \times C$ in $P \setminus K_0$. Now $\tilde{B} = B_0 \cup B_1$ is dense in P, and clearly σ-compact and nowhere countable; it is also nowhere locally compact: if $\emptyset \neq V$ is open in $\tilde{B}$ and locally compact, then $V \cap B_0 = \emptyset$ since B_0 is closed in $\tilde{B}$ and nowhere locally compact; hence $V \subset B_1$, contradicting nowhere local compactness of B_1. Hence $\tilde{B} \approx \boldsymbol{Q} \times C$ by Theorem 2.2(d). By Corollary 3.4, there exists a homeomorphism $h\colon P \to G$ such that $h[\tilde{B}] = B$. Then $K = h[K_0]$ is as required. ■

5. Borel sets which are not strongly σ-complete. We will now prove our main theorems; the argument we use is much like that of Theorem 4.2. However, the situation is more complicated now. Although we can build $\boldsymbol{Q} \times \boldsymbol{P}$ inductively, we can not apply the same method as in Theorem 4.3 to obtain a copy of $\boldsymbol{Q} \times \boldsymbol{P}$ with remainder homeomorphic to T (or S), simply because $\boldsymbol{P}$ is not $\boldsymbol{Q} \times \boldsymbol{P}$ dense homogeneous. We have to make certain in our induction that we define a copy of $\boldsymbol{Q} \times \boldsymbol{P}$ in X with zero-dimensional closure in X.

The proofs of the two main theorems are very similar; hence we will give only one proof (the more complicated one) in full detail, and give a sketch of the other.

5.1. THEOREM. *Let X be compact, and let A be a Borel subset of X which is not the union of a strongly σ-complete and a countable subset. Then A contains a Cantor set K such that $K \cap A \approx S$ and $K \setminus A \approx \boldsymbol{Q} \times \boldsymbol{P}$.*

Proof. Since A is a Borel subset of X, there exists a continuous surjection $\varphi\colon \boldsymbol{P} \to A$. Put $W = \{x \in \boldsymbol{P}$: there exists a neighborhood V_x of x in $\boldsymbol{P}$, a σ-compact subset E_x of X, and a countable subset D_x of A such that $\varphi[V_x] \subset E_x$, and $D_x \cup (E_x \setminus A)$ is σ-compact$\}$. Then W is open in $\boldsymbol{P}$, and there exist countably many open V_i in $\boldsymbol{P}$, σ-compact E_i in X, and countable subsets D_i of A such that $W = \bigcup_{i=1}^{\infty} V_i$, $\varphi[V_i] \subset E_i$, and $D_i \cup (E_i \setminus A)$ is σ-compact for each $i \in N$. Then $E = \bigcup_{i=1}^{\infty} E_i$ is σ-compact, $D = \bigcup_{i=1}^{\infty} D_i$ is a countable subset of A, $D \cup (E \setminus A)$ is σ-compact, and $\varphi[W] \subset E$. Put $F = \boldsymbol{P} \setminus \varphi^{-1}[E \cap A]$. Then F is a G_δ in $\boldsymbol{P}$, hence complete. We claim that F is non-empty. Indeed, if $\varphi^{-1}[E \cap A] = \boldsymbol{P}$, then $A \subset E$, hence $X \setminus A = (X \setminus E) \cup (E \setminus A)$ and $(X \setminus A) \cup D = (X \setminus E) \cup (D \cup (E \setminus A))$ is the union of a complete and a σ-compact space. Thus $A \setminus D$ is the intersection of a complete and a σ-compact space, i.e., $A \setminus D$ is strongly σ-complete (Lemma 2.1); so A is the union of a countable and a strongly σ-complete subset, contradicting our hypothesis on A. So $F \neq \emptyset$. Also, if $\emptyset \neq U$ is open in F, say $U = U' \cap F$ where U' is open in $\boldsymbol{P}$, then $\varphi[U'] = \varphi[U] \cup \varphi[U' \setminus U] \subset \overline{\varphi[U]} \cup E$ which is σ-compact; since $U' \not\subset W$, $N \cup ((\overline{\varphi[U]} \cup E) \setminus A)$ is not σ-compact for any countable $N \subset A$. Since D is a countable subset of A, and $D \cup (E \setminus A)$ is σ-compact, we have that $N \cup (\overline{\varphi[U]} \setminus A)$ is not σ-compact for any countable $N \subset A$. Hence $\overline{\varphi[U]} \cap A$ is not the union of a complete and a countable subset. So by Theorem 4.3, $\overline{\varphi[U]}$ contains a Cantor set K such that $K \cap A \approx \boldsymbol{Q} \times C$ and $K \setminus A \approx \boldsymbol{P}$. Recall that M denotes the set of all finite sequences of natural numbers. We will construct Cantor sets K_s in X, open subsets W_s of F, and finite collections $\mathcal{U}_i$ of open subsets of X, for each $s \in M$ and each $i \in N$, such that the following hold:

(1) $K_s \subset \overline{\varphi[W_s]}$;

(2) for each $n \in N$: $\overline{\varphi[W_{s,n}]} \cap K_s = \emptyset$;

(3) for each $n, m \in N$: $\overline{\varphi[W_{s,n}]} \cap \overline{\varphi[W_{s,m}]} = \emptyset$;

(4) for each $n \in N$: $\overline{W}_{s,n} \subset W_s$ (here closure is taken in F);

(5) $\operatorname{diam}(W_s) \leqslant 2^{-|s|}$ (here the diameter is taken w.r.t. a complete metric for F);

(6) $\operatorname{diam}(\varphi[W_s]) \leqslant 2^{-\nu(s)}$;

(7) for each $n \in N$: $d(K_s, K_{s,n}) \leqslant 2^{1-\nu(s,n)}$;

(8) $K_s \cap A \approx \boldsymbol{Q} \times C$ and $K_s \setminus A \approx \boldsymbol{P}$;

(9) $B_i = \bigcup \{K_s\colon |s| \leqslant i\} \approx C$;

(10) $B_i \subset \bigcup \mathcal{U}_i$;

(11) $\bigcup \mathcal{U}_{i+1} \subset \bigcup \mathcal{U}_i$;

(12) $\{\overline{U}\colon U \in \mathcal{U}_i\}$ is pairwise disjoint;

(13) $\operatorname{diam}(U) < 1/i$ for each $U \in \mathcal{U}_i$.

We proceed by induction on $|s|$ and i. First, put $W_\emptyset = F$; then $\overline{\varphi[W_\emptyset]} \cap A$ is not the union of a countable and a complete subset, so it contains a Cantor set $K_\emptyset$ such that $K_\emptyset \cap A \approx \boldsymbol{Q} \times C$ and $K_\emptyset \setminus A \approx \boldsymbol{P}$. Put $\mathcal{U}_1 = \{X\}$, then (1), (5), (6), (8), (9), (10), (12) and (13) are satisfied since our metrics are bounded by 1. Next, suppose that W_s, K_s and $\mathcal{U}_i$ have been defined for $|s| \leqslant k$ and $i \leqslant k$, in accordance with conditions (1)-(13). Fix $s \in M$ with $|s| = k$; note that by (1), $K_s \subset \overline{\varphi[W_s]}$. We will first prove the following:

CLAIM. *$K_s \cup (\overline{\varphi[W_s]} \setminus \varphi[W_s])$ is nowhere dense in $\overline{\varphi[W_s]}$.*

We distinguish two cases.

Case 1. Let $y \in K_s \cap \varphi[W_s]$, say $y = \varphi(x)$ with $x \in W_s$, and let U be an open neighborhood of y in $\overline{\varphi[W_s]}$. By continuity of $\varphi\colon W_s \to \overline{\varphi[W_s]}$, there exists an open neighborhood V of x in W_s such that $\varphi[V] \subset U \cap \varphi[W_s]$. Suppose $U \cap \varphi[W_s] \subset K_s$. Then $\varphi[V] \subset K_s \cap A$ which is σ-compact by (8). Since V is open in W_s, it is open in F, say $V = V' \cap F$ with V' open in $\boldsymbol{P}$; then $\varphi[V'] = \varphi[V] \cup \varphi[V' \setminus V] \subset (K_s \cap A) \cup E$ which is σ-compact, and $D \cup (((K_s \cap A) \cup E) \setminus A) = D \cup (E \setminus A)$ is σ-compact. But $V' \not\subset W$, a contradiction. Hence $(U \cap \varphi[W_s]) \setminus K_s \neq \emptyset$.

Case 2. Let $y \in \overline{\varphi[W_s]} \setminus \varphi[W_s]$, and let U be an open neighborhood of y in $\overline{\varphi[W_s]}$. Then $U \cap \varphi[W_s] \neq \emptyset$, say $z \in U \cap \varphi[W_s]$. If $z \notin K_s$, we are done; if $z \in K_s$, then $z \in K_s \cap \varphi[W_s]$, so $(U \cap \varphi[W_s]) \setminus K_s \neq \emptyset$ by Case 1.

By the claim, we can apply Lemma 2.5 to K_s, $K_s \cup (\overline{\varphi[W_s]} \setminus \varphi[W_s])$ and $\overline{\varphi[W_s]}$ to obtain a countable discrete subset $D_s = \{y_{s,n}: n \in N\}$ of $\varphi[W_s] \setminus K_s$ such that $\bar{D}_s = D_s \cup K_s$ and $d(y_{s,n}, K_s) \leqslant 2^{-\nu(s,n)}$ for each $n \in N$. Of course, we may assume that $D_s \subset \bigcup \mathscr{U}_k$. The sets $W_{s,n}$ $(n \in N)$ satisfying (2)-(6) are now defined exactly as in the proof of Theorem 4.2, with the additional requirement that $\overline{\varphi[W_{s,n}]} \subset \bigcup \mathscr{U}_k$ for each $n \in N$. Since $\overline{\varphi[W_{s,n}]} \cap A$ is not the union of a complete and a countable subset, $\overline{\varphi[W_{s,n}]}$ contains a Cantor set $K_{s,n}$ such that $K_{s,n} \cap A \approx \boldsymbol{Q} \times C$ and $K_{s,n} \setminus A \approx \boldsymbol{P}$; (7) follows easily. As in the proof of Theorem 4.2, we can show that $\bigcup \{K_{s,n}: |s| \leqslant k, n \in N\} = B_{k+1} \approx C$; note that $B_{k+1} \subset \bigcup \mathscr{U}_k$. Let $\mathscr{V}$ be a cover of B_{k+1} by disjoint clopen subsets of B_{k+1} of diameter less than $1/(k+1)$. By normality, there exist open subsets V' of X (for $V \in \mathscr{V}$), which can be taken to have diameter less than $1/(k+1)$, such that $V' \cap B_{k+1} = V$ and $\mathscr{V}' = \{V': V \in \mathscr{V}\}$ is pairwise disjoint. Again by normality, we can shrink $\mathscr{V}'$ to obtain an open cover $\mathscr{U}_{k+1}$ of B_{k+1} satisfying (10)-(13). This completes the induction. Now put $K = (\bigcup_{i=1}^{\infty} B_i)^-$; we claim that K is as required. Clearly, K is a compact space without isolated points. Also, from (10) and (11) it follows that $K \subset \bigcup \{\bar{U}: U \in \mathscr{U}_i\}$ which is a pairwise disjoint closed cover of K by sets of diameter less than $1/i$; hence $\{\bar{U} \cap K: U \in \mathscr{U}_i, i \in N\}$ is a clopen basis for K, so K is zero-dimensional. Thus $K \approx C$ by Theorem 2.2(a). As in the proof of Theorem 4.2, we have that $K \setminus (\bigcup_{i=1}^{\infty} B_i) \subset \varphi[\boldsymbol{P}] = A$, whence $K \setminus A = \bigcup_{i=1}^{\infty} (B_i \setminus A)$. Fix $i \in N$. Clearly $B_i \setminus A$ is closed in $K \setminus A$; since B_i is nowhere dense in B_{i+1} by (1), (6) and (7), and since $B_i \setminus A$ is dense in B_i by (8), also $B_i \setminus A$ is nowhere dense in $B_{i+1} \setminus A$. From (2), (3) and (8) it follows that $B_i \cap A = \bigcup \{K_s \cap A: |s| \leqslant i\}$ is a countable disjoint union of closed copies of $\boldsymbol{Q} \times C$, whence $B_i \cap A$ is also σ-compact, nowhere countable, and nowhere locally compact (see the argument in the proof of Theorem 4.3, showing that $\tilde{B} \approx \boldsymbol{Q} \times C$) and thus homeomorphic to $\boldsymbol{Q} \times C$ by Theorem 2.2(d); hence $B_i \setminus A \approx \boldsymbol{P}$ since $B_i \cap A$ is dense in B_i. So $K \setminus A \approx \boldsymbol{Q} \times \boldsymbol{P}$ by Theorem 2.2(e). It remains to be shown that $K \cap A \approx S$. By Lemma 2.4, $K \cap A$ is a zero-dimensional nowhere σ-compact space which is the union of a σ-compact and a complete subspace. Now suppose U is open in $K \cap A$; then for some $i \in N$, $U \cap B_i = U \cap B_i \cap A \neq \emptyset$. Since $B_i \cap A \approx \boldsymbol{Q} \times C$, $U \cap B_i \cap A \approx \boldsymbol{Q} \times C$; so U contains a closed copy of $\boldsymbol{Q} \times C$. By Theorem 4.2, U is not the union of a countable and a complete subset, so by Theorem 2.3(a), $K \cap A \approx S$. ∎

5.2. Theorem. *Let X be compact, and let A be a Borel subset of X. Then A is the union of a countable and a strongly σ-complete subset if and only if A does not contain a closed copy of S.*

Proof. Suppose A is the union of a countable and a strongly σ-complete subset, and suppose $S' \subset A$ is a closed copy of S. Then $S' = F \cup \bigcup_{i=1}^{\infty} G_i$, where F is countable, and G_i is a closed and complete subset of $\bigcup_{i=1}^{\infty} G_i$. Then $\bar{G}_i \setminus G_i \subset F$, so $\bar{G}_i$ is the union of a complete and a countable subset; the same is true of $\{x\}$, for each $x \in F$. Hence $S' = \bigcup_{i=1}^{\infty} \bar{G}_i \cup \bigcup_{x \in F} \{x\} = \bigcup_{i=1}^{\infty} A_i$, where each A_i is closed in S' and the union of a countable and a complete subset. From the characterization of S it follows that each A_i is nowhere dense in S', contradicting the fact that S is Baire. The other implication follows immediately from Theorem 5.1. ∎

We now come to our last Hurewicz-type theorem.

5.3. Theorem. *Let X be compact, and let A be a Borel subset of X which is not strongly σ-complete. Then X contains a Cantor set K such that $K \cap A \approx T$ and $K \setminus A \approx \boldsymbol{Q} \times \boldsymbol{P}$.*

Proof. Let $\varphi: \boldsymbol{P} \to A$ be a continuous surjection, and, as in [10], put $W = \{x \in \boldsymbol{P}$: there exists a neighborhood V_x of x in $\boldsymbol{P}$, and a σ-compact subset E_x of X such that $\varphi[V_x] \subset E_x$, and $E_x \setminus A$ is σ-compact$\}$. Then $\varphi[W] \subset E$ for some σ-compact subset E of X with the property that $E \setminus A$ is also σ-compact. Put $F = \boldsymbol{P} \setminus \varphi^{-1}[E \cap A]$. If F is empty, then $X \setminus A = (X \setminus E) \cup (E \setminus A)$ is the union of a complete and a σ-compact subspace and hence A is strongly σ-complete by Lemma 2.1, a contradiction. So $F \neq \emptyset$. If $\emptyset \neq U$ is open in F, say $U = U' \cap F$ where U' is open in $\boldsymbol{P}$, then $\varphi[U'] = \varphi[U] \cup \varphi[U' \setminus U] \subset \overline{\varphi[U]} \cup E$ which is σ-compact; since $U' \not\subset W$, and since $E \setminus A$ is σ-compact, we have that $\overline{\varphi[U]} \setminus A$ is not σ-compact whence $\overline{\varphi[U]} \cap A$ is not complete. So by Theorem 4.1, $\overline{\varphi[U]}$ contains a Cantor set K such that $K \cap A \approx \boldsymbol{Q}$ and $K \setminus A \approx \boldsymbol{P}$. Now define K_s, W_s, and $\mathscr{U}_i$ as in the proof of Theorem 5.1 such that conditions (1)-(7) and (9)-(13) are satisfied, as well as:

(8′) $K_s \cap A \approx \boldsymbol{Q}$ and $K_s \setminus A \approx \boldsymbol{P}$.

(The only major difference with Saint Raymond's proof is the addition of the hypotheses (10)-(13).)

Again, we define $K = (\bigcup_{i=1}^{\infty} B_i)^-$, and prove that $K \approx C$, $K \setminus (\bigcup_{i=1}^{\infty} B_i) \subset \varphi[\boldsymbol{P}] = A$, and $K \setminus A \approx \boldsymbol{Q} \times \boldsymbol{P}$. So all that remains to be shown is that $K \cap A \approx T$. By Lemma 2.3, $K \cap A$ is a zero-dimensional nowhere σ-compact, nowhere complete space. Also,

$$K \cap A = (\bigcup_{i=1}^{\infty} B_i)^- \cap A = K \setminus (\bigcup_{i=1}^{\infty} B_i) \cup \bigcup_{i=1}^{\infty} (B_i \cap A).$$

Since $\bigcup_{i=1}^{\infty} B_i$ is σ-compact, $K \setminus (\bigcup_{i=1}^{\infty} B_i)$ is complete; and $B_i \cap A = \bigcup \{K_s \cap A: |s| \leqslant i\}$ is a countable union of copies of $\boldsymbol{Q}$ by (8′), hence $\bigcup_{i=1}^{\infty} (B_i \cap A)$ is countable. So $K \cap A$ is the union of a countable and a complete subset; by Theorem 2.3(b), $K \cap A \approx T$. ∎

5.4. THEOREM. *Let X be compact, and let A be a Borel subset of X. Then A is strongly σ-complete if and only if A does not contain a closed copy of T.*

Proof. Suppose A is strongly σ-complete, and suppose $T' \subset A$ is a closed copy of T. Then $T' = \bigcup_{i=1}^{\infty} A_i$, where each A_i is complete and closed in T'. Since T is nowhere complete, each A_i is nowhere dense in T', contradicting the fact that T is Baire. The other implication follows immediately from Theorem 5.3. ■

6. Topological characterizations of $Q \times S$ and $Q \times T$. Throughout this section, $\mathscr{K}$ denotes the class of all zero-dimensional spaces which are the union of a strongly σ-complete and a σ-compact subspace, and which are nowhere the union of a countable and a strongly σ-complete subspace, and nowhere the union of a complete and a σ-compact subspace. Similarly, $\mathscr{Q}$ denotes the class of all zero-dimensional spaces which are the union of a strongly σ-complete and a countable subspace, and which are nowhere the union of a complete and a countable subspace, and nowhere strongly σ-complete.

The following theorem shows that both $\mathscr{K}$ and $\mathscr{Q}$ are non-empty.

6.1. THEOREM. *$Q \times S \in \mathscr{K}$ and $Q \times T \in \mathscr{Q}$.*

Proof. Write $S = F \cup G$, where F is σ-compact, and G is complete. Then $Q \times S = \bigcup_{q \in Q} (\{q\} \times F) \cup \bigcup_{q \in Q} (\{q\} \times G)$. Since each $\{q\} \times S$ is closed in $Q \times S$, $\bigcup_{q \in Q} (\{q\} \times G)$ is strongly σ-complete, and $\bigcup_{q \in Q} (\{q\} \times F)$ is σ-compact. Let $\emptyset \neq U$ be open in $Q \times S$, and suppose $\emptyset \neq U_1$ and $\emptyset \neq U_2$ are clopen subsets of Q and S, respectively, such that $U_1 \times U_2 \subset U$. Since $U_1 \approx Q$ and $U_2 \approx S$, U contains a closed copy of $Q \times S$, and thus also a closed copy of S. By Theorem 5.2, U is not the union of a strongly σ-complete and a countable subspace; hence $Q \times S$ is nowhere the union of a strongly σ-complete and a countable subset. In particular, $U_1 \times U_2$ is nowhere the union of a countable and a complete subset, and nowhere σ-compact; so if U is the union of a complete and a σ-compact subset, then so is $U_1 \times U_2$, and hence $U_1 \times U_2 \approx S$. But S is Baire, whereas clearly $Q \times S$ is not. So $Q \times S$ is nowhere the union of a complete and a σ-compact subset. Hence $Q \times S \in \mathscr{K}$. The proof that $Q \times T \in \mathscr{Q}$ is similar; we apply Theorem 5.4 instead of Theorem 5.2 to prove that $Q \times T$ is nowhere strongly σ-complete. ■

The results from Section 5 now enable us to show that the above properties completely characterize $Q \times S$ and $Q \times T$.

6.2. THEOREM. *Up to homeomorphism, $\mathscr{K}$ and $\mathscr{Q}$ each contain only one element.*

Our argument heavily relies on the following instance of a theorem from [8]:

6.3. THEOREM. *Let $A \in \{S, T\}$. Then $Q \times A$ is the unique space which can be written as a countable union of closed subspaces A_i ($i \in N$) such that each A_i is homeomorphic to A and a nowhere dense subset of A_{i+1}.*

Proof of Theorem 6.2. Let $X \in \mathscr{K}$, say $X = \bigcup_{i=1}^{\infty} F_i \cup \bigcup_{i=1}^{\infty} G_i$, where each F_i is compact, and each G_i is complete and closed in $\bigcup_{i=1}^{\infty} G_i$. Then $\bar{G}_i \setminus G_i \subset F = \bigcup_{i=1}^{\infty} F_i$, which is σ-compact. Since G_i is a G_δ in $\bar{G}_i$, $\bar{G}_i \setminus G_i$ is an F_σ in F, hence $\bar{G}_i \setminus G_i$ is also σ-compact. Thus $\bar{G}_i$ is the union of a complete and a σ-compact subset; the same is true of each F_i. Hence $X = \bigcup_{i=1}^{\infty} F_i \cup \bigcup_{i=1}^{\infty} \bar{G}_i = \bigcup_{i=1}^{\infty} A_i$, where each A_i is closed in X and the union of a complete and a σ-compact subset. Since X is zero-dimensional, we can find a disjoint clopen cover $\mathscr{D}$ of $X \setminus A$, such that for each $D \in \mathscr{D}$, $\operatorname{diam}(D) < d(D, A_1)$. Since D is not the union of a countable and a strongly σ-complete subspace, it contains by Theorem 5.2 a closed copy $S(D)$ of S. Put $X_1 = A_1 \cup \bigcup_{D \in \mathscr{D}} S(D)$. Clearly, X_1 is closed in X. Write $S(D) = F(D) \cup G(D)$, where $F(D)$ is σ-compact and $G(D)$ is complete. Then $\bigcup_{D \in \mathscr{D}} F(D)$ is σ-compact, and $\bigcup_{D \in \mathscr{D}} G(D)$ is complete since $\mathscr{D}$ is clopen and disjoint. Since A_1 is also the union of a complete and a σ-compact subspace, so is X_1. To show that $X_1 \approx S$, we must prove that it is nowhere σ-compact, and nowhere the union of a complete and countable subspace. Since $S(D) \approx S$ for each $D \in \mathscr{D}$, and since $S(D)$ is closed in X_1, it suffices to show that $U \cap \bigcup_{D \in \mathscr{D}} S(D) \neq \emptyset$ for each non-empty open U in X_1. So suppose to the contrary that some non-empty open subset U of X_1 is contained in A_1, say $U = U' \cap X_1$ where U' is open in X, and let $x \in U$. Let $\varepsilon > 0$ be such that $B(x, \varepsilon) \subset U'$. Since X is nowhere the union of a complete and a σ-compact space, $B(x, \frac{1}{2}\varepsilon) \not\subset A_1$, whence $B(x, \frac{1}{2}\varepsilon) \cap D \neq \emptyset$ for some $D \in \mathscr{D}$. Since $\operatorname{diam}(D) < (D, A_1) < \frac{1}{2}\varepsilon$ for this D, we have $S(D) \subset D \subset B(x, \varepsilon) \subset U'$, and $S(D) \subset U' \cap X_1 = U \subset A_1$ which is impossible. Thus $X_1 \approx S$, and from the above argument it also follows that A_1 is nowhere dense in X_1. Now replace A_1 by $X_1 \cup A_2$ and construct a closed copy X_2 of S in X such that $X_1 \cup A_2$ is nowhere dense in X_2; in general, we can find closed copies X_i of S in X such that X_i is nowhere dense in X_{i+1} and $A_i \subset X_i$, for each $i \in N$. Then $X = \bigcup_{i=1}^{\infty} X_i$, and $X \approx Q \times S$ by Theorem 6.3.

Now suppose $X \in \mathscr{L}$; as in the proof of Theorem 5.2, we can write $X = \bigcup_{i=1}^{\infty} A_i$, where each A_i is closed in X and the union of a complete and a countable subset. Now follow the same line of argument as above; the closed copies $T(D)$ of T in D can be obtained from Theorem 5.4 and the fact that X is nowhere strongly σ-complete. ■

Using the characterizations of Theorem 2.2 and 2.3, it is easily seen that the properties characterizing $Q \times S$ and $Q \times T$ cannot be weakened; also, they show that $Q \times P$, S, T, $Q \times S$ and $Q \times T$ are pairwise non-homeomorphic.

Appendix: A topological characterization of T. Let $\mathscr{S}$ denote the class of all zero-dimensional spaces which are the union of a σ-compact and a complete subspace, and which are nowhere σ-compact, and nowhere the union of a complete and a countable subset. Let $\mathscr{T}$ denote the class of all zero-dimensional spaces which are the union of a countable and a complete subspace, and which are nowhere σ-compact and nowhere complete.

Theorem 2.3 states, that $\mathscr{S}$ and $\mathscr{T}$ each contain only one element, up to homeomorphism. For the class $\mathscr{S}$ a proof of this fact was given in [9]. A proof for the class $\mathscr{T}$ was given by van Douwen ([4]); since our proof for $\mathscr{T}$ is very similar to that for $\mathscr{S}$, we first sketch the proof from [9].

A.1. LEMMA ([9], Corollary 5.2). *Let $S \subset C$ be dense such that $S \in \mathscr{S}$. Then there is a sequence P_i $(i \in N)$ of closed and complete subspaces of $C \setminus S$ such that for all $i \in N$:*

(1) $P_i \subset P_{i+1}$, *and* $\bigcup_{i=1}^{\infty} P_i = C \setminus S$;

(2) P_i *is nowhere dense in* P_{i+1};

(3) $\bar{P}_i \setminus P_i \approx Q \times C$.

A.2. LEMMA ([9], Theorem 2.3). *Let $K, L \subset C$ be dense copies of $Q \times C$, and let $\varepsilon > 0$ be given. Then there is a homeomorphism $\varphi\colon C \to C$ such that $\varphi[K] = L$ and $d(\varphi, \mathrm{id}) < \varepsilon$.*

A.3. THEOREM ([9], Theorem 5.3). *Let $S_1, S_2 \subset C$ be dense such that $S_1, S_2 \in \mathscr{S}$. Then for each $\varepsilon > 0$ there is a homeomorphism $\varphi\colon C \to C$ such that $\varphi[S_1] = S_2$ and $d(\varphi, \mathrm{id}) < \varepsilon$.*

Proof of Theorem A.3. Write $C \setminus S_1 = \bigcup_{i=1}^{\infty} P_i(1)$, $C \setminus S_2 = \bigcup_{i=1}^{\infty} P_i(2)$, with $P_i(1)$, $P_i(2)$ as in Lemma A.1. Using a Bernstein-type argument, define for each $i \in N$ homeomorphisms $h_i, g_i\colon C \to C$, such that $\varphi = \lim_{n\to\infty} g_n^{-1} \circ h_n \circ \ldots \circ g_1^{-1} \circ h_1$ is a homeomorphism and such that for some sequences of natural numbers $1 = m_0 < m_1 < \ldots$ and $n_1 < n_2 < \ldots$ we have:

(1) $g_i^{-1} \circ h_i \circ g_{i-1}^{-1} \circ h_{i-1} \circ \ldots \circ g_1^{-1} \circ h_1[P_{m_{i-1}}(1)] \subset P_{n_i}(2)$;

(2) $g_i^{-1} \circ h_i \circ g_{i-1}^{-1} \circ h_{i-1} \circ \ldots \circ g_1^{-1} \circ h_1[P_{m_i}(1)] \supset P_{n_i}(2)$;

(3) if $k \geqslant i+1$, then $g_k^{-1} \circ h_k | g_i^{-1} \circ h_i \circ \ldots \circ g_1^{-1} \circ h_1[P_{m_{i-1}}(1)] = \mathrm{id}$.

Then $\varphi[C \setminus S_1] = \varphi[C \setminus S_2]$ and hence $S_1 \approx S_2$. Conditions (1) and (2) can be satisfied using Lemma A.2. ■

Now the proof for $\mathscr{T}$ is exactly the same once we replace $\mathscr{S}$ by $\mathscr{T}$ and $Q \times C$ by Q. Lemma A.2 with Q instead of $Q \times C$ can be easily proved using the fact that C is countable dense homogeneous (see e.g. [5], Exercise 4.3H): if $\varepsilon > 0$ is given, we can write C as a finite disjoint union $\bigcup_{i=1}^{n} C_i$, where each C_i is a clopen subset of C of diameter less than ε; then if K, L are two dense copies of Q in C, we can find autohomeomorphisms h_i of C_i such that $h_i[K \cap C_i] = L \cap C_i$. Then $h = \bigcup_{i=1}^{n} h_i$ is an autohomeomorphism of C, $h[K] = L$, and $d(h, \mathrm{id}) < \varepsilon$.

Hence to complete our proof, we only have to prove the analogue to Lemma A.1.

A.4. LEMMA. *Let $T \subset C$ be dense such that $T \in \mathscr{T}$. Then there is a sequence P_i $(i \in N)$ of closed and complete subspaces of $C \setminus T$ such that for all $i \in N$:*

(1) $P_i \subset P_{i+1}$ *and* $\bigcup_{i=1}^{\infty} P_i = C \setminus T$;

(2) P_i *is nowhere dense in* P_{i+1};

(3) $\bar{P}_i \setminus P_i \approx Q$.

Proof. Since $C \setminus T \approx Q \times P$ by Lemma 2.4, there exist complete closed subspaces E_i of $C \setminus T$ such that $C \setminus T = \bigcup_{i=1}^{\infty} E_i$. Fix $i \in N$. Since $\bar{E}_i \setminus E_i = \bar{E}_i \cap T$ is a closed subset of T, it is the union of a complete subset G_i and a countable subset. Hence $\bar{E}_i \setminus G_i$ is σ-compact, say $\bar{E}_i \setminus G_i = \bigcup_{j=1}^{\infty} K(i,j)$, where each $K(i,j)$ is compact, and $K(i,j) \setminus E_i$ is countable. Put $\{M_n\colon n \in N\} = \{K(i,j) \cap E_i\colon i \in N, j \in N\} \setminus \{\emptyset\}$. Then each M_n is closed in $C \setminus T$ and contained in some E_i, hence complete, and $\bar{M}_n \setminus M_n$ is countable for each $n \in N$. Put $R_1 = M_1$. By Lemma 2.5, there exists a countable discrete subset D of $(C \setminus T) \setminus \bar{R}_1$ such that $\bar{D} = D \cup \bar{R}_1$ (note that R_1 is nowhere dense in $C \setminus T$ since $Q \times P$ is nowhere complete). For each $x \in D$, choose a clopen neighborhood V_x of x in $C \setminus T$ such that $\bar{V}_x \cap \bar{R}_1 = \emptyset$, and such that $\mathrm{diam}(V_x) < d(x, \bar{R}_1)$, and $\{V_x\colon x \in D\}$ is pairwise disjoint. Since $C \setminus T$ is nowhere σ-compact, V_x is a Borel subset of $\bar{V}_x$ which is not an F_σ in $\bar{V}_x$; hence by Theorem 4.1 there exists a copy C_x of the Cantor set in $\bar{V}_x$ such that $C_x \cap V_x \approx P$ and $C_x \setminus V_x \approx Q$. Put $K_1 = \bigcup_{x \in D} C_x \cup \bar{R}_1$. We claim that $K_1 \approx C$. Indeed, suppose $y \in \overline{(\bigcup_{x \in D} C_x)} \setminus (\bigcup_{x \in D} C_x)$, say $y = \lim_{n\to\infty} y_n$, $y_n \in C_{x_n}$. Since $y \notin \bigcup_{x \in D} C_x$, and C_x is closed for each $x \in D$, we may assume that $x_n \neq x_m$ if $n \neq m$. By compactness, $(x_n)_n$ has a convergent subsequence $(x_{n(k)})_k$, say $\lim_{k\to\infty} x_{n(k)} = x$. Then since $\bar{D} = D \cup \bar{R}_1$, x must be in $\bar{R}_1$, and since $\mathrm{diam}(V_{x_{n(k)}}) < d(x_{n(k)}, \bar{R}_1)$, also $y_{n(k)} \to x$. Hence $y \in \bar{R}_1 \subset K_1$, so $\bar{K}_1 = K_1$. Since K_1 clearly does not contain any isolated points, $K_1 \approx C$ by Theorem 2.2(a). Now put $P_1 = K_1 \cap (C \setminus T)$. Then P_1 is complete since $K_1 \setminus P_1 = \bigcup_{x \in D} (C_x \setminus V_x) \cup \bar{R}_1 \setminus R_1 \approx Q$ is an F_σ in K_1; also R_1 is nowhere dense in P_1. Now replace R_1 by $R_2 = P_1 \cup M_2$ and construct P_2 likewise. Proceeding in this manner, we obtain complete subsets P_n of $C \setminus T$ containing M_n such that $R_n = P_{n-1} \cup M_n$ is nowhere dense in P_n. Then $\{P_i\colon i \in N\}$ is as required. ■

A.5. THEOREM. *Let $T_1, T_2 \subset C$ be dense such that $T_1, T_2 \in \mathscr{T}$. Then for each $\varepsilon > 0$ there is a homeomorphism $\varphi\colon C \to C$ such that $\varphi[T_1] = T_2$ and $d(\varphi, \mathrm{id}) < \varepsilon$.*

Proof. Same as the proof of Theorem A.3. ■

From Theorem A.5 it not only follows that $\mathscr{T}$ contains only one element (up to homeomorphism), but also that C is T dense homogeneous. By Lemma A.2 and Theorem A.3, C is also $Q \times C$ dense homogeneous and S dense homogeneous.

Hence, in Theorems 4.3, 5.1, and 5.3, the partition of the Cantor set K in a cop of $Q \times C$ and a copy of P (resp. S and $Q \times P$, resp. T and $Q \times P$) is unique. In other words, given a Cantor set $K = K_1 \cup K_2$, where $K_1 \approx Q \times C$, $K_2 \approx P$, and $K_1 \cap K_2 = \emptyset$, then in *any* compact space X containing a Borel set A which is not the union of a complete and a countable subset, K can be embedded in such a way that $K \cap A = K_1$ and $K \setminus A = K_2$; similarly for the cases of Theorems 5.1 and 5.3.

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Errata to the paper

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Page	For	Insert
22^{19}	3. This is a special case of the preceding properties	3. $B_x(v_1, \ldots, v_k) \in \mathscr{C}$ if and only if $P_x(v_1, \ldots, v_k) \in \mathscr{C}$
39_{15}	fulfilling conditions (M), (U), (H) and	fulfilling conditions (M), (U) and