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Uniqueness results for the $ax+b$ group and related algebraic objects

by

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Abstract. The $ax+b$ and related groups have a unique topology in which they are complete separable metric groups. Several other topological algebraic structures have their topology uniquely determined by their algebraic structure.

Let G be the set of all pairs (a, b), where a is a nonzero complex number and b is a complex number. G is a group with the multiplication $(a_1, b_1)\cdot(a_2, b_2) = (a_1a_2, a_1b_2+b_1)$. G of course can be made into a complete separable metric group in a natural manner. Let G_1 be the subgroup of G for which a is positive real and b is real, let G_2 be the subgroup of G for which a is nonzero real and b is real, and let G_3 be the subgroup of G for which a is of modulus one and b is complex. Each G_i is a closed subgroup of G and thus is a complete separable metric group in a natural manner. It is well known that the field of complex numbers has $2^{\aleph_0}$ discontinuous automorphisms, each of which gives rise to a bizarre topology on G. However, this is not the case for the G_i's. For each positive integer $n\geqslant 1$, let K_n be either G_1, G_2, G_3, or the identity, and let $K = \prod_{n\geqslant 1} K_n$. K is a complete separable metric group in the product topology. The purpose of this note is to prove the following theorem.

THEOREM 1. *Let H be a complete separable metric group and let $\psi\colon H\to K$ be an abstract group isomorphism. Then ψ is a topological isomorphism.*

This theorem seems to be new even if there is only one nontrivial factor in K. The only precedent that I am aware of is Tits ([5], Proposition 6.2), who proved an analogue of Theorem 1 for $K = G_3$ and H a second countable Lie group.

Consider first the case for which $K = G_1$. Let A be the set of all elements of G_1 of the form $(a, 0)$, where a is a positive real number, and let B be the set of all elements of G_1 of the form $(1, b)$, where b is a real number. A and B are maximal abelian subgroups of G_1. Hence, $A_1 = \psi^{-1}(A)$ and $B_1 = \psi^{-1}(B)$ are maximal abelian subgroups of H, and so are closed. Note that

$$[(1, x)|\ x>b] = [(1, b)(a, 0)(1, 1)(a, 0)^{-1}|\ (a, 0)\in A].$$

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Hence,

$$\psi^{-1}([(1,x)|\ x>b]) = [\psi^{-1}((1,b))\cdot p\cdot \psi^{-1}((1,1))\cdot p^{-1}|\ p\in A_1],$$

an analytic set in B_1. From this conclude that the inverse image under ψ of any Borel set in B lies in the σ-field generated by the analytic subsets of B_1. Next, apply a standard argument to show that $\psi\colon B_1\to B$ is a topological isomorphism, as follows. Results from Kuratowski [2] imply that $\psi|B_1$ has the Baire property, and so there exists a residual set B' contained in B_1 such that $\psi|B'$ is continuous. It follows that ψ actually is continuous on all of B_1. To see this, let g_n $(n\geqslant 1)$ and g be elements of B_1 such that $g_n\to g$. The union $g^{-1}(B_1-B')$ and $g_n^{-1}(B_1-B')$ $(n\geqslant 1)$ is a set of first category. Hence, there exists an element h in the complement. Then gh is in B' and g_nh is in B' $(n\geqslant 1)$. But $g_nh\to gh$ and so $\psi(g_nh)\to\psi(gh)$. Hence, $\psi(g_n)\to\psi(g)$. Hence, ψ is continuous on B_1. Souslin's theorem says that $\psi^{-1}|B$ is a Borel mapping, and hence $\psi^{-1}|B$ has the Baire property. We conclude as before that $\psi^{-1}|B$ is continuous. This proves that $\psi\colon B_1\to B$ is a topological isomorphism.

Define $\varrho\colon A\to B$ by $\varrho((a,0)) = (a,0)(1,1)(a,0)^{-1} = (1,a)$. ϱ is continuous and one-to-one. Hence, Souslin's theorem implies that ϱ is a Borel isomorphism onto $[(1,x)|\ x>0]$. Next note that $\psi\colon A_1\to A$ is a Borel mapping, for it is the composition of the Borel mappings $p\to p\psi^{-1}((1,1))p^{-1}\to\psi(p)(1,1)\psi(p)^{-1}\to\varrho^{-1}(\psi(p)(1,1)\psi(p)^{-1}) = \psi(p)$, $A_1\to B_1\to B\to A$. Let τ be the Borel isomorphism $(h,g)\to hg$, $B\times A\to K$, and let τ_1 be the inverse to the Borel isomorphism $(h,g)\to hg$, $B_1\times A_1\to H$. Note that $\psi = \tau\circ(\psi|B_1,\psi|A_1)\circ\tau_1$. Hence, ψ is a Borel mapping. Argue as before that ψ then is a topological isomorphism. This proves Theorem 1 for $K = G_1$.

Consider next the case for which $K = G_2$. It is easy to check that G_1 consists of all of the squares of elements of G_2. Hence, $\psi^{-1}(G_1)$ consists of all the squares of elements of H. Hence, $\psi^{-1}(G_1)$ is an analytic subgroup of H. But there are only two cosets of $\psi^{-1}(G_1)$ in H, each of which must be of second category in H. Hence, Banach ([1], Theoreme 1, p. 21) implies that $\psi^{-1}(G_1)$ must be open and closed in H. But $\psi\colon\psi^{-1}(G_1)\to G_1$ is a topological isomorphism. Hence, $\psi\colon H\to G_2$ is a topological isomorphism. This proves Theorem 1 for $K = G_2$.

Consider next the case for which $K = G_3$. Let T be the set of all elements of G_3 of the form $(t,0)$, where t is a complex number of modulus one, and let C be the set of all elements of G_3 of the form $(1,c)$, where c is a complex number. T and C are maximal abelian subgroups of G_3. Hence, $T_1 = \psi^{-1}(T)$ and $C_1 = \psi^{-1}(C)$ are maximal abelian subgroups of H, and so are closed.

Let $r>0$. Note that

$$\begin{aligned}\psi^{-1}([(1,w+z)|\ |w| = |z| = r]) &= \psi^{-1}([(1,rs_1+rs_2)|\ |s_1| = |s_2| = 1])\\ &= \psi^{-1}([(s_1,0)(1,r)(\bar{s}_1,0)+(s_2,0)(1,r)(\bar{s}_2,0)|\ |s_1|\\ &= |s_2| = 1])\\ &= [t_1\psi^{-1}((1,r))t_1^{-1}t_2\psi^{-1}((1,r))t_2^{-1}|\ t_1, t_2 \text{ are in } T_1]\end{aligned}$$

an analytic subset of C_1. But a simple geometric argument shows that $[w+z|\ |w| = |z| = r]$ is the closed disk about 0 of radius $2r$ in the plane. These disks and their translates generate the Borel structure of the plane. Hence, if B is a Borel subset of C, then $\psi^{-1}(B)$ is in the σ-field generated by the analytic subsets of C_1. Hence, as before, $\psi\colon C_1\to C$ is a topological isomorphism. One can now conclude the proof of Theorem 1 for this case, $K = G_3$, in a manner similar to which one concluded Theorem 1 for the case $K = G_1$.

Define $H_n = \psi^{-1}(K_n)$. Note that each K_n is its own double centralizer in K. Hence, each H_n is also its own double centralizer in H, so each H_n is closed in H, and $\psi\colon H_n\to K_n$ is a topological isomorphism. Let U_n be open in K_n $(1\leqslant n\leqslant N)$, and let $L_N = \prod_{n\geqslant N+1} K_n$ be the intersection of the centralizers of the U_n $(1\leqslant n\leqslant N)$. Hence, $\psi^{-1}(L_N)$, the intersection of the centralizers of the $\psi^{-1}(U_n)$ $(1\leqslant n\leqslant N)$, is a closed subgroup of H. Hence,

$$\psi^{-1}(U_1\times\ldots\times U_N\times L_N) = \psi^{-1}(U_1)\ldots\psi^{-1}(U_N)\psi^{-1}(L_N)$$

is an analytic subset of H. Sets of the form $U_1\times\ldots\times U_N\times L_N$ are a basis for the topology of K, and they generate the Borel structure of K. Therefore, if B is a Borel subset of K, $\psi^{-1}(B)$ is in the σ-field generated by the analytic subsets of H. Hence, as before, $\psi\colon H\to K$ is a topological isomorphism. This proves Theorem 1.

On a topic that appears to be somewhat related to Theorem 1, recall that any ring homomorphism of the reals into itself must either be identically 0 or the identity. The proof is easy, for if the homomorphism is not identically 0, it must fix the rationals, and since it takes squares to squares, it must be order preserving, and thus the identity. The following proposition seems to be a bit more difficult to prove, however.

PROPOSITION 2. *Let R be the ring of real numbers, topologized as usual, let H be a topological ring, whose topology is given by a complete separable metric, and let $\psi\colon H\to R$ be an abstract ring isomorphism. Then ψ is a topological isomorphism.*

Proof.

$$\psi^{-1}([x\geqslant b]) = \psi^{-1}([x = b+y^2|\ y\in R]) = [\psi^{-1}(b)+y^2|\ y\in H],$$

which is obviously an analytic subset of H. Hence, the inverse image under ψ of any Borel subset of R lies in the σ-field generated by the analytic subsets of H. One completes the proof of Proposition 2 by the techniques used in the proof of Theorem 1.

With Proposition 2 in hand, one can go on and show that a whole host of algebraic topological structures have their topology uniquely determined by their algebraic structure. The following theorems are just a hint of the sort of results one can prove in the direction.

For each positive integer n, let K_n be a finite dimensional nonassociate algebra with identity over the reals whose center consists of the reals, or let K_n be the algebra consisting of 0. For example, K_n might be the reals, or the real quaternions, or the full ring of $j\times j$ matrices with entries in the reals or real quaternions, or the 8-dimensional split Cayley algebra over the reals. (See Schafer [4] for a clear introduction to these notions.) Let $K=\prod_{n\geqslant 1}K_n$. Define a nonassociative topological ring in the obvious manner. K in a natural manner is a complete separable metric nonassociative topological ring.

THEOREM 3. *Let L be a complete separable metric nonassociative topological ring, and let $\psi\colon L\to K$ be an abstract nonassociative ring isomorphism. Then ψ is a topological isomorphism.*

This theorem seems to be completely new even if K has only one nontrivial factor. The analogous theorem for nonassociative algebras with identity over the complexes is obviously false since the complex numbers have discontinuous automorphisms.

First, suppose that K has only one nontrivial factor, so that K is a finite dimensional nonassociative algebra with identity over the reals. Let R denote the reals. Center(L) is closed in L, and $\psi\colon$ center$(L)\to R\cdot I=R$ is an abstract ring isomorphism. Hence, Proposition 2 implies that ψ is a topological isomorphism onto $R\cdot I$. Let $I, X_1, \ldots, X_n$ be a basis for K over R, and define $I'=\psi^{-1}(I)$, $X_1'=\psi^{-1}(X_1), \ldots, X_n'=\psi^{-1}(X_n)$. The mapping

$$a_0I+a_1X_1+\ldots+a_nX_n\to\psi^{-1}(a_0I+a_1X_1+\ldots+a_nX_n)$$
$$=\psi^{-1}(a_0)I'+\psi^{-1}(a_1)X_1'+\ldots+\psi^{-1}(a_n)X_n',\ \psi^{-1}\colon K\to L,$$

is a Borel isomorphism of complete separable metric (additive) abelian groups, for it is a composition of the Borel isomorphisms

$$a_0I+a_1X_1+\ldots+a_nX_n\to(a_0,a_1,\ldots,a_n)\to(\psi^{-1}(a_0),\psi^{-1}(a_1),\ldots,\psi^{-1}(a_n))$$
$$\to\psi^{-1}(a_0)I'+\psi^{-1}(a_1)X_1'+\ldots+\psi^{-1}(a_n)X_n',\ K\to R^{n+1}\to(\text{center}(L))^{n+1}\to L.$$

The first mapping is a topological isomorphism by the definition of the topology of K, the second mapping is a topological isomorphism by facts we have just observed, and the third is a Borel isomorphism by Souslin's theorem. Techniques used in the proof of Theorem 1 now show that $\psi\colon L\to K$ is a topological isomorphism.

To handle the general case, for each positive integer $n\geqslant 1$, define P_n as follows. If $K_n=(0)$, set $P_n=0$. If $K_n\neq(0)$, let P_n be that element of K which is the identity of K_n in the nth component and 0 element in the other components. For each positive integer $n\geqslant 1$, define $P_n'=\psi^{-1}(P_n)$ and $L_n=LP_n'$. Each L_n is a closed subset of L since P_n' is a central idempotent, and $\psi\colon L_n\to K_n$ is an algebraic, and therefore a topological, isomorphism.

Let $N\geqslant 1$ be a positive integer. Let U_i be open in K_i $(1\leqslant i\leqslant N)$. Each $\psi^{-1}(U_i)$ is open in L_i and hence is a Borel set in L.

$$\prod_{n\geqslant N+1}K_n=[T\text{ in }K|\ TP_i=0,\ 1\leqslant i\leqslant N].$$

Hence,

$$\psi^{-1}\Big(\prod_{n\geqslant N+1}K_n\Big)=[T\text{ in }L|\ TP_i'=0,\ 1\leqslant i\leqslant N],$$

a closed subset of L. $U_1+\ldots+U_N+\prod_{n\geqslant N+1}K_n$ is a basic open set in K, and

$$\psi^{-1}\Big(U_1+\ldots+U_N+\prod_{n\geqslant N+1}K_n\Big)=\psi^{-1}(U_1)+\ldots+\psi^{-1}(U_N)+\psi^{-1}\Big(\prod_{n\geqslant N+1}K_n\Big)$$

is an analytic subset of L. Hence, if B is a Borel subset of K, then $\psi^{-1}(B)$ is in the σ-field generated by the analytic subsets of L. Hence, as before, $\psi\colon L\to K$ is a topological isomorphism. This proves Theorem 3.

It is clear that the finite dimensional nonassociative algebras that occur in Theorem 3 must in general possess an identity. For example, the analogue of Theorem 3 for the Lie algebra of the 3-dimensional Heisenberg group is false. However, the following two theorems exploit special situations to give an analogue of Theorem 3 even when there is no identity.

View R^3 as a nonassociative ring with the standard addition and with multiplication given by the cross product of 3-vectors. Thus, R^3 has the usual generators $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ with the multiplicative relations: $\boldsymbol{i}\times\boldsymbol{j}=\boldsymbol{k}$, $\boldsymbol{j}\times\boldsymbol{k}=\boldsymbol{i}$, $\boldsymbol{k}\times\boldsymbol{i}=\boldsymbol{j}$, $\boldsymbol{i}\times\boldsymbol{i}=0$, etc. For each positive integer $n\geqslant 1$, let K_n be R^3 with this nonassociative ring structure, or let K_n be 0. Let $K=\prod_{n\geqslant 1}K_n$. K in a natural manner is a complete separable metric nonassociative topological ring.

PROPOSITION 4. *Let L be a complete separable metric nonassociative topological ring, and let $\psi\colon L\to K$ be an abstract nonassociative ring isomorphism. Then ψ is a topological isomorphism.*

First, suppose that K has only one nontrivial factor, so that K is the nonassociative ring R^3. Define $\boldsymbol{i}'=\psi^{-1}(\boldsymbol{i})$, $\boldsymbol{j}'=\psi^{-1}(\boldsymbol{j})$, and $\boldsymbol{k}'=\psi^{-1}(\boldsymbol{k})$. Notice that $R\boldsymbol{i}=[v\text{ in }R^3|\ v\times\boldsymbol{i}=0]$. Hence, $L'=\psi^{-1}(R\boldsymbol{i})$ is closed in L, for

$$L'=[v\text{ in }L|\ v\times'\boldsymbol{i}'=0],$$

where $\times'$ denotes the ring multiplication in L. Put a new ring structure on L' compatible with its topology as follows. Leave addition unaltered, and if v, w are in L', define $v\cdot w=(\boldsymbol{k}'\times' v)\times'(w\times'\boldsymbol{j}')$. If $\psi(v)=a\boldsymbol{i}$ and $\psi(w)=b\boldsymbol{i}$, an easy computation shows that $\psi(v\cdot w)=ab\boldsymbol{i}$. Hence, L' with this new ring structure is isomorphic as an abstract ring with the real numbers. Hence, $\psi\colon L'\to R\boldsymbol{i}$ is a topological isomorphism. Similarly, $\psi\colon\psi^{-1}(R\boldsymbol{j})\to R\boldsymbol{j}$ and $\psi\colon\psi^{-1}(R\boldsymbol{k})\to R\boldsymbol{k}$ are topological isomorphisms of complete separable metric (additive) abelian groups.

Souslin's theorem implies that, as a complete separable metric (additive) abelian group, L is Borel isomorphic to $\psi^{-1}(R\boldsymbol{i})\oplus\psi^{-1}(R\boldsymbol{j})\oplus\psi^{-1}(R\boldsymbol{k})$, which in turn is topologically isomorphic, via ψ, to $R\boldsymbol{i}\oplus R\boldsymbol{j}\oplus R\boldsymbol{k}=R^3$. Hence, $\psi\colon L\to K$ is a Borel isomorphism of complete separable metric (additive) abelian groups, and so, as before, ψ is a topological isomorphism.

To handle the general case, note that each K_n is the intersection of the centralizers of all the K_m's, $m \neq n$. Hence, $\psi^{-1}(K_n)$ is the intersection of the centralizers of all the $\psi^{-1}(K_m)$'s, $m \neq n$, and so $\psi^{-1}(K_n)$ is a closed subring in L. Hence, $\psi\colon \psi^{-1}(K_n) \to K_n$ is a topological isomorphism. Note also that $\psi^{-1}(\prod_{n \geq N+1} K_n)$ is the intersection of the centralizers of the $\psi^{-1}(K_i)$ $(1 \leq i \leq N)$, and so is also closed. One now completes the proof of Proposition 4 in the same manner that one completed the proof of Theorem 3.

Finally, let G_1 and G_3 be as in Theorem 1. Recall that the Lie algebra of G_1 is a 2-dimensional vector space over R with a basis X_1, X_2 so that $[X_1, X_2] = X_2$ and $[X_1, X_1] = [X_2, X_2] = 0$. The Lie algebra of G_3 is a 3-dimensional vector space over R with a basis X_1, X_2, and X_3 so that $[X_1, X_2] = X_3$, $[X_1, X_3] = -X_2$, and $[X_2, X_3] = [X_1, X_1] = [X_2, X_2] = [X_3, X_3] = 0$.

For each integer $n \geq 1$, let K_n be either the Lie algebra of G_1, or the Lie algebra of G_3, or 0. Let $K = \prod_{n \geq 1} K_n$. K in a natural manner is a complete separable metric nonassociative topological ring.

PROPOSITION 5. *Let L be a complete separable metric nonassociative topological ring, and let $\psi\colon L \to K$ be an abstract ring isomorphism. Then ψ is a topological isomorphism.*

First, suppose that K is the Lie algebra of G_1. Let $A = RX_1$, $A' = \psi^{-1}(A)$, $B = RX_2$, and $B' = \psi^{-1}(B)$. Notice that

$$A = [X \text{ in } K \mid [X, X_1] = 0] \quad \text{and} \quad B = [X \text{ in } K \mid [X, X_2] = 0].$$

Hence

$$A' = [Y \text{ in } L \mid [Y, \psi^{-1}(X_1)] = 0] \quad \text{and} \quad B' = [Y \text{ in } L \mid [Y, \psi^{-1}(X_2)] = 0]$$

are closed in L. Observe that

$$\begin{aligned}[aX_2 \mid a \geq b,\ a, b \text{ in } R] &= [(b+c^2)X_2 \mid b, c \text{ in } R]\\ &= bX_2 + [[cX_1, cX_2] \mid c \text{ in } R]\\ &= bX_2 + [[cX_1, dX_2] \mid [cX_1, X_2] - dX_2 = 0,\ c, d \text{ in } R].\end{aligned}$$

Hence,

$$\begin{aligned}&\psi^{-1}([aX_2 \mid a \geq b,\ a, b \text{ in } R])\\ &\quad = \psi^{-1}(bX_2) + [[Y, Z] \mid [Y, \psi^{-1}(X_2)] - Z = 0,\ Y \text{ in } A',\ Z \text{ in } B']\end{aligned}$$

is an analytic subset of B'. As before, this implies that $\psi\colon B' \to B$ is a topological isomorphism of complete separable metric (additive) abelian groups. Observe that the mapping $\tau\colon Z \to [Z, X_2]$, $A \to B$, is a topological isomorphism. Hence, $Y \to [Y, \psi^{-1}(X_2)] \to [\psi(Y), X_2] \to \psi(Y)$, $A' \to B' \to B \to A$, is a Borel isomorphism by Souslin's theorem, for it is the composition of one-to-one Borel mappings. But L is Borel isomorphic to $A' \oplus B'$ and K is Borel isomorphic to $A \oplus B$. Hence, $\psi\colon L \to K$ is a Borel isomorphism of complete separable metric (additive) abelian groups, and so, as before, is a topological isomorphism.

Next, suppose that K is the Lie algebra of G_3. Let $A = RX_1$, $A' = \psi^{-1}(A)$, $B = RX_2$, $B' = \psi^{-1}(B)$, $C = RX_3$, and $C' = \psi^{-1}(C)$. A' is closed in L since A is maximal abelian in K, and $B'+C'$ is closed in L since $B+C$ is maximal abelian in K. Also, since

$$B' = [[Y, \psi^{-1}(X_3)] \mid Y \text{ in } A'] \quad \text{and} \quad C' = [[Y, \psi^{-1}(X_2)] \mid Y \text{ in } A'],$$

B' and C' are analytic (additive) subgroups of L. The natural mapping $A' \times B' \times C' \to A'+B'+C' = L$ is continuous, one-to-one, and onto. Hence, Souslin's theorem implies that this mapping is a Borel isomorphism, and so B' and C' are Borel (additive) subgroups of L. The set $A'+C'$ is a Borel subset of L by Souslin's theorem, and it is a Borel transversal for the quotient space L/B'. Hence, B' is closed (Miller [3], Theorem 1). Similarly, C' is closed.

Notice that

$$\begin{aligned}[aX_2 \mid a \geq b,\ a, b \text{ in } R] &= [(b+c^2)X_2 \mid b, c \text{ in } R]\\ &= bX_2 - [[cX_1, cX_3] \mid c \text{ in } R]\\ &= bX_2 - [[cX_1, dX_3] \mid [cX_1, X_2] - dX_3 = 0,\ c, d \text{ in } R].\end{aligned}$$

Hence

$$\begin{aligned}\psi^{-1}([aX_2 \mid a \geq b,\ a, b \text{ in } R]) = \psi^{-1}(bX_2) - [[Y, Z] \mid [Y, \psi^{-1}(X_2)] - Z = 0,\\ Y \text{ in } A',\ Z \text{ in } C']\end{aligned}$$

is an analytic subset of B'. As before, this implies that $\psi\colon B' \to B$ is a topological isomorphism. As very similar argument implies that $\psi\colon C' \to C$ is a topological isomorphism. The mapping $\tau\colon Y \to [Y, X_2]$, $A \to C$, is a topological isomorphism. The mapping $Y \to [Y, \psi^{-1}(X_2)] \to [\psi(Y), X_2] \to \psi(Y)$, $A' \to C' \to C \to A$, is a composition of one-to-one Borel mappings, and therefore itself is a Borel mapping. Hence, $\psi\colon L = A'+B'+C' \to A+B+C = K$ is a Borel, and therefore a topological, isomorphism of complete separable metric (additive) abelian groups.

One handles the general case and completes the proof of Theorem 5 just as one completed the proof of Theorem 4.

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Resolutions of spaces and proper inverse systems in shape theory

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Abstract. It will be shown that the two notions in shape theory, resolutions of spaces in the sense of S. Mardešić and proper inverse systems in our sense, are essentially equivalent.

1. Introduction and statement of results. Let Top be the category of topological spaces and continuous maps, and Pol its full subcategory of polyhedra. Let us denote by HTop and HPol the homotopy category of Top and Pol respectively.

In the pro-homotopy approach to the shape category of topological spaces, which was introduced in our previous paper [10], one assigns to each topological space X an inverse system in HPol which is associated with X in the sense of [10], while in the approaches of Mardešić–Segal [6] and Fox [2], which are concerned with compact Hausdorff spaces and metric spaces respectively, these authors assign to X inverse systems of ANR's for metric spaces in Top with X as their inverse limit. To prove the equivalence of our approach with those of these authors for the respective cases, we have introduced in [10] the notion of proper inverse systems. Here we recall its definition.

Throughout this paper, let X be a topological space and $\{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ an inverse system in Top, and let $\{p_\lambda\}: X \to \{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ be a morphism in pro-Top, i.e., $p_\lambda: X \to X_\lambda$ is a continuous map for each λ such that $p_\lambda = p_{\lambda\lambda'} p_{\lambda'}$, for $\lambda \leqslant \lambda'$. Let us denote by N the operation of taking the nerve of a cover.

DEFINITION 1 (Morita [10]). $\{p_\lambda\}$ is called *proper* if condition (P) below is satisfied:

(P) For any $\lambda \in \Lambda$, any normal cover $\mathscr{G}$ of X and any normal cover $\mathscr{H}$ of X_λ, there exist a $\mu \in \Lambda$ with $\lambda \leqslant \mu$ and a normal cover $\mathscr{V}$ of X_μ such that $p_\mu^{-1}(\mathscr{V})$ refines $\mathscr{G}$, $\mathscr{V}$ refines $p_{\lambda\mu}^{-1}(\mathscr{H})$ and $N(\mathscr{V})$ is isomorphic to $N(p_\mu^{-1}(\mathscr{V}))$ by the map $V \mapsto p_\mu^{-1}(V)$ for $V \in \mathscr{V}$.

In [10] this definition was described on the assumption that X is an inverse limit, but this assumption was not used actually in the statement of the definition as well as in the proof of [10, Theorem 1.9] and [13, Theorem 3.1]. Thus, it is proved actually by [10, Theorem 1.9] that if $\{p_\lambda\}$ is proper then the inverse system