Extensions of functions defined on product spaces

by

Anna Waśko (Warszawa)

Abstract. A subset A of a space X is P-embedded (M-embedded) in X if every continuous mapping $f: A \to B$ of A into a Banach space B (convex subset B of a Banach space) is extendable over X. A subset A of a space X is π-embedded ($\pi_{\mathscr{Z}}$-embedded where $\mathscr{Z}$ denotes a non-empty class of spaces) if for every completely regular space Z (for every space $Z \in \mathscr{Z}$) and every continuous function $f: A \times Z \to I$ there exists an extension of f over $X \times Z$. It has been proved by M. Starbird and E. Michael that every closed subset of a compact space and every closed subset of a metric space is π-embedded. K. Morita has shown that every locally compact, paracompact and P-embedded subset of a topological space is π-embedded. In connection with these results T. C. Przymusiński raised the following two problems: 1) Is every closed subset of a paracompact p-space π-embedded? 2) Is every Čech-complete paracompact and P-embedded subset of a topological space π-embedded? The first question was also raised by Starbid. In this paper we give an example which provides a negative answer to both questions. We also investigate the relationships between $\pi_{\mathscr{Z}}$-embeddings for various classes of spaces $\mathscr{Z}$ (metric, compact, and paracompact p-spaces) and between $\pi_{\mathscr{Z}}$-embeddings and M-embedding.

§ 0. Introduction. Throughout this paper by a topological space we shall mean a completely regular space and by a function or mapping — a continuous function; by an extension we always mean a continuous extension. Also, all pseudometrics are assumed continuous. The symbols $\mathscr{F}$, $\mathscr{M}$, $\mathscr{C}$, $\mathscr{M} \times \mathscr{C}$ and $\mathscr{P}$ denote the classes of finite spaces, metric spaces, compact spaces, products of a metric and a compact space, and paracompact p-spaces, respectively. By P we will denote the set of all irrational numbers and by Q — the set of all rational numbers in I; and I_Q will denote the Michael line (see [E], Example 5.1.32).

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§ 1. $\pi_{\mathscr{Z}}$-embeddings and π-embedding. Let us recall that a subset A of a space X is P-embedded in X if every pseudometric defined on A is extendable onto X or — equivalently — if every mapping $f: A \to B$ of A into a Banach space B is extendable over X (see [Sh] and [P_1]).

T. C. Przymusiński introduced the notions of $\pi_{\mathscr{Z}}$-embedding and π-embedding. We recall these definitions:

DEFINITION 1.1 [P$_4$]. Let $\mathscr{Z}$ denote a non-empty class of spaces. A subspace A of a space X is *$\pi_{\mathscr{Z}}$-embedded* in X if for every $Z \in \mathscr{Z}$ and for every function $f\colon A \times Z \to I$ there exists an extension $\tilde{f}\colon X \times Z \to I$ of f over $X \times Z$.

DEFINITION 1.2 [P$_3$]. A subspace A of a space X is *π-embedded* in X if it is $\pi_{\mathscr{Z}}$-embedded in X where $\mathscr{Z}$ denotes the class of all spaces.

It is known that $\pi_{\mathscr{M}\cap\mathscr{C}}$-embedding is equivalent to C-embedding ([MH], [P$_1$]), $\pi_{\mathscr{F}}$-embedding is equivalent to C^*-embedding ([P$_4$]) and that $\pi_{\mathscr{C}}$-embedding is equivalent to P-embedding ([P$_1$]). Accordingly, a space X is normal if and only if every closed subset A of X is $\pi_{\mathscr{F}}$-embedded in X, or — equivalently — if and only if every closed subset A of X is $\pi_{\mathscr{M}\cap\mathscr{C}}$-embedded in X. T. E. Gantner in [G] and H. Shapiro in [Sh] have shown that the space X is collectionwise normal if and only if every closed subspace A of X is $\pi_{\mathscr{C}}$-embedded in X. Consequently, it seems interesting to investigate spaces whose every closed subset is π-embedded. Two classes of such spaces were found by M. Starbird and E. Michael, who proved the following theorems (see [S$_1$]).

THEOREM 1.3 (M. Starbird). *Every closed subset of a compact space is π-embedded.* ■

THEOREM 1.4 (E. Michael). *Every closed subset of a metric space is π-embedded.* ■

K. Morita strengthened Theorem 1.3.

THEOREM 1.5 [M$_2$]. *Every locally compact, paracompact and P-embedded subspace of a topological space is π-embedded.* ■

In connection with these results T. C. Przymusiński raised in [P$_3$] the following two problems:

Is every closed subset of a paracompact p-space π-embedded?

Is every Čech-complete, paracompact and P-embedded subset of a topological space π-embedded?

The first problem was also raised by M. Starbird in [S$_1$]. Let us observe that in this problem paracompact p-spaces can be equivalently replaced by products $M \times C$, where M is metric and C is compact. According to Theorem 2.8, the second problem is equivalent to the question whether there exist a paracompact Čech-complete space X and its closed subset F which is not π-embedded in X.

The following theorem, proved by M. Starbird, sheds some light on the first problem.

THEOREM 1.6 [S$_1$]. *Let X be a closed subset of $C \times M$ where C is compact and M is metric. If $M \times Y$ is normal then $X \times Y$ is C^*-embedded in $C \times M \times Y$.* ■

Thus, the first problem is equivalent to the question whether in the above theorem the assumption that $M \times Y$ is a T_4-space can be omitted. The following example shows that this assumption is essential, thus answering the first problem in the negative. The example provides also a negative answer to the second question.

EXAMPLE 1.7. There exists a closed subset X of the space $\beta N \times P$ and a continuous function $\tilde{f}\colon X \times I_Q \to \{0, 1\}$ which is not extendable to a function $F\colon \beta N \times P \times I_Q \to I$. Thus, a closed subset of paracompact p-spaces need not be π-embedded.

Proof. For notational convenience we shall construct our set X in the space $\beta(Q \times N) \times P$, where Q is considered with the discrete topology.

For each $q \in Q \setminus \{O\}$ choose an increasing sequence $\{p_{q,n}\}_{n \in N}$ of irrational numbers converging to q such that the set $\{\langle p_{q,n}, q\rangle\colon q \in Q,\ n \in N\}$ is discrete in $I^2 \setminus \Delta$, where Δ is the diagonal of I^2, and let

$$A = \{\langle q, n, p_{q,n}\rangle\colon q \in Q,\ n \in N\} \subset Q \times N \times P \underset{\text{top}}{=} N \times P$$

and

$$X = \overline{A}^{\beta(Q\times N)\times P} \subset \beta(Q \times N) \times P \underset{\text{top}}{=} \beta N \times P.$$

Then for each $\langle q, n\rangle \in Q \times N$ pick up an open interval $T_{q,n}$ with irrational end-points such that $T_{q,n}$ contains q and does not contain $p_{q,n}$, and the family $\{\{p_{q,n}\} \times T_{q,n}\colon q \in Q,\ n \in N\}$ is discrete in $I^2 \setminus \Delta$. Now we can define the function $f\colon A \times I_Q \to \{0, 1\}$. Let

$$f(q, n, p_{q,n}, t) = \begin{cases} 0, & \text{if } t \notin T_{q,n}, \\ 1, & \text{if } t \in T_{q,n}. \end{cases}$$

One can easily check that f is continuous.

We shall prove that there exists an extension $\tilde{f}\colon X \times I_Q \to \{0, 1\}$ of f over $X \times I_Q$ and there is no extension $F\colon \beta(Q \times N) \times P \times I_Q \to I$ of f over $\beta(Q \times N) \times P \times I_Q$.

At first we shall prove the existence of $\tilde{f}$. Define

$$B_0 = f^{-1}(0) \quad \text{and} \quad B_1 = f^{-1}(1).$$

We have $A \times I_Q = B_0 \cup B_1$. Since

$$X \times I_Q = \overline{A}^{\beta(Q\times N)\times P} \times I_Q = \overline{A \times I_Q}^{\beta(Q\times N)\times P\times I_Q} = \overline{B}_0^{\beta(Q\times N)\times P\times I_Q} \cup \overline{B}_1^{\beta(Q\times N)\times P\times I_Q},$$

it is enough to show that closures of B_0 and B_1 in $\beta(Q \times N) \times P \times I_Q$ are disjoint.

It is obvious that B_0 and B_1 are disjoint open and closed subsets of $A \times I_Q$. Let $\langle z_0, p_0, t_0\rangle \in X \setminus A$ where $z_0 \in \beta(Q \times N) \setminus (Q \times N)$, $p_0 \in P$ and $t_0 \in I_Q$. We are going to find a neighbourhood of $\langle z_0, p_0, t_0\rangle$ whose intersection either with B_0 or with B_1 is empty. There are two possible cases: $t_0 \in P$ or $t_0 \in Q$.

If $t_0 \in P$, put $C_0 = \{\langle q, n\rangle\colon t_0 \notin T_{q,n}\}$ and $C_1 = \{\langle q, n\rangle\colon t_0 \in T_{q,n}\}$. We have $C_0 \cap C_1 = \emptyset$. Consequently $\overline{C}_0^{\beta(Q\times N)} \cap \overline{C}_1^{\beta(Q\times N)} = \emptyset$, and there is a neighbourhood U of z_0 in $\beta(Q \times N)$ such that $U \cap C_0 = \emptyset$ or $U \cap C_1 = \emptyset$. We can assume that $U \cap C_0 = \emptyset$. Then $G = U \times P \times \{t_0\}$ is a neighbourhood of $\langle z_0, p_0, t_0\rangle$ in $\beta(Q \times N) \times P \times I_Q$ and $G \cap B_0 = \emptyset$.

If $t_0 \in Q$, we have $t_0 \neq p_0$ and therefore, by the fact that the family

$$\{\{p_{q,n}\} \times T_{q,n}\colon q \in Q,\ n \in N\}$$

is discrete in $I^2 \setminus \Delta$, there exists a neighbourhood $V \times W$ of $\langle p_0, t_0\rangle$ in $P \times I_Q$ such that the set $C = \{\langle q, n\rangle\colon \{p_{q,n}\} \times T_{q,n} \cap V \times W \neq \emptyset\}$ contains at most one element.

Then $U = (\beta(Q\times N)\setminus C)\times V\times W$ is a neighbourhood of $\langle z_0, p_0, t_0\rangle$ and $U\cap B_1 = \emptyset$.

We have proved that there is an extension $\bar{f}$ of f over $X\times I_Q$.

Now we shall show that there is no extension $F\colon \beta(Q\times N)\times P\times I_Q \to I$ of f over $\beta(Q\times N)\times P\times I_Q$.

Suppose that there is such an extension and let σ be a continuous pseudometric on $P\times I_Q$ defined by the formula

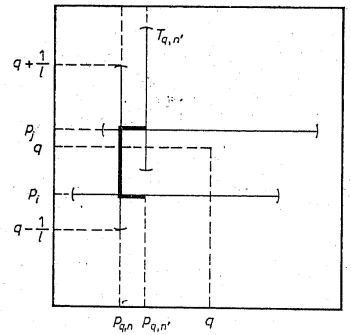
$$\sigma(\langle p, t\rangle, \langle p', t'\rangle) = \sup_{z\in\beta(Q\times N)} |F(z,p,t)-F(z,p',t')|.$$

For each $p\in P$ there exists an $m_p\in N$ such that the σ-diameter of the set

$$(P\cap(p-1/m_p, p+1/m_p))\times\{p\}$$

is less than $\frac{1}{4}$. Therefore, there exist an integer $m\in N$, a rational $q\in Q$ and a sequence $\{p_k\}_{k\in N}$ of irrational numbers such that for every $k\in N$ we have $m_{p_k} = m$ and the sequence $\{p_k\}_{k\in N}$ converges to q.

Let $n\in N$ satisfy $|q-p_{q,n}|<1/2m$. We can assume that for every $k\in N$ we have $\{q, p_{q,n}\}\subset(p_k-1/m,\ p_k+1/m)$. There exists an $l\in N$ such that the σ-diameter of the set $\{p_{q,n}\}\times(q-1/l,\ q+1/l)$ is less than $\frac{1}{4}$ and an $i\in N$ such that $|p_i-q|<1/l$.



Take an $n'>n$ such that $p_i\notin T_{q,n'}$ (such n' exists because the family

$$\{\{p_{q,n}\}\times T_{q,n}\colon\ q\in Q,\ n\in N\}$$

is discrete in $I^2\setminus\Delta$) and a j such that $p_j\in T_{q,n'}\cap(q-1/l,\ q+1/l)$. Thus we have

$$\sigma(\langle p_{q,n'}, p_i\rangle, \langle p_{q,n'}, p_j\rangle)$$
$$\leqslant \sigma(\langle p_{q,n'}, p_i\rangle, \langle p_{q,n}, p_i\rangle)+\sigma(\langle p_{q,n}, p_i\rangle\langle p_{q,n},\ p_j\rangle)+\sigma(\langle p_{q,n}, p_j\rangle, \langle p_{q,n'}, p_j\rangle)<\tfrac{3}{4}<1.$$

On the other hand, we have $f(\langle q, n, p_{q,n'}, p_j\rangle) = 1$ and $f(\langle q, n, p_{q,n'}, p_i\rangle) = 0$; consequently, $\sigma(\langle p_{q,n'}, p_i\rangle, \langle p_{q,n'}, p_j\rangle) = 1$, which yields a contradiction. ∎

Notice that $\beta N\times P$ is not only a paracompact p-space but in fact a Čech-complete Lindelöf space, and that I_Q is a hereditarily paracompact first countable space; thus we have the following corollary.

COROLLARY 1.8. *A closed subset of a Čech-complete Lindelöf space need not be $\pi_{\mathscr{K}}$-embedded, where $\mathscr{K}$ denotes the class of all hereditarily paracompact first countable spaces.* ∎

We have shown that a closed subset of a paracompact p-space need not be π-embedded. This result and Theorems 1.3 and 1.4 lead to the following problem:

PROBLEM 1.9 [P₃]. Characterize those spaces whose every closed subset is π-embedded.

The following problems are also interesting.

PROBLEM 1.10. Characterize those spaces whose every closed subset is $\pi_{\mathscr{P}}$-embedded.

PROBLEM 1.11. Characterize those spaces whose every closed subset is $\pi_{\mathscr{M}}$-embedded.

It is obvious that a closed subset of a paracompact p-space is $\pi_{\mathscr{P}}$-embedded (because the product of two paracompact p-spaces is normal); thus, Example 1.7 shows that $\pi_{\mathscr{P}}$-embedding does not imply π-embedding. Obviously π-embedding implies $\pi_{\mathscr{P}}$-embedding. More generally, if $\mathscr{X}\subset\mathscr{X}'$, then $\pi_{\mathscr{X}'}$-embedding implies $\pi_{\mathscr{X}}$-embedding. For the classes of finite spaces, metric spaces, compact spaces, products of a metric and a compact space, and for paracompact p-spaces we have the following diagram:

$$\begin{tikzcd}[column sep=small, row sep=tiny] & & & & \pi_{\mathscr{C}} \arrow[dr] & & \\ \pi \arrow[r] & \pi_{\mathscr{P}} \arrow[r] & \pi_{\mathscr{M}\times\mathscr{C}} \arrow[r] & \pi_{\mathscr{M}\cup\mathscr{C}} \arrow[ur] \arrow[dr] & & \pi_{\mathscr{M}\cap\mathscr{C}} \arrow[r] & \pi_{\mathscr{F}}. \\ & & & & \pi_{\mathscr{M}} \arrow[ur] & & \end{tikzcd}$$

Most of the arrows in the above diagram cannot be reversed. Some of them can be reversed only for dense subsets. The relationships between various kinds of embeddings for dense subsets are different from those for closed subsets. Existing relationships for dense subsets are described in [P₄]. We are going to discuss them only for closed subsets.

Since $\pi_{\mathscr{F}}$-, $\pi_{\mathscr{M}\cap\mathscr{C}}$- and $\pi_{\mathscr{C}}$-embeddings are equivalent to C^*-, C- and P-embeddings respectively, for closed subsets none of the implications $\pi_{\mathscr{C}}\to\pi_{\mathscr{M}\cap\mathscr{C}}\to\pi_{\mathscr{F}}$ can be reversed. We have already mentioned that the implication $\pi\to\pi_{\mathscr{P}}$ cannot be reversed either. According to Example 2.5 $\pi_{\mathscr{C}}$-embedding does not imply $\pi_{\mathscr{M}}$-embedding. T. C. Przymusiński showed in [P₄] that for dense subsets $\pi_{\mathscr{P}}$-embedding is equivalent to $\pi_{\mathscr{M}\cup\mathscr{C}}$-embedding. In a forthcoming paper we prove, however, that in general $\pi_{\mathscr{M}\times\mathscr{C}}$-embedding does not imply $\pi_{\mathscr{P}}$-embedding we prove also that $\pi_{\mathscr{M}\cup\mathscr{C}}$-embedding implies $\pi_{\mathscr{M}\times\mathscr{C}}$-embedding. Below we prove that for every subset of a topological space $\pi_{\mathscr{M}\cup\mathscr{C}}$-embedding implies some substitute of $\pi_{\mathscr{M}\times\mathscr{C}}$-embedding.

THEOREM 1.12. *If $A\subset X$ is $\pi_{\mathscr{C}}$-embedded and $\pi_{\mathscr{M}}$-embedded in X, M is metric and C compact, and $A\times M$ is normal, then $A\times M\times C$ is C^*-embedded in $X\times M\times C$.*

Proof. Let us first note that it is sufficient to find for a function $f: A\times M\times C\to I$ a function $F: X\times M\times C\to I$ satisfying the inequality $|F(a,m,c)-f(a,m,c)|<\frac{1}{4}$ for every $\langle a,m,c\rangle\in A\times M\times C$. Indeed, the function F yields a $g: X\times M\times C\to[\frac{1}{4},\frac{3}{4}]$ such that $f^{-1}(0)\subset g^{-1}(\frac{1}{4})$ and $f^{-1}(1)\subset g^{-1}(\frac{3}{4})$. Now, using the technique developed in the proof of the Tietze–Uryson theorem (see [E], Theorem 2.1.8), we can show that every function from $A\times M\times C$ to I is extendable over $X\times M\times C$.

Now we are in a position to construct the function F. Let $\mathscr{B}$ be a σ-discrete base in M. Define a pseudometric σ on $A\times M$ by the formula $\sigma(\langle a,m\rangle,\langle a',m'\rangle)=\sup_{z\in C}|f(a,m,z)-f(a',m',z)|$ and take a σ-discrete cover $\mathscr{U}$ of $A\times M$ consisting of balls of diameter less than $\frac{1}{4}$. It follows from [Pa] that the product $A\times M$ is rectangular, thus, there is a σ-locally finite refinement $\mathscr{V}$ of $\mathscr{U}$ consisting of cozero rectangles, i.e., sets of the form $V_1\times V_2$ where V_1 and V_2 are cozero sets in A and M respectively. We can assume that $B\in\mathscr{B}$ for every $V\times B\in\mathscr{V}$. Thus for every $B\in\mathscr{B}$ there exists a σ-locally finite family $\mathscr{V}_B$ consisting of sets functionally open in A such that $\mathscr{V}=\{V\times B: B\in\mathscr{B}, V\in\mathscr{V}_B\}$. Since A is P-embedded in X, for every $B\in\mathscr{B}$ there is a σ-locally finite family $\mathscr{W}_B$ of sets functionally open in X such that $\mathscr{W}_{B|A}\prec\mathscr{V}_B$ and $\bigcup\mathscr{W}_{B|A}=\bigcup\mathscr{V}_B$. Define $\mathscr{W}=\{W\times B: B\in\mathscr{B}, W\in\mathscr{W}_B\}$. Then $\bigcup\mathscr{W}$ is functionally open in $X\times M$ and contains $A\times M$. At the same time $A\times M$ is C-embedded in $X\times M$, and thus there exists a functionally open set G such that $(X\times M)\setminus\bigcup\mathscr{W}\subset G\subset(X\times M)\setminus(A\times M)$. Let $\mathscr{G}=\mathscr{W}\cup\{G\}$. The family $\mathscr{G}$ is a σ-locally finite covering of $X\times M$ consisting of functionally open sets. Moreover, $\mathscr{G}_{|A\times M}$ is a refinement of $\mathscr{U}$.

Let $\{f_s\}_{s\in S}$ be a locally finite partition of unity subordinated to $\mathscr{G}$ and let $S_0=\{s\in S: f_s^{-1}((0,1])\cap(A\times M)\neq\emptyset\}$. For each $s\in S_0$ choose $\langle x_s,y_s\rangle\in f_s^{-1}((0,1])\cap(A\times M)$ and let $g_s: C\to I$ be obtained by letting

$$g_s(z)=\begin{cases} f(x_s,y_s,z), & \text{if } s\in S_0,\\ 0, & \text{if } s\in S\setminus S_0.\end{cases}$$

Define $F: X\times M\times C\to I$ by the formula $F(x,y,z)=\sum_{s\in S}f_s(x,y)\cdot g_s(z)$. Then for every $\langle x,y\rangle\in A\times M$ we have $|F(x,y,z)-f(x,y,z)|<\frac{1}{4}$. ■

Corollary 1.13. *If $A\subset X$ is $\pi_{\mathscr{M}\cup\mathscr{C}}$-embedded in X, M and C are a metric and a compact space respectively, P is a closed subset of $M\times C$, and $A\times M$ is normal, then $A\times P$ is C^*-embedded in $X\times M\times C$.*

Proof. Let $f: A\times P\to I$ be an arbitrary function. Since $A\times M$ is normal, using Theorem 1.6 we can extend f over $A\times M\times C$ and then using Theorem 1.12 extend it further over $X\times M\times C$. ■

Corollary 1.14. *A closed subspace of a perfectly normal space is P-embedded if and only if it is $\pi_{\mathscr{P}}$-embedded.*

Proof. Let A be a closed subset of a perfectly normal space X. Since for every metrizable space M the product $X\times M$ is perfectly normal ([M_1]), A is $\pi_{\mathscr{M}}$-embedded in X. Applying Corollary 1.13, we obtain our corollary. ■

Since a closed subset of a collectionwise normal space is P-embedded ([P_1]), Corollary 1.14 implies the following

Corollary 1.15. *A closed subset of a perfect collectionwise normal space is $\pi_{\mathscr{P}}$-embedded.*

§ 2. Relationship between $\pi_{\mathscr{X}}$-embeddings and M-embedding. Strengthening the notion of P-embedding, L. Sennott introduced the notion of M-embedding.

Definition 2.1 [Se_1]. A subset A of a space X is *M-embedded* in X if every mapping $f: A\to K$ of X into a convex subset K of a Banach space B is extendable over X.

Moreover, she gave characterizations of M-embedding in terms of P-embedding and pseudometrics:

Theorem 2.2 [Se_1]. *For every subset A of a space X the following conditions are equivalent:*

(i) *A is M-embedded in X;*

(ii) *A is P-embedded in X and for every pseudometric ϱ on X there is a functionally closed set F such that*

$$A\subset F\subset\{x\in X: \text{there exists an } a\in A \text{ satisfying } \varrho(x,a)=0\};$$

(iii) *For every pseudometric ϱ on A there exists such an extension $\tilde{\varrho}$ of ϱ over X that $\tilde{\varrho}(x,A)=0$ if and only if there exists an $a\in A$ satisfying $\tilde{\varrho}(x,a)=0$.*

Obviously, M-embedding implies P-embedding; Example 2.4 from [H] shows that M-embedding is not equivalent to P-embedding. It is natural to ask about the place of M-embedding in the diagram on page 31. A partial answer was given in [Se_2].

Theorem 2.3 [Se_2]. *Every closed and $\pi_{\mathscr{M}\cup\mathscr{C}}$-embedded subset of a normal space is M-embedded.* ■

We can strengthen this result to the following

Theorem 2.4. *Every $\pi_{\mathscr{M}\cup\mathscr{C}}$-embedded subset of a topological space is M-embedded.*

Proof. It is enough to show that the theorem holds for dense as well as for closed subsets.

At first we assume that A is a dense subset of a space X. Let ϱ be an arbitrary pseudometric on A and $\tilde{\varrho}$ an extension of ϱ over X. We shall verify that for every $x\in X$ there is an $a\in A$ such that $\tilde{\varrho}(x,a)=0$. Assume the contrary. Then there is an $x_0\in X$ such that for every $a\in A$ we have $\tilde{\varrho}(x_0,a)>0$. Then the function $f: A\to R$ obtained by letting $f(x)=1/\tilde{\varrho}(x_0,a)$ is well defined and has no extension, because for every sequence $\{a_n\}_{n\in N}$ of elements of A converging to x_0 in the $\tilde{\varrho}$-topology we have $f(a_n)\to\infty$. A contradiction.

Notice that the above observation implies that the notions of P-embedding and M-embedding coincide for dense subsets.

Now, let us assume that A is a closed subset of a space X, and let ϱ be a pseudometric on X. We shall find a functionally open set U such that

$$A \subset X \setminus U \subset T = \{x \in X\colon \text{ there exists an } a \in A \text{ satisfying } \varrho(x, a) = 0\}.$$

Let (M, ϱ) be a metric space associated with the pseudometric space $(X \setminus T, \varrho)$ and let $p\colon X \setminus T \to M$ be a natural quotient mapping. Define

$$\Delta = \{\langle x, p(x)\rangle\colon x \in X \setminus T\} \subset X \times M.$$

Obviously Δ is functionally closed in $X \times M$ and disjoint from $A \times M$. Hence, since $A \times M$ is C-embedded in $X \times M$, there exists an $f\colon X \times M \to I$ such that $\Delta = f^{-1}(\{0\})$ and $A \times M \subset f^{-1}(\{1\})$.

Take a σ-discrete base $\{B_s\}_{s \in S}$ in X with ϱ-topology, and let

$$S_0 = \{s \in S\colon B_s \setminus T \neq \emptyset\}.$$

For each $s \in S_0$ choose a $b_s \in B_s \cap (X \setminus T)$. Let $V_s = \{x \in X\colon f(x, p(b_s)) < \frac{1}{2}\}$ and let $U_s = B_s \cap V_s$. Then the sets U_s are functionally open in X and contained in $X \setminus A$. Moreover, the family $\{U_s\}_{s \in S_0}$ is σ-discrete in X. Hence, the union $U = \bigcup_{s \in S_0} U_s$ is functionally open and contained in $X \setminus A$.

It remains to verify that $X \setminus T \subset U$.

Let $x_0 \in X \setminus T$. Then $\langle x_0, p(x_0)\rangle \in \Delta$, $f(\langle x_0, p(x_0)\rangle) = 0$ and there exists an $\varepsilon > 0$ such that $\{x_0\} \times B(p(x_0), \varepsilon) \subset f^{-1}([0, \frac{1}{2}))$. Then $f(\langle x_0, p(b_s)\rangle) < \frac{1}{2}$, and $x_0 \in V_s$. Hence $x_0 \in U_s = B_s \cap V_s$. ■

We can now include M-embedding into our diagram:

$$\begin{tikzcd}[column sep=small] & & & & M \arrow[r] & \pi_{\mathscr{C}} = P \arrow[dr] & \\ \pi \arrow[r] & \pi_{\mathscr{P}} \arrow[r] & \pi_{\mathscr{M}\times\mathscr{C}} \arrow[r] & \pi_{\mathscr{M}\cup\mathscr{C}} \arrow[ur] \arrow[dr] & & & \pi_{\mathscr{M}\cap\mathscr{C}} = C \arrow[r] & \pi_{\mathscr{F}} = C^* \\ & & & & \pi_{\mathscr{M}} \arrow[urr] & & \end{tikzcd}$$

One can ask whether the converse to the above theorem is true. L. Sennott posed this problem in [Se$_2$]. She also proved there that if A is M-embedded in X then $A \times Y$ is M-embedded in $X \times Y$ for any locally compact paracompact space Y. As we shall show below, the assumption that Y is locally compact cannot be omitted, and so the answer to the above question is negative.

EXAMPLE 2.5. There exists a Lindelöf space X containing a closed G_δ-subset which is M-embedded but is not $\pi_{\mathscr{M}}$-embedded in X.

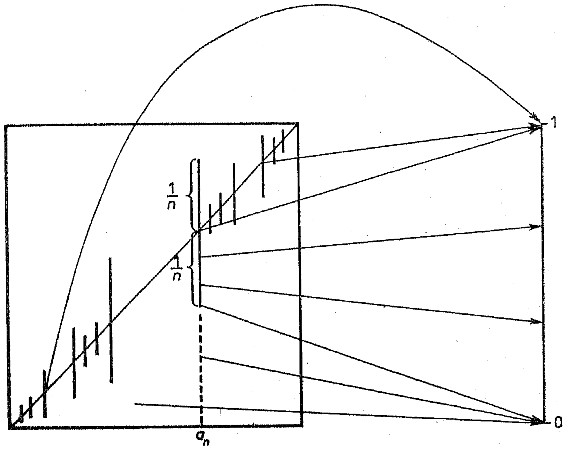
Proof. Let us represent the interval I as the union of two disjoint sets B and D such that $|B| = |D| = \mathfrak{c}$ and every compact space contained either in B or in D is countable (see [K], Chapter III, § 40). Assume that $Q \subset B$. Let $X = I^2$ be the square where points of the form $\langle b, 0\rangle$, for $b \in B$, have neighbourhoods as in the Niemytzki plane (see [E], Example 1.2.4) and all other points have the usual Euclidean neighbourhoods. The subspace $I \times \{0\}$ of X is homeomorphic to the space I_D (see [E], Example 5.1.22), and we shall identify $I \times \{0\}$ with I_D.

It is obvious that I_D is a closed G_δ-set in X. Thus, I_D is M-embedded in X (see [P$_2$]). Now we shall prove that I_D is not $\pi_{\mathscr{M}}$-embedded in X.

Let $\langle a_1, a_1\rangle, \langle a_2, a_2\rangle, \ldots$ denote the end-points of all intervals removed from $\Delta \subset I^2$ in the construction of the Cantor set $\{\langle c, c\rangle \in \Delta\colon c \in C\} \subset I^2$ on Δ.

Let $A = \{a_n\colon n \in N\}$ and define the function $f\colon I_D \times B \to I$, where B has the topology of a subspace of I, by the formula

$$f(\langle x, y\rangle) = \begin{cases} \max\{0, 1 - n|x - y|\}, & \text{if } x = a_n, \\ 0, & \text{if } x \in I \setminus A. \end{cases}$$



Obviously, f is continuous. We shall show that f is not extendable over $X \times B$.

Suppose that $\tilde{f}\colon X \times B \to I$ is an extension of f. We cannot represent the set $B \cap C$ as a countable union of subsets nonwhere dense in C, and so we cannot represent the set $(B \setminus A) \cap C$ as such a union either. Therefore there are an integer $n \in N$ and a set U open in C such that the set

$$\{x \in B \cap U\colon B(\langle x, 1/n\rangle, 1/n) \times (x - 1/n, x + 1/n) \subset \tilde{f}^{-1}([0, \tfrac{1}{2}))\}$$

is dense in U (where $B(\langle x, 1/n\rangle, 1/n)$ is the disc of radius $1/n$ tangent to $I \times \{0\}$ at $\langle x, 0\rangle$).

Take an $a \in A$; every neighbourhood V of the point $\langle\langle a, 0\rangle, a\rangle \in X \times B$ intersects the set $\tilde{f}^{-1}([0, \frac{1}{2}))$ and at the same time we have $\tilde{f}(\langle\langle a, 0\rangle, a\rangle) = f(\langle a, a\rangle) = 1$, which yields a contradiction. ■

It seems interesting to characterize M-embedding for some particular classes of spaces, for example paracompact p-spaces or metric spaces. It turns out that in these cases M-embedding is equivalent to $\pi_{\mathscr{P}}$-embedding and π-embedding, respectively. This fact is a corollary to the theorem that we are now going to prove.

THEOREM 2.6. *For a subset A of X the following conditions are equivalent:*

(i) *A is a paracompact p-space and A is M-embedded in X;*

(ii) *There is a paracompact p-space Z such that $X \subset Z$ and A is closed in Z.*

Moreover, if in (i) we assume in addition that $A = M \times C$ for a metric M and a compact C, then in (ii) we can claim that Z is a product of a metric space M' and a compact space C', where $M = \overline{M} \subset M'$, $C = \overline{C} \subset C'$, and that there is a homeomorphic embedding $h\colon X \to Z = M' \times C'$ such that $h_{|M\times C} = \mathrm{id}_{M'|M} \times \mathrm{id}_{C'|C}$: $M \times C \to M' \times C'$.

Proof. Clearly (ii) implies (i). It remains to prove that (i) implies (ii). We shall first prove the second part of our theorem.

Let $A = M \times C \subset X$ and let A be M-embedded in X. Since A is paracompact, there is no space containing A as a dense and P-embedded subspace (see [E], Problem 8.5.13.b), and A is closed in X. Take a metric σ on M and define a pseudometric ϱ on A by letting $\varrho(\langle m_1, c_1\rangle, \langle m_2, c_2\rangle) = \sigma(m_1, m_2)$. Extend ϱ to such a pseudometric $\tilde{\varrho}\colon X \times X \to R$ that

(*) $\tilde{\varrho}(x, A) = 0$ if and only if there exists an $a \in A$ satisfying $\tilde{\varrho}(x, a) = 0$.

Denote by (M', ϱ') the metric space associated with the pseudometric space $(X, \tilde{\varrho})$ and by $q\colon X \to M'$ the natural quotient mapping. Of course, $A/\tilde{\varrho} = M$ and, by virtue of (*), $M = \overline{M} \subset M'$.

Let R be the equivalence relation on X defined by letting

$x_1 R x_2$ if and only if $x_1 = x_2$ or $x_1 = \langle m_1, c_1\rangle \in A$, $x_2 = \langle m_2, c_2\rangle \in A$ and $c_1 = c_2$.

Let $Y = X/R$. Then we have $A/R = C \subset Y$. We shall show that Y is completely regular.

If $x \in Y \setminus A/R$ and U is a neighbourhood of x in Y then of course there exists a continuous function $f\colon Y \to I$ such that $f(x) = 1$ and $f^{-1}((0, 1]) \subset U$.

Let $y_0 = [c_0] \in A/R \subset Y$ and let $U \subset Y$ be a neighbourhood of y_0 in Y. Then there exist open V, $W \subset C$ such that $c_0 \in V \subset \overline{V} \subset W \subset C$ and $M \times W \subset p^{-1}(U) \cap A$, where $p\colon X \to Y$ is the natural quotient mapping. For each $\langle m, c\rangle \in M \times \overline{V}$ take a functionally open $G_{\langle m,c\rangle}$ such that $G_{\langle m,c\rangle} \subset p^{-1}(U)$. The set $M \times \overline{V}$ is paracompact and M-embedded in X, and thus there exists a locally finite family $\mathcal{G}$ of functionally open in X sets such that the union $\bigcup \mathcal{G}$ is contained in $p^{-1}(U)$ and for each $G \in \mathcal{G}$ one of the sets $G_{\langle m,c\rangle}$ contains G. Therefore we have $M \times V \subset \bigcup \mathcal{G} \subset p^{-1}(U)$, where $M \times \overline{V}$ is M-embedded and $\bigcup \mathcal{G}$ is functionally open in X. Consequently, there is a continuous function $f_1\colon X \to I$ such that $M \times \overline{V} \subset f_1^{-1}(\{1\})$ and $\bigcup \mathcal{G} = f_1^{-1}((0, 1])$. Now, take the function $f_2\colon C \to I$ such that $f_2(c_0) = 1$ and $f_2^{-1}((0, 1]) \subset V$. Define $f_3\colon A \to I$ by the formula $f_3(\langle m, c\rangle) = f_2(c)$ and let $\tilde{f}_3\colon X \to I$ be an extension of f_3 over X. The function $f_4 = \tilde{f}_3 f_1\colon X \to I$ is continuous and constant on the fibres of p. We also have $M \times \{c_0\} \subset f_4^{-1}(\{1\})$ and $f_4^{-1}((0, 1]) \subset \bigcup \mathcal{G} \subset p^{-1}(U)$. The function $f\colon Y \to I$ determined by the equality $f_4 = f \circ p$ is continuous, $f(y_0) = 1$ and $f^{-1}((0, 1]) \subset U$. The proof that Y is completely regular is concluded.

Therefore, there exists a compact space C' such that $Y \subset C'$; hence, we have $A/R = C = \overline{C} \subset Y \subset C'$. We also know that $M = \overline{M} \subset M'$.

Let $h\colon X \to Z = M' \times C'$ be the function obtained by letting $h(x) = \langle q(x), p(x)\rangle$. It is not difficult to verify that h is a homeomorphic embedding. Moreover, $h_{|M\times C} = \mathrm{id}_{M'|M} \times \mathrm{id}_{C'|C}$. We have proved the second part of our theorem.

Now, let $A = \overline{A} \subset M \times C$ be an arbitrary paracompact p-space M-embedded in X. Obviously, A is closed in X. Let $X' = X \cup_{i_A} M \times C$ where $i_A\colon A \to A$ is the identity mapping (see [E], page 127). Then $M \times C$ is closed in X'. We shall show that $M \times C$ is M-embedded in X'.

Let ϱ be a pseudometric on $M \times C$. Denote by ϱ' the restriction of ϱ to A and by $\tilde{\varrho}'$ an extension of ϱ' over X satisfying $\tilde{\varrho}'(x, A) = 0$ if and only if there exists an $a \in A$ such that $\tilde{\varrho}'(x, a) = 0$. For $x, y \in X'$ define

$$\tilde{\varrho}(x, y) = \begin{cases} \varrho(x, y), & \text{if } x, y \in M \times C, \\ \tilde{\varrho}'(x, y), & \text{if } x, y \in X, \\ \inf\limits_{z \in A} \big(\tilde{\varrho}'(x, z) + \varrho(z, y)\big), & \text{if } x \in X \text{ and } y \in M \times C. \end{cases}$$

One can easily check that $\tilde{\varrho}$ is a pseudometric and that it satisfies the condition: $\tilde{\varrho}(x, M \times C) = 0$ if and only if there exists an $\langle m, c\rangle \in M \times C$ such that $\tilde{\varrho}(x, \langle m, c\rangle) = 0$.

We have shown that $M \times C$ is M-embedded in X'. As we have already proved, there is a paracompact space Z such that $X' \subset Z$ and $M \times C = \overline{M \times C} \subset Z$. Since $X \subset X'$ and $A = \overline{A} \subset M \times C$, we have $X \subset Z$ and $A = \overline{A} \subset Z$. The proof is complete. ∎

Remark 2.7. If one does not need the second part of the above theorem, the first part can be proved in a simpler way. The following proof was communicated to the author by T. C. Przymusiński.

Let $f\colon A \to M$ be a perfect mapping onto a metric space M and let $\tilde{M}$ be an absolute retract for metric spaces such that $M = \overline{M} \subset \tilde{M}$ (see [B], Chapter III, Theorem 8.1). There is an extension $\tilde{f}\colon X \to \tilde{M}$ of f over X. Then $h = \tilde{f} \triangle i\colon X \to \tilde{M} \times I^{\tau}$, where $i\colon X \to I^{\tau}$ is an embedding, is also an embedding. Now, let $Z = \tilde{M} \times I^{\tau}$. It suffices to show that $h(A) = \overline{h(A)} \subset Z$. But $h_{|A} = f \triangle i_{|A}\colon A \to M \times I^{\tau}$ and f is perfect, hence $h_{|A}$ is perfect and $h_{|A}(A) = \overline{h_{|A}(A)} \subset M \times I^{\tau}$. At the same time $M \times I^{\tau} = \overline{M \times I^{\tau}} \subset \tilde{M} \times I^{\tau}$, so that $h(A) = h_{|A}(A) = \overline{h_{|A}(A)} \subset \tilde{M} \times I^{\tau}$. ∎

It is known (see [F]) that a topological space A is paracompact and Čech-complete if and only if there exists a perfect mapping of A onto a completely metrizable space. Arguing as in Remark 2.7 and assuming in addition that M is a completely metrizable space and $\tilde{M}$ is a Banach space, we obtain the following theorem:

THEOREM 2.8. *For a subset A of X the following conditions are equivalent:*

(i) *A is a paracompact Čech-complete space and A is P-embedded in X;*

(ii) *There is a paracompact Čech-complete space Z such that $X \subset Z$ and A is closed in Z.* ∎

Since a closed subset of a paracompact p-space X is $\pi_{\mathcal{P}}$-embedded in X, Theorems 2.6 and 2.8 imply the following two corollaries:

COROLLARY 2.9. *A paracompact p-space $A \subset X$ is M-embedded in X if and only if it is $\pi_{\mathscr{P}}$-embedded in X.* ■

COROLLARY 2.10 [Se$_1$]. *A paracompact Čech-complete space $A \subset X$ is P-embedded in X if and only if it is $\pi_{\mathscr{P}}$-embedded in X.* ■

The next corollary is a consequence of Theorem 2.6 and the fact that the subset $M \times C$ of a space $M' \times C'$, where $M' \in \mathscr{M}$, $C' \in \mathscr{C}$, $M = \overline{M} \subset M'$, and $C = \overline{C} \subset C'$, is π-embedded in $M' \times C'$.

COROLLARY 2.11. *The product $M \times C \subset X$ of a metric space M and a compact space C is M-embedded in X if and only if it is π-embedded in X.*

COROLLARY 2.12. *A metric space M is M-embedded in X if and only if it is π-embedded in X.*

We have found classes of spaces for which $\pi_{\mathscr{P}}$-embedding is equivalent to M-embedding and a class of spaces for which π-embedding is equivalent to M-embedding. This leads to the following problems:

PROBLEM 2.13. Characterize spaces for which $\pi_{\mathscr{P}}$-embedding is equivalent to M-embedding.

PROBLEM 2.14. Characterize spaces for which π-embedding is equivalent to M-embedding.

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DEPARTMENT OF MATHEMATICS
AGRICULTURE AND PEDAGOGICAL UNIVERSITY
08-110 Siedlce
ul. Nowotki 19/21
Poland

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