Investigating the ANR-property of metric spaces

by

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Abstract. A characterization of ANR-spaces is established and is applied to show, among other things, that if $X \in \mathrm{ANR}$ then the family $\mathscr{F}_k(X)$ of all non-empty subsets of X consisting of at most k points, topologized by the Hausdorff metric, is an ANR-space for each $k \in N \cup \{\infty\}$, where $\mathscr{F}_\infty(X) = \bigcup_{k=1}^{\infty} \mathscr{F}_k(X)$.

This answers affirmatively a problem of Borsuk.

Let $\{\mathscr{U}_n\}$ be a sequence of open covers of a metric space X. Let $\mathscr{U} = \bigcup_{n \in N} \mathscr{U}_n$. By $\mathscr{N}(\mathscr{U})$ we denote the nerve of $\mathscr{U}$. We write $K \prec \{\mathscr{U}_n\}$ iff K is a subcomplex of $\mathscr{N}(\mathscr{U})$ and for each $\sigma \in K$ we have $\sigma \subset \mathscr{U}_n \cup \mathscr{U}_{n+1}$ for some $n \in N$ (recall that simplices in K are finite subsets of $\mathscr{U} = \bigcup_{n \in N} \mathscr{U}_n$). For each $\sigma \in K$ we put

$$n(\sigma) = \max\{n \in N \colon \sigma \subset \mathscr{U}_n \cup \mathscr{U}_{n+1}\}.$$

In this paper we show that a metric space $X \in \mathrm{ANR}$ if and only if there exists a sequence of open covers $\{\mathscr{U}_n\}$ of X such that for each $K \prec \{\mathscr{U}_n\}$ and for each selection $f\colon K^0 \to X$ there is a map $g\colon K \to X$ such that for any sequence $\{\sigma_k\}$ of simplices of K with $n(\sigma_k) \to \infty$ we have

$$\delta(\sigma_k) = \sup\{d(g(x), f(V)) \colon x \in \sigma_k,\ V \in \sigma_k^0\} \to 0.$$

We then give the following applications of this fact.

In § 2 we show that if $X \in \mathrm{ANR}$ then the symmetric powers $\mathscr{F}_k(X)$ of X are ANR-spaces for all $k \in N$. This provides a positive answer to a problem of Borsuk [Bo]. Let us note that in the compact case the result was established by Jaworowski [J].

In § 3 we prove the following fact which extends the earlier result of Bessaga and Pełczyński [BP2]: If X is a complete metrizable space then the space $M(X)$ of all measurable functions of $[0, 1]$ into X is homeomorphic to a Hilbert space. Let us note that this result was pointed out to us by Toruńczyk [T3].

Further applications of our characterization are given in [N].

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§ 1. A characterization of ANR-spaces. In this section we prove the following theorem

1-1. THEOREM. *For a metric space X the following conditions are equivalent*

(i) $X \in$ ANR.

(ii) *There exists a sequence of open covers $\{\mathscr{U}_n\}$ of X such that for each $K \prec \{\mathscr{U}_n\}$ and for each selection $f\colon K^0 \to X$ there is an extension $g\colon K \to X$ such that for any sequence $\{\sigma_k\}$ of simplices of K with $n(\sigma_k) \to \infty$ we have* $\operatorname{diam} g(\sigma_k) \to 0$.

(iii) *There exists a sequence of open covers $\{\mathscr{U}_n\}$ of X such that for each $K \prec \{\mathscr{U}_n\}$ and for each selection $f\colon K^0 \to X$ there is a map $g\colon K \to X$ such that for any sequence $\{\sigma_k\}$ of simplices of K with $n(\sigma_k) \to \infty$ we have*

$$\delta(\sigma_k) = \sup\{d(g(x), f(V))\colon x \in \sigma_k,\ V \in \sigma_k^0\} \to 0 .$$

Proof. (i) $\Rightarrow$ (ii) Assume that $X \in$ ANR. Consider X as a closed subset of a convex set Z lying in a Banach space. Let W be a neighbourhood of X in Z and let $r\colon W \to X$ be a retraction. For each $n \in N$ take a cover $\mathscr{V}_n$ of X consisting of open convex sets in W such that

(1) $\operatorname{conv} V \subset W$ and $\operatorname{diam} r(\operatorname{conv} V) < 2^{-n}$ for each $V \in \operatorname{st}\mathscr{V}_n$.

(2) $\mathscr{V}_{n+1} \prec \mathscr{V}_n$ for each $n \in N$.

Let us put

$$\mathscr{U}_n = \{V \cap X\colon V \in \mathscr{V}_n\} .$$

Now let $K \prec \{\mathscr{U}_n\}$ and let $f\colon K^0 \to X$ be a selection. For each simplex $\sigma = \langle V_1, \ldots, V_k\rangle \in K$ we define $g|\sigma$ by the formula

$$g(x) = r\Big(\sum_{i=1}^{k} t_i f(V_i)\Big) \quad \text{for each } x = \sum_{i=1}^{k} t_i V_i .$$

Then $g|K^0 = f$ and from (1)(2) we get

$$d(g(x), g(V_1)) = d\Big(rf(V_1), r\Big(\sum_{i=1}^{k} t_i f(V_i)\Big)\Big) < 2^{-n(\sigma)} .$$

Therefore $\operatorname{diam} g(\sigma) < 2^{-n(\sigma)+1}$ for each $\sigma \in K$.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i) Consider X as a closed subset of a metric space Z. We shall show that X is a retract of a neighbourhood of X in Z.

For each open set $U \subset X$ we put

$$\operatorname{Ext} U = \{x \in Z\colon d(x, U) < d(x, X \setminus U)\} , \quad \text{see [K]} .$$

It is easy to see, [K], that

(3) $$\operatorname{Ext} U \cap V = \operatorname{Ext} U \cap \operatorname{Ext} V .$$

The proof of the implication is based on the following fact.

1-2. FACT. *For any sequence of open covers $\{\mathscr{U}_n\}$ of X there exist a sequence of neighbourhoods $\{W_n\}$ of X in Z and a locally finite open cover $\mathscr{V}$ of $W_1 \setminus X$ with the following properties*

(i) $d(x, X) < 1/n$ *for each* $x \in W_n$,

(ii) $\overline{W}_{n+1} \subset W_n$ *for each* $n \in N$,

(iii) *If $V \in \mathscr{V}$ and $V \cap \overline{W}_n \neq \emptyset$ then $V \subset W_{n-1}$ and there exist $\varphi(V) \in \mathscr{U}_n$ and a point $a(\varphi(V)) \in \varphi(V)$ such that $V \subset \operatorname{Ext}\varphi(V)$ and*

$$d(x, a(\varphi(V))) < 5d(x, X) \quad \text{for each } x \in V.$$

Proof. The proof follows from the proofs of Lemmas 4-3, 4-4 and 4-5 of [Hu] (see [Hu], p. 127–128).

Now let us pass to the proof of the implication (iii) $\Rightarrow$ (i) of Theorem 1-1.

Assume that there is a sequence of open covers $\{\mathscr{U}_n\}$ as in 1-1 (iii). Using 1-2 we take a sequence of open neighbourhoods $\{W_n\}$ of X in Z and an open cover $\mathscr{V}$ of $W_1 \setminus X$ satisfying the conditions 1-2 (i)-(iii). We will show that X is a retract of W_1.

For each $V \in \mathscr{V}$, put $n(V) = \sup\{n\colon V \cap \overline{W}_n \neq \emptyset\}$.

By 1-2 (iii) there is a $\varphi(V) \in \mathscr{U}_{n(V)}$ and $a(\varphi(V)) \in \varphi(V)$ such that $V \subset \operatorname{Ext}\varphi(V)$ and

$$d(x, a(\varphi(V))) < 5d(x, X) \quad \text{for each } x \in V.$$

Let us put

$$K^0 = \{\varphi(V)\colon V \in \mathscr{V}\} \subset \mathscr{U} = \bigcup_{n \in N} \mathscr{U}_n .$$

We define a simplical complex K with vertices K^0 by letting

$$\sigma = \langle \varphi(V_1), \ldots, \varphi(V_p)\rangle \in K \quad \text{iff} \quad \langle V_1, \ldots, V_p\rangle \in \mathscr{N}(\mathscr{V}) .$$

Then from (3) and 1-2 (iii) we get

$$\bigcap_{i=1}^{p} \varphi(V_i) \neq \emptyset \quad \text{whenever} \quad \langle V_1, \ldots, V_p\rangle \in \mathscr{N}(\mathscr{V}) .$$

Therefore K is a subcomplex of $\mathscr{N}(\mathscr{U})$. Let us show that $K \prec \{\mathscr{U}_n\}$. For each simplex $\sigma = \langle \varphi(V_1), \ldots, \varphi(V_p)\rangle \in K$ we have $\bigcap_{i=1}^{p} V_i \neq \emptyset$. Thus there is a $k \in N$ such that

$$V_i \cap (\overline{W}_k \setminus W_{k+1}) \neq \emptyset \quad \text{for each } i = 1, \ldots, p .$$

From 1-2 (iii) it follows that

$$k \leqslant n(V_i) \leqslant k+1 \quad \text{for each } i = 1, \ldots, p .$$

Therefore
$$\varphi(V_i) \in \mathcal{U}_k \cup \mathcal{U}_{k+1} \quad \text{for each } i = 1, \dots, p.$$
Consequently $K \prec \{\mathcal{U}_n\}$.

We now define a selection $f\colon K^0 \to X$ by the formula
$$f(\varphi(V)) = a(\varphi(V)) \quad \text{for each } \varphi(V) \in K^0.$$
From 1-2 (iii) it follows that f is well defined. By hypothesis there is a map $g\colon K \to X$ satisfying the condition 1-1 (iii). We define a retraction $r\colon W_1 \to X$ by the formula
$$r(x) = \begin{cases} x & \text{if } x \in X, \\ g\tilde{\varphi}h(x) & \text{if } x \in W_1 \setminus X \end{cases}$$
where $h\colon W_1 \setminus X \to \mathcal{N}(\mathcal{V})$ is the canonical map and $\tilde{\varphi}\colon \mathcal{N}(\mathcal{V}) \to K$ is the simplical map induced by φ. Let us show that r is continuous.

For each $x \in W_1 \setminus X$, say $x \in \overline{W}_{n(x)} \setminus W_{n(x)+1}$, let $\sigma = \langle V_1, \dots, V_p\rangle$ be a simplex of $\mathcal{N}(\mathcal{V})$ containing $h(x)$. It is easy to see that
$$\varphi(\sigma) \subset \mathcal{U}_{n(x)} \cup \mathcal{U}_{n(x)+1},$$
where
$$\varphi(\sigma) = \langle \varphi(V_1), \dots, \varphi(V_p)\rangle \in K.$$
Thus we have $n(\varphi(\sigma)) \geqslant n(x)$. Consequently
$$d(x, r(x)) = d(x, g\tilde{\varphi}h(x)) \leqslant d(x, a\varphi(V_1) + d(f\varphi(V_1), g\tilde{\varphi}h(x)))$$
$$\leqslant 5d(x, X) + \delta(\varphi(\sigma)).$$
Since $n(\varphi(\sigma)) \geqslant n(x) \to \infty$ as $x \to x_0 \in X$ we infer that r is continuous. This completes the proof of Theorem 1-1.

1-3. Remark. Let us note that if X is separable then the nerve $\mathcal{N}(\mathcal{V})$ of $\mathcal{V}$ can be chosen to be locally finite. Therefore from the proof of Theorem 1-1 it follows that a separable metric space X is an ANR iff the condition 1-1 (iii) is satisfied for any locally finite simplical complex $K \prec \{\mathcal{U}_n\}$.

§ 2. Hyperspaces of finite sets of an ANR-spaces. For a metric space X let 2^X denote the hyperspace of all non-empty compact sets in X topologized by the Hausdorff metric
$$d(A, B) = \max\{\max_{a \in A}\min_{b \in B} d(a, b), \max_{b \in B}\min_{a \in A} d(a, b)\} \quad \text{for} \quad A, B \in 2^X.$$
For each $k \in N$ let us put
$$\mathcal{F}_k(X) = \{A \in 2^X\colon \operatorname{card} A \leqslant k\}, \qquad \mathcal{F}_\infty(X) = \bigcup_{k \in N} \mathcal{F}_k(X).$$
Borsuk [Bo] (see [Bo] p. 215 Problem 4-2) asked whether the functors $\mathcal{F}_k$, $k \in N$ preserve the property of being ANR-spaces. Jaworowski [J] has shown that the answer to this question is positive if X is compact and $k < \infty$ (see [F] for a new proof of this fact). In this section we resolve Borsuk's problem in the general case.

2-1. Theorem. *If $X \in$ ANR then $\mathcal{F}_k(X) \in$ ANR* for each $k \in N \cup \{\infty\}$.

2-2. Remark. It is easy to see that if X is contractible then $\mathcal{F}_k(X)$ is contractible for each $k \in N \cup \{\infty\}$.

Therefore Theorem 2-1 gives

2-3. Corollary. *If $X \in$ AR then $\mathcal{F}_k(X) \in$ AR for each $k \in N \cup \{\infty\}$.*

Proof of Theorem 2-1. Consider X as a closed subset of a convex set Z lying in a Banach space. Let W be a neighbourhood of X in Z and let $r\colon W \to X$ be a retraction. For each $n \in N$ take a cover $\mathcal{V}_n$ of X consisting of open convex sets in W such that

(4) $$\operatorname{conv} V \subset W \quad \text{for each } V \in \operatorname{st}\mathcal{V}_n.$$

(5) $$\max\{\operatorname{diam}\operatorname{conv} V, \operatorname{diam} r(\operatorname{conv} V)\} < 2^{-n} \quad \text{for each } V \in \operatorname{st}\mathcal{V}_n.$$

(6) $$\mathcal{V}_{n+1} \prec \mathcal{V}_n \quad \text{for each } n \in N.$$

Let us put
$$\mathcal{U}_n = \{V \cap X\colon V \in \mathcal{V}_n\}$$
and for each finite family of open sets $\{U_1, \dots, U_q\}$ of X we let
$$S(U_1, \dots, U_q) = \{A \in \mathcal{F}_k(X)\colon A \subset \bigcup_{i=1}^{q} U_i \text{ and } A \cap U_i \neq \emptyset \text{ for } i = 1, \dots, q\},$$
$$\tilde{\mathcal{U}}_n^k = \{S(U_1, \dots, U_q)\colon U_i \in \mathcal{U}_n \text{ and if } U_i \neq U_j \text{ then } \operatorname{dist}(U_i, U_j) \geqslant 4.2^{-n}\},$$
$$\mathcal{U}_n^k = \bigcup_{i \geqslant n} \tilde{\mathcal{U}}_i^k.$$
We shall show that the sequence $\{\mathcal{U}_n^k\}$ satisfies the condition 1-1 (iii) for $\mathcal{F}_k(X)$.

Let $K \prec \{\mathcal{U}_n^k\}$ and let $f\colon K^0 \to \mathcal{F}_k(X)$ be an arbitrary selection. For each $V = S(U_1, \dots, U_p) \in K^0$ take a set $\{a_1(V), \dots, a_r(V)\} \subset f(V)$ such that
$$\{a_1(V), \dots, a_r(V)\} \cap U_i$$
is a one point set for each $i = 1, \dots, p$. Let us put
$$g(V) = \{a_1(V), \dots, a_r(V)\}$$
and for each simplex $\sigma = \langle V_1, \dots, V_p\rangle \in K$ with $V_i = S(U_1^i, \dots, U_{k_i}^i)$, write
$$A(\sigma) = \{\{a_1, \dots, a_p\}\colon a_i \in g(V_i) \cap U^i,\ U^i \in \{U_1^i, \dots, U_{k_i}^i\} \text{ for } i = 1, \dots, p \text{ and } \bigcap_{i=1}^{p} U^i \neq \emptyset\}$$
Note that for each simplex $\sigma = \langle V_1, \dots, V_p\rangle \in K$, for each $i \in \{1, \dots, p\}$ and for each $a_i \in g(V_i)$ there exists $\{a_1, \dots, a_p\} \in A(\sigma)$ such that $a_i \in \{a_1, \dots, a_p\}$.

Let us show that $\operatorname{card} A(\sigma) \leqslant k$ for each $\sigma \in K$. In fact, put

$$n_0 = \max\{n: \tilde{\mathscr{U}}_n^k \cap \sigma^0 \neq \emptyset\}$$

and let $V_i = S(U_1^i, \ldots, U_{k_i}^i) \in \tilde{\mathscr{U}}_{n_0}^k$. Since $\operatorname{card} g(V_i) \leqslant k$, it suffices to show that for each $a_i \in g(V_i)$ there exists unique $\{a_1, \ldots, a_p\} \in A(\sigma)$ such that $a_i \in \{a_1, \ldots, a_p\}$.

If it is not the case, then there would exist two distinct members $\{a_1, \ldots, a_p\}$ and $\{a_1', \ldots, a_p'\}$ of $A(\sigma)$ such that

$$a_i \in \{a_1, \ldots, a_p\} \cap \{a_1', \ldots, a_p'\}.$$

We may assume that $a_1 \neq a_1'$ and $a_1, a_1' \in g(V_1) = g(S(U_1^1, \ldots, U_{k_1}^1))$. Let $a_1 \in U_j^1$, $a_1' \in U_{j'}^1$ and $a_i \in U_l^i$. Then we have

$$U_l^i \cap U_j^1 \neq \emptyset \quad \text{and} \quad U_l^i \cap U_{j'}^1 \neq \emptyset.$$

Note that if $S(U_1^1, \ldots, U_{k_1}^1) \in \tilde{\mathscr{U}}_{n_1}^k$ then

$$\operatorname{dist}(U_j^1, U_{j'}^1) \geqslant 4.2^{-n_1} \geqslant 4.2^{-n_0}.$$

Since $\operatorname{diam} U_l^i < 2^{-n_0}$, the above is impossible.

We now define $g: K \to \mathscr{F}_k(X)$ by the formula

$$g(x) = \{r(\sum_{i=1}^{p} \lambda_i a_i): \{a_1, \ldots, a_p\} \in A(\sigma)\}$$

for each $x = \sum_{i=1}^{p} \lambda_i V_i \in \sigma$. It is easy to see that g is continuous.

Note that for each simplex $\sigma = \langle V_1, \ldots, V_p \rangle \in K$ and $x \in \sigma$ we have

$$d(g(x), f(V_i)) \leqslant d(g(x), g(V_i)) + d(g(V_i), f(V_i)) \leqslant 2^{-n(\sigma)+1} + 2^{-n(\sigma)+1} = 2^{-n(\sigma)+2}.$$

for each $i = 1, \ldots, p$. Therefore

$$\delta(\sigma) = \sup\{d(g(x), f(V)): x \in \sigma,\ V \in \sigma^0\} \leqslant 2^{-n(\sigma)+2}$$

for each $\sigma \in K$. Thus by Theorem 1-1 we have $\mathscr{F}_k(X) \in \mathrm{ANR}$.

This completes the proof of the theorem.

From Theorem 2-1 we also get

2-4 Corollary. *$\mathscr{F}_k(l_2) \cong l_2$ for each $k \in N$.*

Proof. Since $\mathscr{F}_k(l_2) \in \mathrm{AR}$, by [DT] it suffices to establish the following fact

(***) For each $\varepsilon > 0$ there exists a $\delta > 0$ such that for every compact set $\mathscr{K} \subset \mathscr{F}_k(l_2)$ there is an ε-homotopy $h_t: \mathscr{K} \to \mathscr{F}_k(l_2)$ such that $h_0 = \mathrm{id}$ and $\operatorname{dist}(h_1(\mathscr{K}), \mathscr{K}) \geqslant \delta$.

Proof of (***). Given $\varepsilon > 0$. Since $B(\varepsilon) = \{x \in l_2: \|x\| \leqslant \varepsilon\}$ is not compact, there is a $\delta > 0$ such that no compact set of l_2 is a δ-net for $B(\varepsilon)$.

Now, given a compact set $\mathscr{K} \subset \mathscr{F}_k(l_2)$, put

$$K = \bigcup\{A: A \in \mathscr{K}\} \subset l_2, \quad K^* = K - K = \{x - y: x, y \in K\} \subset l_2.$$

Then K^* is compact. Take $a \in B(\varepsilon)$ such that $d(a, K^*) \geqslant \delta$. We define a homotopy $h_t: \mathscr{K} \to \mathscr{F}_k(l_2)$ by the formula

$$h_t(A) = A + ta \quad \text{for each } A \in \mathscr{K} \text{ and } t \in [0, 1].$$

Obviously h satisfies the desired condition.

§ 3. Spaces of measurable functions. Let X be a metrizable space. By $M(X)$ we denote the space of all measurable functions of $[0, 1]$ into X equipped with the topology of convergence in measure. We identify $f \equiv g$ iff

$$|\{t \in [0, 1]: f(t) \neq g(t)\}| = 0.$$

Here $|A|$ denotes the Lebesgue measure of A in $[0, 1]$.

Bessaga and Pełczyński [BP1] [BP2] showed that $M(X) \cong l_2$ iff X is a complete separable metrizable space having more than one point. Here we have

3-1. Theorem. *$M(X)$ is homeomorphic to a Hilbert space for any complete metrizable space X.*

Let us note that Theorem 3-1 was mentioned by Toruńczyk [T3].

Since $M(X)$ is homeomorphic to the countable Cartesian product of itself [T3], by [T4] in order to prove Theorem 3-1 it suffices to establish the following fact

3-2. Proposition. *$M(X) \in \mathrm{AR}$ for any metrizable space X.*

Proof. For the reader's convenience we first present the proof in the separable case. The same idea will be used in the proof of the general case.

Let d be a compatible metric of X bounded by 1. Then the formula

$$d(f, g) = \int_0^1 d(f(t), g(t))\,dt \quad \text{for} \quad f, g \in M(X)$$

defines a compatible metric of $M(X)$.

Since $M(X)$ is contractible, it suffices to show that $M(X) \in \mathrm{ANR}$. Let us verify the condition 1-1 (iii). Take a sequence $\{\mathscr{U}_n\}$ of open covers of $M(X)$ such that $\operatorname{diam} U < 2^{-n}$ for each $U \in \mathscr{U}_n$. Let $K \prec \{\mathscr{U}_n\}$ be a locally finite simplical complex and let $f: K^0 \to M(X)$ be a selection. Take a map $g_0: K^0 \to M(X)$ such that $g_0(V)$ is piecewise constant for each $V \in K^0$ and

$$(7) \qquad d(f(V), g_0(V)) < 2^{-n(V)} \quad \text{for each } V \in K^0$$

where $n(V) = \sup\{n: V \in \mathscr{U}_n\}$.

For each simplex $\sigma \in K$, let $A(\sigma)$ denote the set of all vertices of simplices with σ as a face. Since K is locally finite, $A(\sigma)$ is finite for each $\sigma \in K$.

We shall define inductively a sequence of maps $g_n\colon K^{(n)}\to M(X)$ with the following properties

(8) $g_n|K^{(n-1)}=g_{n-1}$ for each $n\geqslant 1$,

(9) for each $\sigma\in K^{(n)}$ there exists an $m(\sigma)\in N$ such that for each $x\in\sigma\cup A(\sigma)$ there exist intervals $A_1,\dots,A_k$, $k\leqslant m(\sigma)$ such that $\bigcup_{i=1}^{k}A_i=[0,1]$, $A_i^0\cap A_j^0=\emptyset$ for $i\neq j$ and $g_n(x)|A_i$ is constant for each $i=1,\dots,k$,

(10) for each $\sigma\in K^{(n)}$, for each $h\in g_0(A(\sigma))$ and for each $x\in\sigma$ we have

$$d(h,g_n(x))\leqslant\max\{d(h,f_0(V))\colon V\in\sigma^0\}+(1-2^{-n})\operatorname{diam}g_0(\sigma^0).$$

Obviously, g_0 satisfies the conditions (9), (10). Assume that g_{n-1} has been defined with the properties (8)–(10). Let us define $g_n\colon K^{(n)}\to M(X)$ as follows. For each $\sigma\in K^{(n)}$ take $k(\sigma)\in N$ such that

$$(11)\quad k(\sigma)\geqslant n+1+\log_2\frac{\sum\{m(V)\colon V\in A(\sigma)\}+\max\{m(\sigma')\colon \sigma' \text{ is a face of } \sigma\}}{\operatorname{diam}g_0(\sigma^0)}.$$

Put

$$\Delta(k(\sigma),i)=(i2^{-k(\sigma)},(i+1)2^{-k(\sigma)})\quad\text{for}\quad i=0,\dots,2^{-k(\sigma)}-1.$$

Let c be an interior point of the simplex σ. We put

$$g_n(c)=g_{n-1}(V_0)=g_0(V_0)$$

where V_0 is a vertex of σ.

Note that for each $x\in\sigma$, $x\neq c$ there exist a unique $s\in[0,1]$ and $y\in\dot\sigma$ (the boundary of σ) such that $x=sc+(1-s)y$. We define $g_n(x)$ as follows: If $g_n(c)|\Delta(k(\sigma),i)$ and $g_{n-1}(y)|\Delta(k(\sigma),i)$ are constant then we put

$$(12)\qquad g_n(x)(t)=\begin{cases}g_n(c)(t) & \text{if } t\in[i2^{-k(\sigma)},(i+s)2^{-k(\sigma)}),\\ g_{n-1}(y)(t) & \text{if } t\in[(i+s)2^{-k(\sigma)},(i+1)2^{-k(\sigma)}).\end{cases}$$

Otherwise we subdivide $\Delta(k(\sigma),i)$ into the family of subintervals $\{\Delta\}$ such that $g_n(c)|_\Delta$ and $g_{n-1}(y)|_\Delta$ are constant and that each $\Delta\in\{\Delta\}$ is maximal, that is, if $\Delta'\supsetneqq\Delta$ then either $g_n(c)|_{\Delta'}$ or $g_{n-1}(y)|_{\Delta'}$ is not constant. We define $g_n(x)|_\Delta$ by the formula (12) with $\Delta(k(\sigma),i)$ replaced by Δ. Obviously g_n satisfies the conditions (8), (9). Let us check (10).

Given $x\in\sigma$ with $x=sc+(1-s)y$ for some $s\in[0,1]$, $y\in\dot\sigma$, put

$$\bar\Delta=\bigcup\{\Delta(k(\sigma),i)\subset[0,1]\colon g_{n-1}(z)|\Delta(k(\sigma),i)\text{ is not constant for some } z\in A(\sigma)\cup\cup\{y\}\},$$

$$\tilde\Delta=[0,1]\setminus\bar\Delta.$$

We subdivide $\tilde\Delta=\Delta^*\cup\Delta^{**}$ so that

$$g_n(x)|_{\Delta^*}=g_n(c)\quad\text{and}\quad g_n(x)|_{\Delta^{**}}=g_{n-1}(y).$$

Let us note that $\bar\Delta$, $\tilde\Delta$, Δ^*, Δ^{**} depend on σ and y. Then for each $h\in g_0(A(\sigma))$ we have

$$\begin{aligned}d(h,g_n(x))&=\int_0^1 d(h(t),g_n(x)(t))\,dt\\&=\int_{\Delta^*}d(h(t),g_n(x)(t))\,dt+\int_{\Delta^{**}}d(h(t),g_n(x)(t))\,dt+\int_{\bar\Delta}d(h(t),g_n(x)(t))\,dt\\&=s\int_{\tilde\Delta}d(h(t),g_n(c)(t))\,dt+(1-s)\int_{\tilde\Delta}d(h(t),g_{n-1}(y)(t))\,dt+\\&\quad+\int_{\bar\Delta}d(h(t),g_n(x)(t))\,dt\\&=s\int_0^1 d(h(t),g_n(c)(t))\,dt+(1-s)\int_0^1 d(h(t),g_{n-1}(y)(t))\,dt+\\&\quad+\int_{\bar\Delta}\{d(h(t),g_n(x)(t))-sd(h(t),g_n(c)(t))-(1-s)d(h(t),g_{n-1}(y)(t))\}\,dt\\&\leqslant sd(h,g_n(c))+(1-s)d(h,g_{n-1}(y))+2|\bar\Delta|\\&\leqslant\max\{d(h,g_n(c)),d(h,g_{n-1}(y))\}+2|\bar\Delta|.\end{aligned}$$

Note that

$$|\bar\Delta|\leqslant 2^{-k(\sigma)}(\sum\{m(V)\colon V\in A(\sigma)\}+\max\{m(\sigma')\colon \sigma' \text{ is a face of } \sigma\}).$$

Therefore from (11) we have

$$|\bar\Delta|\leqslant 2^{-n-1}\operatorname{diam}g_0(\sigma^0).$$

Let $\sigma'\in K^{(n-1)}$ denote a face of σ containing y. Then by inductive assumption for every $h\in g_0(A(\sigma'))$ we get

$$\begin{aligned}d(h,g_{n-1}(y))&\leqslant\max\{d(h,g_0(V))\colon V\in\sigma'^0\}+(1-2^{-n+1})\operatorname{diam}g_0(\sigma'^0)\\&\leqslant\max\{d(h,g_0(V))\colon V\in\sigma^0\}+(1-2^{-n+1})\operatorname{diam}g_0(\sigma^0).\end{aligned}$$

Since $A(\sigma')\supset A(\sigma)$, for $h\in g_0(A(\sigma))$ we obtain

$$\begin{aligned}d(h,g_n(x))&\leqslant\max\{d(h,g_n(c)),d(h,g_{n-1}(y))\}+2|\bar\Delta|\\&\leqslant\max\{d(h,g_0(V))\colon V\in\sigma^0\}+(1-2^{-n})\operatorname{diam}g_0(\sigma^0).\end{aligned}$$

Hence the condition (16) holds.

Finally we define $g\colon K\to M(X)$ by the formula

$$g(x)=\lim_{n\to\infty}g_n(x)\quad\text{for each } x\in K.$$

Then $g|K^0 = g_0$ and for each $x \in \sigma \in K$ and $V \in \sigma^0$ from (7), (10) we get

$$d(g(x), f(V)) \leqslant d(g(x), g_0(V)) + d(g_0(V), f(V)) \leqslant 2\operatorname{diam} g_0(\sigma^0) + 2^{-n(V)} \leqslant 5.2^{-n(\sigma)}.$$

Hence

$$\delta(\sigma) = \sup\{d(g(x), f(V)) \colon x \in \sigma,\ V \in \sigma^0\} \leqslant 5.2^{-n(\sigma)}.$$

Hence by Theorem 1-1 we have $M(X) \in \mathrm{ANR}$.

The general case. In the non-separable case, the simplical complex K is not locally finite, therefore the set $A(\sigma)$ is, in general, infinite. However, we can provide a new metric $\bar{d}$ on $M(X)$ for which the condition (10) holds true for all elements $h \in M(X)$. The metric $\bar{d}$ is defined as follows: For $f, g \in M(X)$, write

$$\omega(k, i)(f, g) = \sup_{x \in X} \Big| \int_{\Delta(k,i)} d(f(t), x)\,dt - \int_{\Delta(k,i)} d(g(t), x)\,dt \Big|$$

where $\Delta(k, i) = [i2^{-k}, (i+1)2^{-k})$ for $i = 0, \ldots, 2^k - 1$,

$$d_k(f, g) = \sum_{i=0}^{2^k-1} \omega(k, i)(f, g),$$

$$\bar{d}(f, g) = \sum_{k=1}^{\infty} 2^{-k} d_k(f, g).$$

It is easy to see that $\bar{d}$ is a compatible metric on $M(X)$.

Let us note that, when using the metric $\bar{d}$ instead of d, the sequence $\{g_n\}$ constructed in the proof of the separable case satisfies conditions (9) for each $x \in \sigma$ and (10) for each $h \in M(X)$.

In fact, we take

$$k(\sigma) > n + 3 + \log_2 \frac{\max\{m(\sigma') \colon \sigma' \text{ is a face of } \sigma\}}{\operatorname{diam} g_0(\sigma^0)} \tag{11*}$$

Then, for each $k \leqslant k(\sigma)$ and for each $z \in X$ we have

$$\int_{\Delta(k,i)} d(g_n(x)(t), z)\,dt = s \int_{\Delta(k,i)} d(g_n(c)(t), z)\,dt + (1-s) \int_{\Delta(k,i)} d(g_{n-1}(y)(t), z)\,dt + \int_{\bar{\Delta}(k,i)} \{d(g_n(x)(t), z) - s d(g_n(c)(t), z) - (1-s) d(g_{n-1}(y)(t), z)\}\,dt$$

where

$$\bar{\Delta}(k, i) = \bigcup \{\Delta(k(\sigma), i) \subset \Delta(k, i) \colon \varphi|\Delta(k(\sigma), i) \text{ is not constant for some } \varphi \in \{g_{n-1}(y), g_n(c)\}\}.$$

Hence, using the condition (11*) we obtain, for each $k \leqslant k(\sigma)$

$$d_k(g_n(x), h) \leqslant s d_k(g_n(c), h) + (1-s) d_k(g_{n-1}(y), h) + 2^{-n-1} \operatorname{diam} g_0(\sigma^0).$$

Therefore

$$\begin{aligned} d(g_n(x), h) &= \sum_{k=1}^{\infty} 2^{-k} d_k(g_n(x), h) \\ &= \sum_{k \leqslant k(\sigma)} 2^{-k} d_k(g_n(x), h) + \sum_{k > k(\sigma)} 2^{-k} d_k(g_n(x), h) \\ &\leqslant s d(g_n(c), h) + (1-s) d(g_{n-1}(y), h) + 2^{-n-1} \operatorname{diam} g_0(\sigma^0) + 2^{-k(\sigma)} \\ &\leqslant \max\{d(g_n(c), h), d(g_{n-1}(y), h)\} + 2^{-n} \operatorname{diam} g_0(\sigma^0). \end{aligned}$$

Consequently using the inductive hypothesis we get condition (10).

3-3. Remark. The proof of Proposition 3-2 also shows that the space $M_c(X)$ consisting of piecewise constant functions in $M(X)$ is an AR for any metrizable space X.

Added in proof. After this paper has been accepted for publication V. V. Fedorchuk kindly informed us that for $k < \infty$ Theorem 2-1 has been obtained earlier by M. R. Cauty (C. R. Acad. Sc. Paris 276 (1973), pp. 359–361). However, as in the proof of Jaworowski, the proof of Cauty uses the fact that if pairwise intersections of ANR's are ANR's then a finite union of them is also an ANR's (see Fedorchuk, Soviet Math. Dokl. 22 (1980), pp. 849–853).

EXAMPLE Let $X = B \setminus \bigcup \{B_i \colon i \in N\}$ where $B = \{z \in C \colon \|z\| \leqslant 1\} \times I$ is the unite ball in R^3 and $\{B_i\}$ is a null-sequence disjont balls in B centered at points of $0 \times 0 \times [0, 1]$. Then $X \notin \mathrm{ANR}$ but X can be written as the union of three AR's X_i such that $X_i \cap X_j \in \mathrm{AR}$ for $i \neq j$ (take $X_i = X \cap \{(s, t) \in B \colon \arg(s) \in [\frac{4}{3}\pi i, \frac{4}{3}\pi(i+1)]\}$).

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Uniqueness results for the $ax+b$ group and related algebraic objects

by

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Abstract. The $ax+b$ and related groups have a unique topology in which they are complete separable metric groups. Several other topological algebraic structures have their topology uniquely determined by their algebraic structure.

Let G be the set of all pairs (a, b), where a is a nonzero complex number and b is a complex number. G is a group with the multiplication $(a_1, b_1)\cdot(a_2, b_2) = (a_1a_2, a_1b_2+b_1)$. G of course can be made into a complete separable metric group in a natural manner. Let G_1 be the subgroup of G for which a is positive real and b is real, let G_2 be the subgroup of G for which a is nonzero real and b is real, and let G_3 be the subgroup of G for which a is of modulus one and b is complex. Each G_i is a closed subgroup of G and thus is a complete separable metric group in a natural manner. It is well known that the field of complex numbers has $2^{\aleph_0}$ discontinuous automorphisms, each of which gives rise to a bizarre topology on G. However, this is not the case for the G_i's. For each positive integer $n\geqslant 1$, let K_n be either G_1, G_2, G_3, or the identity, and let $K = \prod_{n\geqslant 1} K_n$. K is a complete separable metric group in the product topology. The purpose of this note is to prove the following theorem.

THEOREM 1. *Let H be a complete separable metric group and let $\psi\colon H\to K$ be an abstract group isomorphism. Then ψ is a topological isomorphism.*

This theorem seems to be new even if there is only one nontrivial factor in K. The only precedent that I am aware of is Tits ([5], Proposition 6.2), who proved an analogue of Theorem 1 for $K = G_3$ and H a second countable Lie group.

Consider first the case for which $K = G_1$. Let A be the set of all elements of G_1 of the form $(a, 0)$, where a is a positive real number, and let B be the set of all elements of G_1 of the form $(1, b)$, where b is a real number. A and B are maximal abelian subgroups of G_1. Hence, $A_1 = \psi^{-1}(A)$ and $B_1 = \psi^{-1}(B)$ are maximal abelian subgroups of H, and so are closed. Note that

$$[(1, x)|\ x>b] = [(1, b)(a, 0)(1, 1)(a, 0)^{-1}|\ (a, 0)\in A].$$

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