Resolutions of spaces and proper inverse systems 
in shape theory

by

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Abstract. It will be shown that the two notions in shape theory, resolutions of spaces in the sense of S. Mardešić and proper inverse systems in our sense, are essentially equivalent.

1. Introduction and statement of results. Let Top be the category of topological spaces and continuous maps, and Pol its full subcategory of polyhedra. Let us denote by HTop and HPol the homotopy category of Top and Pol respectively.

In the pro-homotopy approach to the shape category of topological spaces, which was introduced in our previous paper [10], one assigns to each topological space X an inverse system in HPol which is associated with X in the sense of [10], while in the approaches of Mardešić-Segal [6] and Fox [2], which are concerned with compact Hausdorff spaces and metric spaces respectively, these authors assign to X inverse systems of ANR’s for metric spaces in Top with X as their inverse limit. To prove the equivalence of our approach with those of these authors for the respective cases, we have introduced in [10] the notion of proper inverse systems. Here we recall its definition.

Throughout this paper, let X be a topological space and \( \{ X_i, p_{ij}, A \} \) an inverse system in Top, and let \( \{ p_i \} : X \rightarrow \{ X_i, p_{ij}, A \} \) be a morphism in pro-Top, i.e., \( p_i : X \rightarrow X_i \), is a continuous map for each \( i \) such that \( p_i = p_{ij} p_k \), for \( k \leq i \). Let us denote by \( N \) the operation of taking the nerve of a cover.

**Definition 1** (Morita [10]). \( \{ p_i \} \) is called proper if condition (P) below is satisfied:

(P) For any \( \lambda \in A \), any normal cover \( \mathcal{U} \) of \( X \) and any normal cover \( \mathcal{V} \) of \( X_i \), there exist \( \mu \in A \) with \( \lambda \leq \mu \) and a normal cover \( \mathcal{W} \) of \( X_\mu \) such that \( p_{\mu}^{-1}(\mathcal{V}) \) refines \( \mathcal{W} \), \( \mathcal{V} \) refines \( p_{\mu}^{-1}(\mathcal{W}) \) and \( N(\mathcal{V}) \) is isomorphic to \( N(p_{\mu}^{-1}(\mathcal{V})) \) by the map \( V \mapsto p_{\mu}^{-1}(V) \) for \( V \in \mathcal{V} \).

In [10] this definition was described on the assumption that \( X \) is an inverse limit, but this assumption was not used actually in the statement of the definition as well as in the proof of [10, Theorem 1.9] and [13, Theorem 3.1]. Thus, it is proved actually by [10, Theorem 1.9] that if \( \{ p_i \} \) is proper then the inverse system
The following theorem, which is the main theorem in this paper, clarifies the interrelation among the above notions and contains the results I and II above as immediate corollaries.

**THEOREM 1.** Let \( \{ p_{\lambda} \} : X \rightarrow \{ X_{\lambda}, p_{\lambda}, A \} \) be a morphism in \( \text{pro}-\text{Top} \). Then the following conditions are equivalent:

(a) \( \{ p_{\lambda} \} \) is a resolution of \( X \).
(b) \( \{ p_{\lambda} \} \) is weakly complemented.
(c) \( \{ p_{\lambda} \} \) is proper.
(d) \( \{ p_{\lambda} \} \) is weakly proper.

Let \( \tau \) be the Tychonoff functor which is the reflector from \( \text{Top} \) to its full subcategory of Tychonoff spaces (cf. [11, §1]); that is, \( \tau(X) \) is a Tychonoff space for every topological space \( X \), and there is a natural transformation \( \Phi \) from the identity functor to \( \tau \) such that \( \Phi(X) : X \rightarrow \tau(X) \) is a homeomorphism whenever \( X \) is Tychonoff; here we write \( \Phi(X) \) instead of \( \Phi_{X} \) in [11, §1]. Then we have

**THEOREM 2.** If \( \{ p_{\lambda} \} : X \rightarrow \{ X_{\lambda}, p_{\lambda}, A \} \) is a resolution of \( X \), then the morphisms \( \{ \tau(\lambda) \} : \tau(X) \rightarrow \{ \tau(X_{\lambda}), \tau(p_{\lambda}), A \} \) is also a resolution of \( \tau(X) \).

Let \( \mu \) be the covariant functor from the category of Tychonoff spaces and continuous maps to its full subcategory of topologically complete spaces which assigns to each Tychonoff space \( X \) the completion of \( X \) with respect to the finest uniformity of \( X \). Then Theorem 3 below holds.

**THEOREM 3.** If \( \{ p_{\lambda} \} : X \rightarrow \{ X_{\lambda}, p_{\lambda}, A \} \) is a resolution of \( X \) and if \( X \) and each \( X_{\lambda} \) are Tychonoff spaces, then \( \{ \mu(p_{\lambda}) \} : \mu(X) \rightarrow \{ \mu(X_{\lambda}), \mu(p_{\lambda}), A \} \) is a resolution of \( \mu(X) \).

The following theorem generalizes Morita [13, Theorem 3.3] and Mardešić [4, Theorem 7] as far as Tychonoff spaces are concerned, and also [13, Theorem 3.4] by Theorem 3 above.

**THEOREM 4.** Let \( \{ p_{\lambda} \} : X \rightarrow \{ X_{\lambda}, p_{\lambda}, A \} \) be a resolution of \( X \). If \( X \) and each \( X_{\lambda} \) are Tychonoff spaces and if \( X \) is topologically complete, then \( X = \lim \{ X_{\lambda}, p_{\lambda}, A \} \).

**2. Proof of Theorem 1.**

Before proceeding to the proof of Theorem 1 we shall first prove Lemma 1 below.

**LEMMA 1.** Let \( \mathcal{U} \) be a normal cover of a topological space \( Y \) and let \( A \) be a subset of \( Y \). Then there is a locally finite cover \( \mathcal{V} \) of \( Y \) by clopen-sets such that

1) \( \mathcal{V} \) refines \( \mathcal{U} \).
2) the nerve of the cover \( \{ W \in \mathcal{V} : W \cap A \neq \emptyset \} \) of \( \text{St}(A, \mathcal{U}) \) is isomorphic to the nerve of the cover \( \{ W \in \mathcal{V} : W \cap A \neq \emptyset \} \) of \( A \) by the map \( \mathcal{W} \rightarrow W \cap A \).

**Proof.** Since \( \mathcal{U} \) is a normal cover of \( Y \), there exist a metric space \( T \), a continuous map \( h : Y \rightarrow T \) and a locally finite open cover \( \mathcal{V} = \{ V_{\alpha} \cap \alpha \in \Omega \} \) of \( T \) such that
$k^{-1}(\emptyset)$ refines $\mathcal{U}$. Let $\mathcal{O}' = \{ a \in \mathcal{O} \mid G_a \cap h(A) \neq \emptyset \}$. Since $T$ is metric, there exist open sets $H_a$, $a \in \mathcal{O}'$, such that

1. $G_a \cap Clh(A) = H_a \cap Clh(A)$ for $a \in \mathcal{O}'$;
2. $G_a \cap Clh(A) = H_a \cap Clh(A)$ for $a \in \mathcal{O}'$;
3. $\{ H_a \cap Clh(A) \mid a \in \mathcal{O}' \} \neq \emptyset$ whenever $\gamma$ is a finite subset of $\mathcal{O}'$.

Let us put $K_a = (T - Clh(A)) \cap G_a$ for $a \in \mathcal{O}$.

Then $\mathcal{W} = \{ H_a, K_a \mid a \in \mathcal{O}, a \in \mathcal{O}' \}$ is a locally finite open cover of $T$, and hence $\mathcal{W}$ is a locally finite normal cover of $Y$ by corozets-sets which is a refinement of $\mathcal{U}$. Since we have $\{ W \in \mathcal{W} \mid \forall a \neq \emptyset \} = (h^{-1}(H_a) \cap \{ a \in \mathcal{O}' \})$, the nerve of $\{ h^{-1}(H_a) \cap \{ a \in \mathcal{O}' \} \}$ is isomorphic to $\mathcal{W}$, and hence from (3) we know that $\mathcal{W}$ has the properties required by Lemma 1.

Proof of Theorem 1. (a) $\Rightarrow$ (b). The proof of [4, Theorem 6] is available for the present case, because every binary normal cover $\{ U_a, U_b \}$ of a topological space $Y$ admits a continuous map $g: Y \to I$ (where $I = [0, 1]$) such that $g(j) = 0$ or 1 according as $y \in Y - U_a$ or $y \in U_b$.

(b) $\Rightarrow$ (a). Assume (b). Let $\lambda \in A$, and let $\mathcal{F}$ be any open covers of $X$ and $X_a$ respectively, respectively. Then by (B1) there exist a $\mu \in A$ with $\lambda \in \mu$ and a normal cover $\mathcal{V}$ of $X_a$ such that $\mathcal{V}$ refines $p_a^{\mu}(\mathcal{W})$ and $p_a^{\mu}(\mathcal{W})$ refines $\mathcal{F}$.

By Lemma 1 there exists a locally finite cover $\mathcal{F}^{*}$ of $X_a$ by corozets-sets such that $\mathcal{F}^{*}$ refines $\mathcal{F}$, and the nerve of the cover $\{ W \in \mathcal{F}^{*} \mid W \cap p_a(X) \neq \emptyset \}$ of $St(p_a(X), \mathcal{F})$ is isomorphic to the nerve of the cover $\{ W \in \mathcal{W} \mid W \cap p_a(X) \neq \emptyset \}$. By the correspondence $W \mapsto W \cap p_a(X)$.

Since $\{ St(p_a(X), \mathcal{F}) \mid X_a - Clp_a(X) \}$ is a normal cover of $X_a$ by $\mathcal{B}^{*}$, there exists a $\mu \in A$ with $\lambda \in \mu$ such that $p_a(X_a) = St(p_a(X), \mathcal{F})$. Now, let us put $\mathcal{Y} = \{ p_a(X) \mid W \in \mathcal{F}^{*}, W \cap p_a(X) \neq \emptyset \}$. Then, $\mathcal{Y}$ is a locally finite cover of $X_a$ by corozets-sets and hence it is a normal cover of $X_a$ by [9, Theorem 2]. Moreover, the nerve of $\{ V \cap p_a(X) \mid V \in \mathcal{Y} \}$ is isomorphic to the nerve of $\{ V \cap p_a(X) \mid V \in \mathcal{F} \}$ of $p_a(X)$ by the correspondence $V \mapsto V \cap p_a(X)$, since $p_a(p_a(X)) = p_a(p_a(X)) = St(p_a(X), \mathcal{F})$. Then, the nerve of $\mathcal{Y}$ is isomorphic to the nerve of $p_a^{\mu}(\mathcal{W})$ under the map $p_a^{\mu}$.

This proves (c).

(c) $\Rightarrow$ (d). Obvious from the definitions.

(d) $\Rightarrow$ (a). Assume (d). Then (B1) follows immediately from Definition 1.

Let $\lambda \in A$ and let $U$ be an open neighborhood of $Clp_a(X)$ in $X_a$ such that $\mathcal{W} = \{ U, X_a - Clp_a(X) \}$ is a normal cover of $X_a$. Then by Definition 1 there exist a $\mu \in A$ with $\lambda \in \mu$ and a normal cover $\mathcal{V}$ of $X_a$ such that $\mathcal{V}$ refines $p_a^{\mu}(\mathcal{W})$ and $V \cap p_a(X) \neq \emptyset$ for each $V \in \mathcal{V}$. Since $\mathcal{V}$ refines $p_a^{\mu}(\mathcal{W})$, we have either $V \cap p_a^{\mu}(U) \neq \emptyset$ or $V \cap p_a^{\mu}(X_a - Clp_a(X)) = X_a - Clp_a(X)$, and hence we must have $V \cap p_a^{\mu}(U)$ for each $V \in \mathcal{V}$. Therefore we have $p_a(X_a) = U$. This proves (B2) and hence (b) holds.

(b) $\Rightarrow$ (a). Assume (b). To prove (R1), let $p$ be a polyhedron, $\mathcal{W}$ an open cover of $P$ and $f: X \to P$ a continuous map. Let $L$ be a triangulation of $P$ such that the cover $\mathcal{W}$ of $P$ by open stars of vertices of $L$ refines $\mathcal{W}$. Then, since (b) $\Rightarrow$ (c) has been proved above, by (c) we know that there exist an $a \in A$ and a locally finite normal cover $\mathcal{V}$ of $X_a$ such that $p_a^{\mu}(\mathcal{W})$ refines $f^{-1}(\mathcal{W})$, and the nerve $K$ of $\mathcal{V}$ is isomorphic to the nerve of $p_a^{\mu}(\mathcal{W})$ under the map $p_a^{\mu}$.

For each vertex $v$ of $K$, let us choose a vertex $g(v)$ of $L$ so that $p_a^{\mu}(v) = f^{-1}(\{ g(v) \} \times L)$.

On the other hand, since $\mathcal{V}$ is a locally finite normal cover of $X_a$, there is a continuous map $\psi: X_a \to \{ K \}$ such that $\psi^{-1}(St(p_a^{\mu}(X_a), \mathcal{F})) 

Now, let $x \in X_a$ be a vertex of $\mathcal{F}$, $e \in \mathcal{E}$, and $y \in \mathcal{V}$. Then $\mathcal{V} = \{ V \in \mathcal{V} \mid \psi^{-1}(St(p_a^{\mu}(X_a), \mathcal{F})) \subseteq V \}$. Since $\mathcal{V}$ is a continuous map $\psi: X_a \to \{ K \}$, we have $x \in p_a^{\mu}(V_1) \in \mathcal{F}^{-1}(St(g(V_1), L))$. Then, $\psi(x) = g(V_1)$, and $\psi^{-1}(St(g(V_1), L))$. Therefore, we put $h = g: X_a \to L$, and set $h$ as $\mathcal{W}$-near. This proves (R1).

To prove (R2), let $\lambda \in A$ and let $\mathcal{F}$ be any open cover of a polyhedron $P$. Let $\mathcal{V}$ be a star-refinement of $\mathcal{F}$. Assume that $p \in \mathcal{F}$ and $p_2 \in \mathcal{F}$ are $\mathcal{W}$-near for two continuous maps $f, f_2: X_a \to P$. We put $\mathcal{W}$ = $\{ f^{-1}(V) \cap f_2^{-1}(V) \mid V \}$. Then $\mathcal{W}$ is a normal cover of $X_a$. Since $\{ St(p(X), \mathcal{W}) \mid X_a - Clp(X) \}$ is a normal cover of $X_a$, by (B2) there exists a $\mu \in A$ with $\lambda \in \mu$ such that $p_a(X_a) = St(p_a(X), \mathcal{W})$.

Now, let $x \in X_a$. Then there is a $V \in \mathcal{F}$ such that $p_a(x) \in V \in \mathcal{F}$.

Let $p_1(x) \in V_1$, for a point $x_0 \in X$. Suppose that $V_0 \cap f^{-1}(V) \cap f_2^{-1}(V) = V_1$. Then, $V \cap p_a(X) = \emptyset$. Since $X_a$ and $f_2$ are $\mathcal{F}$-near, there is $x_0 \in V_0$ such that $f_2(p(x_0)) = V_0$ for $i = 1, 2$. Then $f(p_1(x_0)) \in V_1$. Then, $V \cap p_a(X) = \emptyset$ for $i = 1, 2$. Hence $\mathcal{W}$ is a normal cover of $X_a$.

This shows that $f_2$ is $\mathcal{W}$-near. Hence (R2) holds. Thus the proof of (b) $\Rightarrow$ (a) is completed.

3. Proof of Theorems 2, 3 and 4. Throughout this section let us assume that $\{ p \}: X_a \to \{ X_a, p_U, A \}$ is a resolution of $X$; by Theorem 1 we will use this assumption in the form of Definition 1.

Proof of Theorem 2. It is obvious that $\{ \psi(x) \}: \{ X_a \} \to \{ (X_a, \mathcal{F}_a), \mathcal{F}_a \}$. A is a morphism in pro-Top. Let $\lambda \in A$ and let $\mathcal{F}$ and $\mathcal{F}_a$ be any normal covers of $\mathcal{V}(X)$ and $\mathcal{V}(X_a)$ respectively. Then $\phi(X)^{-1}(\mathcal{F})$ and $\phi(X_a)^{-1}(\mathcal{F}_a)$ are normal covers...
of $X$ and $X_0$ respectively (for the notations used here, cf. the introduction). By assumption (cf. also the proof of Theorem 1), there exist a $v \in A$ with $\lambda \leq v$ and a locally finite cofin set-cover $\mathfrak{V}$ of $X$, such that $p_0^{-1}(\mathfrak{V})$ refines $\Phi(X)^{-1}(\mathfrak{V})$. Then $p_0^{-1}(\Phi(X)^{-1}(\mathfrak{V}))$ refines $\mathfrak{V}$ of $X_0$ for each $v \in \mathfrak{V}$. Then by [11, Lemma 1], there is a normal cover $\mathfrak{V}$ of $X$, such that $\mathfrak{V}$ refines $\Phi(X)^{-1}(\mathfrak{V})$. Then it follows from the naturality that $\tau(p_0^{-1}(\mathfrak{V}))$ refines $\mathfrak{V}$, and $\mathfrak{W}$ refines $\tau(p_0^{-1}(\mathfrak{V}))$ for each $w \in \mathfrak{W}$. Thus, $\{\tau(p_0^{-1}(\mathfrak{V})) : (X_0, \tau(p_0^{-1}(\mathfrak{V})), \lambda) \}$ is weakly proper. This proves Theorem 2.

Now, let $Y$ be a Tychonoff space. For any open set $G$ of $Y$, let us put

$$G^* = \{\mu(Y) \subseteq \mathfrak{U} \cap Y \subseteq G\}$$

and for any normal cover $\mathfrak{V}$ of $Y$, let us put $\mathfrak{V}^* = \{G^* : G \subseteq \mathfrak{V}\}$. Then Lemmas 2 and 3 below are known.

**Lemma 2.** For any normal cover $\mathfrak{V}$ of $Y$, $\mathfrak{V}^*$ is a normal cover of $\mu(Y)$ (cf. [8, II, Theorem 1]).

**Lemma 3.** Any normal cover of $\mu(Y)$ is refined by $\mathfrak{V}$ for some normal cover $\mathfrak{V}$ of $Y$ (cf. [13, Proof of Theorem 3.4]).

We need only one more lemma.

**Lemma 4.** Let $Y \rightarrow Z$ be a continuous map between Tychonoff spaces $Y$ and $Z$. Then for any open set $V$ of $Z$ and any normal cover $\mathfrak{W}$ of $Z$ we have

$$\mu(f)^{-1}(V) \subseteq \mu(f)^{-1}(V) \subseteq \mu(f)^{-1}(V) \subseteq \mu(f)^{-1}(V).$$

**Proof.** Since $Y \subseteq \mu(Y) \subseteq \mathfrak{U} \subseteq \mathfrak{U} \subseteq \mathfrak{U}$, the first inclusion relation is obvious.

The second holds since we have

$$\mu(f)^{-1}(V) = \mu(f)^{-1}(V) \subseteq \mu(f)^{-1}(V) \subseteq \mu(f)^{-1}(V).$$

We are now ready to prove Theorem 3.

**Proof of Theorem 3.** Assume that $X$ and each $X_0$ are Tychonoff spaces. Let $\lambda \in A$, and let $\mathfrak{W}$ and $\mathfrak{W}'$ be any normal covers of $X$ and $X_0$ respectively. Let $\mathfrak{W}'$ be a star-refinement of $\mathfrak{W}$.

Then, since $\{X_0\}$ is weakly proper, there exist $v \in A$ with $\lambda \leq v$ and a normal cover $\mathfrak{V}$ of $X$, such that $\tau(p_0^{-1}(\mathfrak{V}))$ refines $\mathfrak{W}$, and $\tau(p_0^{-1}(\mathfrak{V})) \subseteq \Phi(X)^{-1}(\mathfrak{V})$ for each $v \in \mathfrak{V}$. Then by Lemma 4, $\mathfrak{V}$ refines $\mu(p_0(\mathfrak{W}))$ and $\mu(p_0^{-1}(\mathfrak{W}))$ refines $\mathfrak{W}$. Since $\mu(p_0(\mathfrak{W})) \subseteq \mathfrak{V}$, this completes the proof of Theorem 3 by Lemma 3.

**Proof of Theorem 4.** Let $\{q_0\}: Y \rightarrow \{X_0, p_0^{-1}(U) \} \subseteq C(Y)$, where $\lambda \in A$, and $\mathcal{U}$ ranges over all open neighborhoods of $\mathfrak{U}$. Then let $\mathfrak{W}$ be any normal cover of $X$, and let $p_0^{-1}(\mathcal{U}) \subseteq C(Y)$. Then

$$\mathfrak{W} = \{U, X_0 \setminus q_0(U)\}.$$

is a normal cover of $X_0$ since $X_0$ is Tychonoff. Hence by assumption there exist $v \in A$ with $\lambda \leq v$ and a normal cover $\mathfrak{V}$ of $X$, such that $p_0^{-1}(\mathfrak{V})$ refines $\mathfrak{V}$, $\mathfrak{V}$ refines $p_0^{-1}(\mathfrak{V})$ refines $\mathfrak{W}$, and $\mathfrak{W} \subseteq \Phi(X)^{-1}(\mathfrak{V})$ for each $\mathfrak{V} \in \mathfrak{W}$. Then $\tau(p_0^{-1}(\mathcal{U}))$ is weakly proper. Since $\mu(p_0^{-1}(\mathcal{U})) \subseteq \Phi(X)^{-1}(\mathfrak{V})$ for each $\mathfrak{V} \in \mathfrak{W}$, it follows that $\Lambda(\mathfrak{V})$ is a Cauchy family with respect to the finest uniformity of $X$. Since $X$ is complete with respect to this uniformity, the intersection $\Lambda(\mathfrak{V})$ consists of a single point, which we denote by $y(\mathfrak{V})$. Then $p_0^{-1}(\mathcal{U}) \subseteq \Phi(X)^{-1}(\mathfrak{V})$ for each $\mathfrak{V} \in \mathfrak{W}$. Hence $\mu(p_0^{-1}(\mathcal{U})) \subseteq \Phi(X)^{-1}(\mathfrak{V})$ for each $\mathfrak{V} \in \mathfrak{W}$.

4. **Remark.** Concerning conditions (B1) and (B2), we have clearly (B1) $\Rightarrow$ (B2). However, the reverse implication does not hold as will be shown below.

Let $X$ be a closed subset of a Tychonoff space $Y$ which is $\mathcal{P}$-embedded in $Y$. Let $\{X_0, p_0^{-1}(U) \} \subseteq C(Y)$ be an inverse system whose totality $X_0$ forms the totality of all the open neighborhoods $U$ of $X$ such that $\{U, Y \setminus X\}$ is a normal cover of $Y$ and whose bonding maps $p_0^{-1}(U)$ are inclusion maps, and let $\{q_0\}: X \rightarrow \{X_0, p_0^{-1}(U) \}$ be a morphism in pro-Top which consists of inclusion maps.

**Lemma 5.** Let $\{p_0\}$ be defined as above. Then $\{p_0\}$ satisfies (B1) and (B2). If there exists a closed subset $B$ of $Y$ such that $X \setminus B$ is disjoint but cannot be separated by open sets, then (B2) does not hold.

**Proof.** Clearly, (B1) and (B2) are satisfied. Suppose that there is a closed $B$ of $Y$ stated in the lemma. Let us put $U = B - B$, and let $y_0$ be any point of $B$. Since $\{B - B, Y \setminus B\}$ is a normal cover of $Y$, we have $\{B - B, Y \setminus B\} \subseteq \{X_0, p_0^{-1}(U) \} \subseteq C(Y)$.

We are now ready to prove Theorem 3.

Then $X = \{X_0, p_0^{-1}(U) \} \subseteq C(Y)$.

**Example.** Let $W(\omega_1 + 1)$ be the space of all ordinals less than $\omega_1 + 1$ with the order topology, where $\omega_1$ is the first uncountable ordinal. Let us put $Y = W(\omega_1 + 1) \times W(\omega_1 + 1) \setminus \{\omega_1, \omega_1\}$, $X = \{\omega_1, \omega_1\} \subseteq \omega_1$.

Then $X$ is closed and $\mathcal{P}$-embedded in $Y$, and by Lemma 5, $\{p_0\}$ constructed above satisfies (B1) and (B2), but not (B2).

**References**


Borel sets in compact spaces: some Hurewicz type theorems

by

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Abstract. Let \( X \) be a compact metric space, and let \( A \) be a Borel subset of \( X \). We identify two subspaces \( S \) and \( T \) of the Cantor set, and prove that:

1. \( A \) is not the union of a complete and a countable subset if and only if \( X \) contains a Cantor set \( K \) such that \( K \cap S \approx \mathbb{Q} \) and \( K \cap T \not\approx \mathbb{Q} \).
2. \( A \) is not strongly \( \alpha \)-complete if and only if \( X \) contains a Cantor set \( K \) such that \( K \cap S \approx \mathbb{Q} \times \mathbb{P} \) and \( K \cap T \not\approx \mathbb{Q} \times \mathbb{P} \).
3. \( A \) is not the union of a strongly \( \alpha \)-complete and a countable subset if and only if \( X \) contains a Cantor set \( K \) such that \( K \cap S \approx \mathbb{Q} \times \mathbb{P} \) and \( K \cap T \approx \mathbb{Q} \times \mathbb{P} \).

As an application, we give topological characterizations of \( \mathbb{Q} \times S \) and \( \mathbb{Q} \times T \).

1. Introduction.

All spaces under discussion are separable metric spaces.

In his 1928 paper [6], Hurewicz proved that a Borel subset \( A \) of a compact space \( X \) is not a \( G_\delta \) in \( X \) (i.e. \( A \) is not topologically complete) if and only if there exists a compact subset \( K \) of \( X \) such that \( K \cap A \not\approx \mathbb{Q} \) (the rationals) and \( K \cap A \not\approx \mathbb{P} \) (the irrationals). A theorem of the same type was proved in 1978 by Saint Raymond ([10]): he showed, among others, that a Borel subset \( A \) of a compact space \( X \) is not the union of any \( G_\delta \) and \( G_\delta \) in \( X \) (i.e. \( A \) is not the union of a \( \sigma \)-compact and a topologically complete subspace) if and only if there exists a compact subspace \( K \) of \( X \) such that \( K \cap A \not\approx \mathbb{Q} \times \mathbb{P} \). However, he did not prove anything concerning \( K \cdot A \).

In the light of Hurewicz's result, this suggests an obvious question; in this paper, we will answer this question, and prove some more "Hurewicz-type" theorems.

We identify a certain zero-dimensional space \( T \), which can easily be visualized as the remainder of \( \mathbb{Q} \times \mathbb{P} \) in some compactification of \( \mathbb{Q} \times \mathbb{P} \), and we prove that a Borel subset \( A \) of a compact space \( X \) is not the union of a \( \sigma \)-compact and a topologically complete subspace if and only if there exists a Cantor set \( K \) in \( X \) such that \( K \cap A \not\approx \mathbb{Q} \times \mathbb{P} \) and \( K \cdot A \not\approx T \). This theorem can also be stated in a slightly different way. Call a subset \( Y \) of a space \( X \) strongly \( \sigma \)-complete if \( Y = \bigcup \{ Y_i : i \in \mathbb{N} \} \), where each \( Y_i \) is topologically complete and closed in \( Y \); it is easily seen that a subset \( Y \) of a compact space \( X \) is strongly \( \sigma \)-complete if and only if \( Y \) is the intersection

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