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Resolutions of spaces and proper inverse systems in shape theory

by

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Abstract. It will be shown that the two notions in shape theory, resolutions of spaces in the sense of S. Mardešić and proper inverse systems in our sense, are essentially equivalent.

1. Introduction and statement of results. Let Top be the category of topological spaces and continuous maps, and Pol its full subcategory of polyhedra. Let us denote by HTop and HPol the homotopy category of Top and Pol respectively.

In the pro-homotopy approach to the shape category of topological spaces, which was introduced in our previous paper [10], one assigns to each topological space X an inverse system in HPol which is associated with X in the sense of [10], while in the approaches of Mardešić–Segal [6] and Fox [2], which are concerned with compact Hausdorff spaces and metric spaces respectively, these authors assign to X inverse systems of ANR's for metric spaces in Top with X as their inverse limit. To prove the equivalence of our approach with those of these authors for the respective cases, we have introduced in [10] the notion of proper inverse systems. Here we recall its definition.

Throughout this paper, let X be a topological space and $\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ an inverse system in Top, and let $\{p_\lambda\}: X \to \{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ be a morphism in pro-Top, i.e., $p_\lambda: X \to X_\lambda$ is a continuous map for each λ such that $p_\lambda = p_{\lambda\lambda'} p_{\lambda'}$, for $\lambda \leqslant \lambda'$. Let us denote by N the operation of taking the nerve of a cover.

DEFINITION 1 (Morita [10]). $\{p_\lambda\}$ is called *proper* if condition (P) below is satisfied:

(P) For any $\lambda \in \Lambda$, any normal cover $\mathscr{G}$ of X and any normal cover $\mathscr{H}$ of X_λ, there exist a $\mu \in \Lambda$ with $\lambda \leqslant \mu$ and a normal cover $\mathscr{V}$ of X_μ such that $p_\mu^{-1}(\mathscr{V})$ refines $\mathscr{G}$, $\mathscr{V}$ refines $p_{\lambda\mu}^{-1}(\mathscr{H})$ and $N(\mathscr{V})$ is isomorphic to $N(p_\mu^{-1}(\mathscr{V}))$ by the map $V \mapsto p_\mu^{-1}(V)$ for $V \in \mathscr{V}$.

In [10] this definition was described on the assumption that X is an inverse limit, but this assumption was not used actually in the statement of the definition as well as in the proof of [10, Theorem 1.9] and [13, Theorem 3.1]. Thus, it is proved actually by [10, Theorem 1.9] that if $\{p_\lambda\}$ is proper then the inverse system

$\{X_\lambda, [p_{\lambda\lambda'}], \Lambda\}$ in HTop, which is obtained by the application of the homotopy functor, is associated with X.

Definition 1′ below is a modification of Definition 1, which, however, remains to be equivalent as will be shown later, and is included here for the sake of comparison.

DEFINITION 1′. We shall say that $\{p_\lambda\}$ is *weakly proper* if (P′) is satisfied:

(P′) For any $\lambda \in \Lambda$, any normal cover $\mathscr{G}$ of X and any normal cover $\mathscr{H}$ of X_λ there exist a $\mu \in \Lambda$ with $\lambda \leqslant \mu$ and a normal cover $\mathscr{V}$ of X_μ such that $p_\mu^{-1}(\mathscr{V})$ refines $\mathscr{G}$, $\mathscr{V}$ refines $p_{\lambda\mu}^{-1}(\mathscr{H})$ and $p_\mu^{-1}(V) \neq \emptyset$ for each $V \in \mathscr{V}$.

DEFINITION 2 (Bacon [1]). $\{p_\lambda\}$ is called *complemented* if (B_1) and (B_2) below are satisfied:

(B_1) For any normal cover $\mathscr{G}$ of X there exist a $\lambda \in \Lambda$ and a normal cover $\mathscr{U}$ of X_λ such that $p_\lambda^{-1}(\mathscr{U})$ refines $\mathscr{G}$.

(B_2) For any λ and any open neighborhood U of $p_\lambda(X)$ in X_λ there exists a $\mu \in \Lambda$ with $\lambda \leqslant \mu$ such that $p_{\lambda\mu}(X_\mu) \subset U$.

The condition (B_2) was weakened to (B_2') below by Mardešić [4].

(B_2') For any $\lambda \in \Lambda$ and any open neighborhood U of $\mathrm{Cl}\,p_\lambda(X)$ there exists a $\mu \in \Lambda$ with $\lambda \leqslant \mu$ such that $p_{\lambda\mu}(X_\mu) \subset U$.

Here we make a further modification of (B_2') which is really distinct from (B_2') as will be shown later:

(B_2^*) For any $\lambda \in \Lambda$ and any open neighborhood U of $\mathrm{Cl}\,p_\lambda(X)$ such that $\{U, X_\lambda - \mathrm{Cl}\,p_\lambda(X)\}$ is a normal cover of X_λ, there exists a $\mu \in \Lambda$ with $\lambda \leqslant \mu$ such that $p_{\lambda\mu}(X_\mu) \subset U$.

DEFINITION 2*. We shall say that $\{p_\lambda\}$ is *weakly complemented* if conditions (B_1) and (B_2^*) are satisfied.

Recently Mardešić [4] gave the following definition.

DEFINITION 3 (Mardešić [4]). $\{p_\lambda\}$ is called a *resolution of* X if conditions (R_1) and (R_2) below are satisfied:

(R_1) For any polyhedron P, any open cover $\mathscr{U}$ of P and any continuous map $f\colon X \to P$ there exist a $\lambda \in \Lambda$ and a continuous map $h\colon X_\lambda \to P$ such that f and hp_λ are $\mathscr{U}$-near.

(R_2) For any polyhedron P and any open cover $\mathscr{U}$ of P there exists a normal cover $\mathscr{V}$ of P such that if $\lambda \in \Lambda$ and fp_λ and gp_λ are $\mathscr{V}$-near for continuous maps $f, g\colon X_\lambda \to P$, then there exists a $\mu \in \Lambda$ with $\lambda \leqslant \mu$ such that $fp_{\lambda\mu}$ and $gp_{\lambda\mu}$ are $\mathscr{U}$-near.

Concerning the above notions the following results have been obtained hitherto.

I. If $\{p_\lambda\}$ is complemented, then $\{p_\lambda\}$ is proper (Morita [13, Theorem 3.1]).

II. If $\{p_\lambda\}$ satisfies (B_1) and (B_2'), then $\{p_\lambda\}$ is a resolution, and conversely, if $\{p_\lambda\}$ is a resolution and each X_λ is a normal space, then $\{p_\lambda\}$ satisfies (B_1) and (B_2') (Mardešić [4, Theorems 5 and 6]).

The following theorem, which is the main theorem in this paper, clarifies the interrelation among the above notions and contains the results I and II above as immediate corollaries.

THEOREM 1. *Let* $\{p_\lambda\}\colon X \to \{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ *be a morphism in pro-Top. Then the following conditions are equivalent:*

(a) $\{p_\lambda\}$ *is a resolution of* X.

(b) $\{p_\lambda\}$ *is weakly complemented.*

(c) $\{p_\lambda\}$ *is proper.*

(d) $\{p_\lambda\}$ *is weakly proper.*

Let τ be the Tychonoff functor which is the reflector from Top to its full subcategory of Tychonoff spaces (cf. [11, § 1]); that is, $\tau(X)$ is a Tychonoff space for every topological space X, and there is a natural transformation Φ from the identity functor to τ such that $\Phi(X)\colon X \to \tau(X)$ is a homeomorphism whenever X is Tychonoff; here we write $\Phi(X)$ instead of Φ_X in [11, § 1]. Then we have

THEOREM 2. *If* $\{p_\lambda\}\colon X \to \{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ *is a resolution of* X, *then the morphism* $\{\tau(p_\lambda)\}\colon \tau(X) \to \{\tau(X_\lambda), \tau(p_{\lambda\lambda'}), \Lambda\}$ *is also a resolution of* $\tau(X)$.

Let μ be the covariant functor from the category of Tychonoff spaces and continuous maps to its full subcategory of topologically complete spaces which assigns to each Tychonoff space X the completion of X with respect to the finest uniformity of X. Then Theorem 3 below holds.

THEOREM 3. *If* $\{p_\lambda\}\colon X \to \{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ *is a resolution of* X *and if* X *and each* X_λ *are Tychonoff spaces, then* $\{\mu(p_\lambda)\}\colon \mu(X) \to \{\mu(X_\lambda), \mu(p_{\lambda\lambda'}), \Lambda\}$ *is a resolution of* $\mu(X)$.

The following theorem generalizes Morita [13, Theorem 3.3] and Mardešić [4, Theorem 7] as far as Tychonoff spaces are concerned, and also [13, Theorem 3.4] by Theorem 3 above.

THEOREM 4. *Let* $\{p_\lambda\}\colon X \to \{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ *be a resolution of* X. *If* X *and each* X_λ *are Tychonoff spaces and if* X *is topologically complete, then* $X = \varprojlim\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$.

2. Proof of Theorem 1.

Before proceeding to the proof of Theorem 1 we shall first prove Lemma 1 below.

LEMMA 1. *Let* $\mathscr{U}$ *be a normal cover of a topological space* Y *and let* A *be a subset of* Y. *Then there is a locally finite cover* $\mathscr{W}$ *of* Y *by cozero-sets such that*

1) $\mathscr{W}$ *refines* $\mathscr{U}$,

2) *the nerve of the cover* $\{W \in \mathscr{W} \mid W \cap A \neq \emptyset\}$ *of* $\mathrm{St}(A, \mathscr{W})$ *is isomorphic to the nerve of the cover* $\{W \cap A \mid W \in \mathscr{W}, W \cap A \neq \emptyset\}$ *of* A *by the map* $W \mapsto W \cap A$.

Proof. Since $\mathscr{U}$ is a normal cover of Y, there exist a metric space T, a continuous map $h\colon Y \to T$ and a locally finite open cover $\mathscr{G} = \{G_\alpha \mid \alpha \in \Omega\}$ of T such that

$h^{-1}(\mathscr{G})$ refines $\mathscr{U}$. Let $\Omega' = \{\alpha \in \Omega |\ G_\alpha \cap h(A) \neq \emptyset\}$. Since T is metric, there exist open sets H_α, $\alpha \in \Omega'$, such that

$$(1) \qquad G_\alpha \cap \operatorname{Cl} h(A) \subset H_\alpha \subset G_\alpha \quad \text{for} \quad \alpha \in \Omega',$$

$$(2) \qquad G_\alpha \cap \operatorname{Cl} h(A) = H_\alpha \cap \operatorname{Cl} h(A) \quad \text{for} \quad \alpha \in \Omega',$$

$$(3) \qquad \bigcap \{H_\alpha \cap \operatorname{Cl} h(A) |\ \alpha \in \gamma\} = \emptyset \Rightarrow \bigcap \{H_\alpha |\ \alpha \in \gamma\} = \emptyset$$
whenever γ is a finite subset of Ω'.

Let us put

$$K_\alpha = (T - \operatorname{Cl} h(A)) \cap G_\alpha \quad \text{for} \quad \alpha \in \Omega.$$

Then $\mathscr{H} = \{H_{\alpha'}, K_\alpha |\ \alpha' \in \Omega', \alpha \in \Omega\}$ is a locally finite open cover of T, and hence $\mathscr{W} = h^{-1}(\mathscr{H})$ is a locally finite normal cover of Y by cozero-sets which is a refinement of $\mathscr{U}$. Since we have $\{W \in \mathscr{W} |\ W \cap A \neq \emptyset\} = \{h^{-1}(H_{\alpha'}) |\ \alpha' \in \Omega'\}$, the nerve of $\{h^{-1}(H_{\alpha'}) \cap A |\ \alpha' \in \Omega'\}$ and $\{h^{-1}(H_{\alpha'}) |\ \alpha' \in \Omega'\}$ are isomorphic by virtue of (3). Thus, $\mathscr{W}$ has the properties required by Lemma 1.

Proof of Theorem 1. (a) $\Rightarrow$ (b). The proof of [4, Theorem 6] is available for the present case, because every binary normal cover $\{U_0, U_1\}$ of a topological space Y admits a continuous map $g\colon Y \to I$ (where $I = [0, 1]$) such that $g(y) = 0$ or 1 according as $y \in Y - U_0$ or $Y - U_1$.

(b) $\Rightarrow$ (c). Assume (b). Let $\lambda \in \Lambda$, and let $\mathscr{G}$ and $\mathscr{H}$ be any normal covers of X and X_λ respectively. Then by (B_1) there exist a $\mu \in \Lambda$ with $\lambda \leqslant \mu$ and a normal cover $\mathscr{U}$ of X_μ such that $\mathscr{U}$ refines $p_{\lambda\mu}^{-1}(\mathscr{H})$ and $p_\mu^{-1}(\mathscr{U})$ refines $\mathscr{G}$.

By Lemma 1 there exists a locally finite cover $\mathscr{W}$ of X_μ by cozero-sets such that $\mathscr{W}$ refines $\mathscr{U}$, and that the nerve of the cover $\{W \in \mathscr{W} |\ W \cap p_\mu(X) \neq \emptyset\}$ of $\operatorname{St}(p_\mu(X), \mathscr{W})$ is isomorphic to the nerve of the cover $\{W \cap p_\mu(X) |\ W \in \mathscr{W}, W \cap p_\mu(X) \neq \emptyset\}$ of $p_\mu(X)$ by the correspondence $W \mapsto W \cap p_\mu(X)$.

Since $\{\operatorname{St}(p_\mu(X), \mathscr{W}), X_\mu - \operatorname{Cl} p_\mu(X)\}$ is a normal cover of X_μ, by (B_2^*) there exists a $\nu \in \Lambda$ with $\mu \leqslant \nu$ such that $p_{\mu\nu}(X_\nu) \subset \operatorname{St}(p_\mu(X), \mathscr{W})$. Now, let us put

$$\mathscr{V} = \{p_{\mu\nu}^{-1}(W) |\ W \in \mathscr{W}, W \cap p_\mu(X) \neq \emptyset\}.$$

Then $\mathscr{V}$ is a locally finite cover of X_ν by cozero-sets and hence it is a normal cover of X_ν by [9, Theorem 1.2]. Moreover, the nerve of $\mathscr{V}$ is isomorphic to the nerve of $\{V \cap p_\nu(X) |\ V \in \mathscr{V}\}$ of $p_\nu(X)$ by the correspondence $V \mapsto V \cap p_\nu(X)$, since $p_{\mu\nu}(p_\nu(X)) = p_\mu(X) \subset p_{\mu\nu}(X_\nu) \subset \operatorname{St}(p_\mu(X), \mathscr{W})$. Thus, $\mathscr{V}$ refines $p_{\lambda\nu}^{-1}(\mathscr{H})$, $p_\nu^{-1}(\mathscr{V})$ refines $\mathscr{G}$ and the nerve of $\mathscr{V}$ is isomorphic to the nerve of $p_\nu^{-1}(\mathscr{V})$ under the map p_ν^{-1}. This proves (c).

(c) $\Rightarrow$ (d). Obvious from the definitions.

(d) $\Rightarrow$ (b). Assume (d). Then (B_1) follows immediately from Definition 1′. Let $\lambda \in \Lambda$ and let U be an open neighborhood of $\operatorname{Cl} p_\lambda(X)$ in X_λ such that $\mathscr{H} = \{U, X_\lambda - \operatorname{Cl} p_\lambda(X)\}$ is a normal cover of X_λ. Then by Definition 1′ there exist a $\mu \in \Lambda$ with $\lambda \leqslant \mu$ and a normal cover $\mathscr{V}$ of X_μ such that $\mathscr{V}$ refines $p_{\lambda\mu}^{-1}(\mathscr{H})$ and $V \cap p_\mu(X) \neq \emptyset$ for each $V \in \mathscr{V}$. Since $\mathscr{V}$ refines $p_{\lambda\mu}^{-1}(\mathscr{H})$, we have either $V \subset p_{\lambda\mu}^{-1}(U)$ or $V \subset p_{\lambda\mu}^{-1}(X_\lambda - \operatorname{Cl} p_\lambda(X)) \subset X_\mu - \operatorname{Cl} p_\mu(X)$, and hence we must have $V \subset p_{\lambda\mu}^{-1}(U)$ for each $V \in \mathscr{V}$. Therefore we have $p_{\lambda\mu}(X_\mu) \subset U$. This proves ($B_2^*$) and hence (b) holds.

(b) $\Rightarrow$ (a). Assume (b). To prove (R_1), let P be a polyhedron, $\mathscr{U}$ an open cover of P and $f\colon X \to P$ a continuous map. Let L be a triangulation of P such that the cover $\mathscr{L}$ of P by open stars of vertices of L refines $\mathscr{U}$. Then, since (b) $\Rightarrow$ (c) has been proved above, by (c) there exist a $\lambda \in \Lambda$ and a locally finite normal cover $\mathscr{V}$ of X_λ such that $p_\lambda^{-1}(\mathscr{V})$ refines $f^{-1}(\mathscr{L})$, and that the nerve K of $\mathscr{V}$ is isomorphic to the nerve of $p_\lambda^{-1}(\mathscr{V})$ under the map p_λ^{-1}.

For each vertex V of K, let us choose a vertex $g(V)$ of L so that $p_\lambda^{-1}(V) \subset f^{-1}(\operatorname{St}(g(V), L))$. Then $g\colon K \to L$ is a simplicial map, because, for $V_i \in \mathscr{V}$, $i = 0, 1, \ldots, n$, we have

$$\bigcap \{V_i |\ i = 0, 1, \ldots, n\} \neq \emptyset \Rightarrow \bigcap \{p_\lambda^{-1}(V_i) |\ i = 0, 1, \ldots, n\} \neq \emptyset$$
$$\Rightarrow \bigcap \{f^{-1}(\operatorname{St}(g(V_i), L)) |\ i = 0, 1, \ldots, n\} \neq \emptyset.$$

On the other hand, since $\mathscr{V}$ is a locally finite normal cover of X_λ, there is a continuous map $\varphi\colon X_\lambda \to |K|$ such that $\varphi^{-1}(\operatorname{St}(V, K)) \subset V$ for each $V \in \mathscr{V}$.

Now, let $x \in X$. Then there is a $V_0 \in \mathscr{V}$ such that $p_\lambda(x) \in \varphi^{-1}(\operatorname{St}(V_0, K))$. Since $\varphi^{-1}(\operatorname{St}(V_0, K)) \subset V_0$, we have $x \in p_\lambda^{-1}(V_0) \subset f^{-1}(\operatorname{St}(g(V_0), L))$, and hence $f(x) \in \operatorname{St}(g(V_0), L)$. On the other hand, since $g\colon K \to L$ is a simplicial map, we have $g\varphi p_\lambda(x) \in g(\operatorname{St}(V_0, K)) \subset \operatorname{St}(g(V_0), L)$. Thus, $\{f(x), g\varphi p_\lambda(x)\} \subset \operatorname{St}(g(V_0), L)$. Therefore, if we put $h = g\varphi\colon X_\lambda \to P$, then f and hp_λ are $\mathscr{U}$-near. This proves (R_1).

To prove (R_2), let $\lambda \in \Lambda$ and let $\mathscr{U}$ be any open cover of a polyhedron P. Let $\mathscr{V}$ be a star-refinement of $\mathscr{U}$. Suppose that $f_1 p_\lambda$ and $f_2 p_\lambda$ are $\mathscr{V}$-near for two continuous maps $f_1, f_2\colon X_\lambda \to P$. Let us put $\mathscr{W} = f_1^{-1}(\mathscr{V}) \wedge f_2^{-1}(\mathscr{V})$, i.e., $\mathscr{W} = \{f_1^{-1}(V_1) \cap f_2^{-1}(V_2) |\ V_1, V_2 \in \mathscr{V}\}$. Then $\mathscr{W}$ is a normal cover of X_λ. Since $\{\operatorname{St}(p_\lambda(X), \mathscr{W}), X_\lambda - \operatorname{Cl} p_\lambda(X)\}$ is a normal cover of X_λ, by (B_2^*) there exists a $\mu \in \Lambda$ with $\lambda \leqslant \mu$ such that $p_{\lambda\mu}(X_\mu) \subset \operatorname{St}(p_\lambda(X), \mathscr{W})$.

Now, let $x \in X_\mu$. Then there is a $W_0 \in \mathscr{W}$ such that $p_{\lambda\mu}(x) \in W_0$, $W_0 \cap p_\lambda(X) \neq \emptyset$. Let $p_\lambda(x_0) \in W_0$ for a point $x_0 \in X$. Suppose that $W_0 = f_1^{-1}(V_1) \cap f_2^{-1}(V_2)$ with $V_1, V_2 \in \mathscr{V}$. Since $f_1 p_\lambda$ and $f_2 p_\lambda$ are $\mathscr{V}$-near, there is $V_0 \in \mathscr{V}$ such that $f_i p_\lambda(x_0) \in V_0$ for $i = 1, 2$. Since $f_i p_\lambda(x_0) \in V_i$, we have $V_0 \cap V_i \neq \emptyset$ for $i = 1, 2$. Hence

$$f_i p_{\lambda\mu}(x) \in V_i \subset \operatorname{St}(V_0, \mathscr{V}).$$

This shows that $f_1 p_{\lambda\mu}$ and $f_2 p_{\lambda\mu}$ are $\mathscr{U}$-near. Hence (R_2) holds. Thus the proof of (b) $\Rightarrow$ (a) is completed.

3. Proof of Theorems 2, 3 and 4. Throughout this section let us assume that $\{p_\lambda\}\colon X \to \{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ is a resolution of X; by Theorem 1 we will use this assumption in the form of Definition 1′.

Proof of Theorem 2. It is obvious that $\{\tau(p_\lambda)\}\colon \tau(X) \to \{\tau(X_\lambda), \tau(p_{\lambda\lambda'}), \Lambda\}$ is a morphism in pro-Top. Let $\lambda \in \Lambda$ and let $\mathscr{G}$ and $\mathscr{H}$ be any normal covers of $\tau(X)$ and $\tau(X_\lambda)$ respectively. Then $\Phi(X)^{-1}(\mathscr{G})$ and $\Phi(X_\lambda)^{-1}(\mathscr{H})$ are normal covers.

of X and X_λ respectively (for the notations used here, cf. the introduction). By assumption (cf. also the proof of Theorem 1), there exist a $\nu \in \Lambda$ with $\lambda \leqslant \nu$ and a locally finite cozero-set cover $\mathscr{V}$ of X_ν such that $p_\nu^{-1}(\mathscr{V})$ refines $\Phi(X)^{-1}(\mathscr{G})$, $\mathscr{V}$ refines $p_{\lambda\nu}^{-1}(\Phi(X_\lambda)^{-1}(\mathscr{H}))$ and $p_\nu(X) \cap V \neq \emptyset$ for each $V \in \mathscr{V}$. Then by [11, Lemma 1.1] and [9, Theorem 1.1] there is a normal cover $\mathscr{W}$ of $\tau(X_\nu)$ such that $\mathscr{V} = \Phi(X_\nu)^{-1}(\mathscr{W})$. Then it follows from the naturality that $\tau(p_\nu)^{-1}(\mathscr{W})$ refines $\mathscr{G}$, $\mathscr{W}$ refines $\tau(p_{\lambda\nu})^{-1}(\mathscr{H})$ and $\tau(p_\nu)(\tau(X)) \cap W \neq \emptyset$ for each $W \in \mathscr{W}$. Thus, $\{\tau(p_\lambda)\}: \tau(X) \to \{\tau(X_\lambda), \tau(p_{\lambda\lambda'}), \Lambda\}$ is weakly proper. This proves Theorem 2.

Now, let Y be a Tychonoff space. For any open set G of Y, let us put

$$G^* = \mu(Y) - \mathrm{Cl}_{\mu(Y)}(Y-G)$$

and for any normal cover $\mathscr{G}$ of Y let us put $\mathscr{G}^* = \{G^* | \ G \in \mathscr{G}\}$. Then Lemmas 2 and 3 below are known.

Lemma 2. *For any normal cover $\mathscr{G}$ of Y, $\mathscr{G}^*$ is a normal cover of $\mu(Y)$* (cf. [8, II, Theorem 1]).

Lemma 3. *Any normal cover of $\mu(Y)$ is refined by $\mathscr{G}^*$ for some normal cover $\mathscr{G}$ of Y* (cf. [13, Proof of Theorem 3.4]).

We need one more lemma.

Lemma 4. *Let $f: Y \to Z$ be a continuous map between Tychonoff spaces Y and Z. Then for any open set V of Z and any normal cover $\mathscr{W}$ of Z we have*

$$\mu(f)^{-1}(V^*) \subset [f^{-1}(V)]^* \subset \mu(f)^{-1}(\mathrm{St}(V, \mathscr{W})^*).$$

Proof. Since $Y \cap \mu(f)^{-1}(V^*) = f^{-1}(V)$, the first inclusion relation is obvious. The second holds since we have

$$[f^{-1}(V)]^* \subset \mathrm{Cl}_{\mu(Y)} f^{-1}(V) \subset \mathrm{St}(f^{-1}(V), \mu(f)^{-1}(\mathscr{W}^*))$$
$$\subset \mu(f)^{-1}(\mathrm{St}(V, \mathscr{W}^*)) \subset \mu(f)^{-1}(\mathrm{St}(V, \mathscr{W})^*).$$

We are now ready to prove Theorem 3.

Proof of Theorem 3. Assume that X and each X_λ are Tychonoff spaces. Let $\lambda \in \Lambda$, and let $\mathscr{G}$ and $\mathscr{H}$ be any normal covers of X and X_λ respectively. Let $\mathscr{K}$ be a star-refinement of $\mathscr{H}$. Then, since $\{p_\lambda\}$ is weakly proper, there exist $\nu \in \Lambda$ with $\lambda \leqslant \nu$ and a normal cover $\mathscr{V}$ of X_ν such that $\mathscr{V}$ refines $p_{\lambda\nu}^{-1}(\mathscr{K})$, $p_\nu^{-1}(\mathscr{V})$ refines $\mathscr{G}$, and $p_\nu(X) \cap V \neq \emptyset$ for each $V \in \mathscr{V}$. Then by Lemma 4 $\mathscr{V}^*$ refines $\mu(p_{\lambda\nu})^{-1}(\mathscr{H}^*)$ and $\mu(p_\nu)^{-1}(\mathscr{V}^*)$ refines $\mathscr{G}^*$. Since

$$\mu(p_\nu)(\mu(X)) \cap V^* \supset p_\nu(X) \cap V \neq \emptyset,$$

this completes the proof of Theorem 3 by Lemma 3.

Proof of Theorem 4. Let $\{q_\lambda\}: Y \to \{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ be any morphism in pro-Top. For each point y of Y, let us denote by $C(y)$ the collection of all the sets $p_\lambda^{-1}(U)$, where $\lambda \in \Lambda$ and U ranges over all open neighborhoods of $q_\lambda(y)$.

Let $\mathscr{G}$ be any normal cover of X, and let $p_\lambda^{-1}(U) \in C(y)$. Then

$$\mathscr{H} = \{U, X_\lambda - q_\lambda(y)\}$$

is a normal cover of X_λ since X_λ is Tychonoff. Hence by assumption there exist a $\nu \in \Lambda$ with $\lambda \leqslant \nu$ and a normal cover $\mathscr{V}$ of X_ν such that $p_\nu^{-1}(\mathscr{V})$ refines $\mathscr{G}$, $\mathscr{V}$ refines $p_{\lambda\nu}^{-1}(\mathscr{H})$ and $p_\nu^{-1}(V) \neq \emptyset$ for each $V \in \mathscr{V}$. Let $q_\nu(y) \in V$ for some $V \in \mathscr{V}$. Then $V \subset p_{\lambda\nu}^{-1}(U)$ and hence $p_\lambda^{-1}(U) \supset p_\nu^{-1}(V) \neq \emptyset$. Since $p_\nu^{-1}(V) \in C(y)$ and $p_\nu^{-1}(V) \subset G$ for some $G \in \mathscr{G}$, it follows that $C(y)$ is a Cauchy family with respect to the finest uniformity of X. Since X is complete with respect to this uniformity, the intersection $\bigcap C(y)$ consists of a single point, which we denote by $g(y)$. Then $p_\lambda g = q_\lambda$ for $\lambda \in \Lambda$. Let G be any open neighborhood of $g(y)$. Since $\{G, X - g(y)\}$ is a normal cover of X there is $p_\lambda^{-1}(U) \in C(y)$ such that $g(y) \in p_\lambda^{-1}(U) \subset G$. Hence $y \in g^{-1} p_\lambda^{-1}(U)$ $(= q_\lambda^{-1}(U)) \subset g^{-1}(G)$. Thus, g is continuous. If $h: Y \to X$ is a map such that $p_\lambda h = q_\lambda$ for each $\lambda \in \Lambda$, then $h(y) \in \bigcap \{p_\lambda^{-1} q_\lambda(y) | \ \lambda \in \Lambda\} = g(y)$, and hence $h = g$. Therefore $X = \varprojlim \{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$.

4. Remark. Concerning conditions (B_2') and (B_2^*), we have clearly $(\mathrm{B}_2') \Rightarrow (\mathrm{B}_2^*)$. However, the reverse implication does not hold as will be shown below.

Let X be a closed subset of a Tychonoff space Y which is P-embedded in Y. Let $\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ be an inverse system whose terms X_λ form the totality of all the open neighborhoods U of X such that $\{U, Y-X\}$ is a normal cover of Y and whose bonding maps $p_{\lambda\lambda'}$ are inclusion maps, and let $\{p_\lambda\}: X \to \{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ be a morphism in pro-Top which consists of inclusion maps.

Lemma 5. *Let $\{p_\lambda\}$ be defined as above. Then $\{p_\lambda\}$ satisfies (B_1) and (B_2^*). If there exists a closed subset B of Y such that X and B are disjoint but cannot be separated by open sets, then (B_2') does not hold.*

Proof. Clearly, (B_1) and (B_2^*) are satisfied. Suppose that there is a closed B of Y stated in the lemma. Let us put $U = Y - B$, and let y_0 be any point of B. Since $\{Y - y_0, Y - X\}$ is a normal cover of Y, we have $Y - y_0 = X_\lambda$ for some λ, and $\mathrm{Cl}\, p_\lambda(X) = X \subset U \subset Y - y_0$. Since $\{V, Y-X\}$ is not a normal cover of Y for any open set V of Y such that $X \subset V \subset U$, there is no X_μ with $\lambda \leqslant \mu$ such that $p_{\lambda\mu}(X_\mu) \subset U$. Therefore (B_2') does not hold.

Example. Let $W(\omega_1 + 1)$ be the space of all ordinals less than $\omega_1 + 1$ with the order topology, where ω_1 is the first uncountable ordinal. Let us put

$$Y = W(\omega_1+1) \times W(\omega_1+1) - \{(\omega_1, \omega_1)\},$$
$$X = \{(\alpha, \omega_1) | \ \alpha < \omega_1\}.$$

Then X is closed and P-embedded in Y, and by Lemma 5 $\{p_\lambda\}$ constructed above satisfies (B_1) and (B_2^*), but not (B_2').

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Borel sets in compact spaces: some Hurewicz type theorems

by

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Abstract. Let X be a compact metric space, and let A be a Borel subset of X. We identify two subspaces S and T of the Cantor set, and prove that:

(1) A is not the union of a complete and a countable subset if and only if X contains a Cantor set K such that $K\setminus A\approx \boldsymbol{P}$ and $K\cap A\approx \boldsymbol{Q}\times \boldsymbol{C}$.

(2) A is not strongly σ-complete if and only if X contains a Cantor set K such that $K\setminus A\approx \boldsymbol{Q}\times \boldsymbol{P}$ and $K\cap A\approx T$.

(3) A is not the union of a strongly σ-complete and a countable subset if and only if X contains a Cantor set K such that $K\setminus A\approx \boldsymbol{Q}\times \boldsymbol{P}$ and $K\cap A\approx S$.

As an application, we give topological characterizations of $\boldsymbol{Q}\times S$ and $\boldsymbol{Q}\times T$.

1. Introduction.

All spaces under discussion are separable metric.

In his 1928 paper [6], Hurewicz proved that a Borel subset A of a compact space X is not a G_δ in X (i.e. A is not topologically complete) if and only if there exists a compact subset K of X such that $K\cap A\approx \boldsymbol{Q}$ (the rationals) and $K\setminus A\approx \boldsymbol{P}$ (the irrationals). A theorem of the same type was proved in 1978 by Saint Raymond ([10]): he showed, among others, that a Borel subset A of a compact space X is not the union of an F_σ and a G_δ of X (i.e. A is not the union of a σ-compact and a topologically complete subspace) if and only if there exists a compact subspace K of X such that $K\cap A\approx \boldsymbol{Q}\times \boldsymbol{P}$. However, he did not prove anything concerning $K\setminus A$. In the light of Hurewicz's result, this suggests an obvious question; in this paper, we will answer this question, and prove some more "Hurewicz-type" theorems.

We identify a certain zero-dimensional space T, which can easily be visualized as the remainder of $\boldsymbol{Q}\times \boldsymbol{P}$ in some compactification of $\boldsymbol{Q}\times \boldsymbol{P}$, and we prove that a Borel subset A of a compact space X is not the union of a σ-compact and a topologically complete subspace if and only if there exists a Cantor set K in X such that $K\cap A\approx \boldsymbol{Q}\times \boldsymbol{P}$ and $K\setminus A\approx T$. This theorem can also be stated in a slightly different way. Call a subset Y of a space X *strongly σ-complete* if $Y=\bigcup\{Y_i\colon i\in N\}$, where each Y_i is topologically complete and closed in Y; it is easily seen that a subset Y of a compact space X is strongly σ-complete if and only if Y is the intersection