Entropy of transformations of the unit interval

by

Anna Zdunik (Warszawa)

Abstract. The topological entropy in one parameter families $a \mapsto f_a = a \cdot f$ is studied. An example is constructed showing that the entropy can be decreasing for some values of the parameter even in very regular families. However, there is a countable set of kneading invariants which can be attained in the family only once.

I. Introduction. Let $\mathscr{A}$ be a family of continuous maps from the unit interval into itself such that every $f \in \mathscr{A}$ is concave, $f(0) = f(1) = 0$, the point $\frac{1}{2}$ is the unique critical point of f and f is symmetric $\left(f(\frac{1}{2}-y) = f(\frac{1}{2}+y)\right)$.

We consider one-parameter families of maps $a \mapsto f_a = a \cdot f$ or $f_a = a+f$ such that all f_a are in $\mathscr{A}$.

Define a function $h(\cdot)$ as $h(a) = h(f_a)$ where $h(f_a)$ is the topological entropy of f_a.

We are interested in the following problem: Is h a non-decreasing function of a?

Some results in this direction have been obtained recently. First Hofbauer proved for the family $a \mapsto ax(1-x)$ that there is a countable set of values of h which can be attained only once (see [4]). Recently Douady and Hubbard [2] gave a proof of monotonicity of function h for this family.

Matsumoto [5] considered families $a \mapsto a \cdot f$ with arbitrary $f \in \mathscr{A}$ and then families such that all f_a have no homtervals. In both cases he obtained results similar to those in Hofbauer's paper but for other values of h.

In this paper I give an example showing that h can be decreasing for some values of the parameter. Then I prove some partial results about the monotonicity of this function.

II. Example. We shall construct a piecewise linear map $f \in \mathscr{A}$ such that, for small and positive ε, $h\left((1+\varepsilon) \cdot f\right) < h(f)$ and $h(f+\varepsilon) < h(f)$ (where $(f+\varepsilon)(x) = f(x)+\varepsilon$).

The map f has a fixed point q and a point p of period two such that $f(p) < q < p$ (Fig. 1). The slopes are:

β_1 on the interval $\left(f^2(\frac{1}{2}), (\frac{1}{2})\right)$,

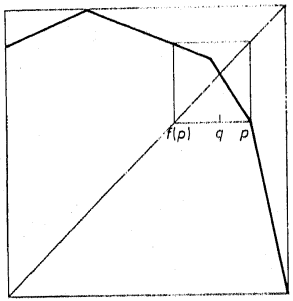

Fig. 1. Graph of the function f

β_1 on $(\frac{1}{2}, r)$ (for some $f(p)<r<p$),

β_2 on (r, p) and α on $(p, f(\frac{1}{2}))$,

where the parameters α, β_1, β_2 are chosen so that $\beta_1<1$, $\beta_1\cdot\beta_2<1$, $\beta_2>1$, $\alpha\cdot\beta_1>1$ and $f^3(\frac{1}{2}) = p$.

Since $\beta_2<1/\beta_1$, the slope at the fixed point q is larger than one $\big(f'(q) = -\beta_2\big)$ and it is repelling (see Fig. 2).

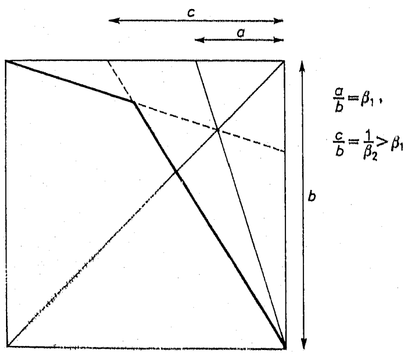

Fig. 2

Our map is now defined on the interval $[f^2(\frac{1}{2}), f(\frac{1}{2})]$. It can be extended to a piecewise linear and concave transformation from $[0, 1]$ into itself such that $f(0) = f(1) = 0$.

It follows from the construction that the kneading invariant (see [1]) of f is equal to RLR^∞.

We shall prove the following proposition:

PROPOSITION. *There exists an* $\varepsilon_0>0$ *such that for every* $\varepsilon\in(0, \varepsilon_0)$

$$h\big((1+\varepsilon)\cdot f\big)<h(f) \quad \text{and} \quad h(f+\varepsilon)<h(f).$$

Proof. Let $g = (1+\varepsilon)\cdot f$. Then

$$g(p)-f(p) = \varepsilon\cdot f(p),\ g^2(p)-f^2(p) = \varepsilon\big(p-(1+\varepsilon)\beta_1\cdot f(p)\big).$$

It follows that $g^2(p)>p$ for ε sufficiently small. Let s be such that $g(s) = p$ and $s>\frac{1}{2}$. Then $g(p)<s$ (Fig. 3).

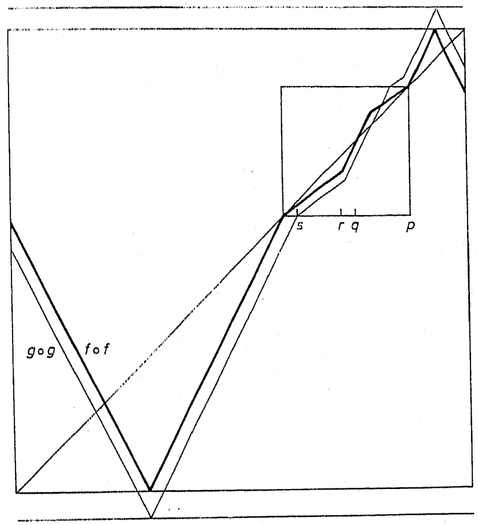

Fig. 3

The absolute value of $(g\circ g)'(x)$ is larger than one for $x\in(\frac{1}{2}, s)$ and smaller than one for $x\in(s, r)$ (see Fig. 3).

Furthermore

$$\begin{aligned} g^2(q)-f^2(q) &= g\big(g(q)\big)-g\big(f(q)\big)+g\big(f(q)\big)-f\big(f(q)\big) \\ &= -(1+\varepsilon)\beta_2\varepsilon q+\varepsilon q = \varepsilon q\big(1-(1+\varepsilon)\beta_2\big). \end{aligned}$$

Hence $g^2(q)<q$ if ε is positive and small enough.

It follows that the map g has no fixed points in the interval $(\frac{1}{2}, q)$. If ε is small enough, then the point $g^4(\frac{1}{2})$ lies close to $f(p)$. Hence there exists a $k_0>2$ such that for $2<(k-1)\leqslant k_0$:

$$\tfrac{1}{2}<g^{2k}(\tfrac{1}{2})<g^{2(k-1)}(\tfrac{1}{2}), \quad g^{2k-1}(\tfrac{1}{2})>\tfrac{1}{2} \quad \text{and} \quad g^{2k_0}(\tfrac{1}{2})\leqslant\tfrac{1}{2}.$$

Thus the kneading invariant $\underline{K}(g) = K_i$ is such that $K_1 = R$, $K_2 = L$, $K_i = R$ for $3 \leqslant i < 2k_0 - 1$, $K_{2k_0} = L$ or C and therefore $\underline{K}(g)$ is smaller than $\underline{K}(f) = \mathrm{RLR}^\infty$.

The map f has an entropy equal to $\log(\sqrt{2})$, and it follows from the results of Guckenheimer [3] that $h(g) < h(f)$.

The argument and estimations for a map $f+\varepsilon$ are quite similar.

Remark. Modifying this example, one can construct a smooth and concave function F such that $\underline{K}(F) = \underline{K}(f)$ and F has the same property as f.

The point p is also periodic for F and $F'(p) = -1/\beta_1$, $F'(F(p)) = -\beta_1$, $F^2(p) = p$. The map F has a fixed point q_F in $[F(p), p]$ and $F'(q_F) < -1$.

Let us consider a family $\varepsilon \mapsto (1+\varepsilon)F = F_\varepsilon$, $|\varepsilon| < \varepsilon_1$. Then $F_\varepsilon(p) - F(p) = \varepsilon F(p)$,

$$F_\varepsilon^2(p) - F^2(p) = F_\varepsilon(F_\varepsilon(p)) - F_\varepsilon(F(p)) + F_\varepsilon(F(p)) - F^2(p)$$
$$= -\varepsilon\beta_1, (\varepsilon) F(p) + \varepsilon p = \varepsilon(p - \beta_1, (\varepsilon) F(p))$$

where $\beta_1(\varepsilon) = -F_\varepsilon'(t)$ for some t lying between $F(p)$ and $F_\varepsilon(p)$.

Since $\lim_{\varepsilon \to 0} \beta_1(\varepsilon) = \beta_1$, we obtain

$$\frac{\partial}{\partial \varepsilon} F_\varepsilon^2(p)|_{\varepsilon=0} = \lim_{\varepsilon \to 0} \frac{F_\varepsilon^2(p) - F^2(p)}{\varepsilon} > 0 .$$

Moreover, if F_ε^2 has a fixed point $p_\varepsilon \in (\frac{1}{2}, q_F)$, then $\lim_{\varepsilon \to 0} p_\varepsilon = F(p)$.

We shall use the following technical lemma:

LEMMA. *Let us consider a smooth family of smooth maps $a \mapsto f_a$, where f_a maps the unit interval into itself. If for some $p \in (0, 1)$ and a_0*

$$f_{a_0}^2(p) = p , \quad \frac{\partial f_{a_0}^2}{\partial x}(p) = 1 , \quad \frac{\partial f_{a_0}^2}{\partial x^2}(p) > 0 , \quad \frac{\partial}{\partial a} f_a^2(p)|_{a=a_0} > 0 ,$$

then there exist a $\varepsilon_0 > 0$ and a neighbourhood U of p such that for $\varepsilon < \varepsilon_0$ and $a \in [a_0, a_0+\varepsilon]$ f_a has no periodic points of period two in U.

We omit the easy proof.

Since the conditions of the lemma are fulfilled for both $F_\varepsilon = (1+\varepsilon)F$ and $F = F + \varepsilon$, F_ε^2 has no fixed points in $(\frac{1}{2}, q_F)$ for ε small and positive. The same arguments as those in the proposition show that then $h(F_\varepsilon) < h(F)$.

Remark. In our example F has a homterval and a bifurcation occurs at $\varepsilon = 0$. For some $\varepsilon < 0$, $|F_\varepsilon^2|'$ has a non-zero local minimum. This is impossible for a map with a negative Schwarzian derivative (see [7]). One can check that, if a map g has no homtervals and the same kneading invariant as F, then for a family $g = g + \varepsilon$ the function $h(\varepsilon)$ is increasing at $\varepsilon = 0$.

III. Some partial results. Now we shall show a fact similar to Hofbauer's result but in a more general case.

THEOREM. *Let $f \in \mathscr{A} \cap C^2$, $\frac{d^2 f}{dx^2} < 0$ for $x \in [0, 1]$, $f(\frac{1}{2}) = 1$. Consider the family $f_a = a \cdot f$ and fix a positive integer $n > 0$. If $f_{a_n}^n(\frac{1}{2}) = \frac{1}{2}$ and $\underline{K}(f_{a_n}) = \mathrm{RLL} \dots \mathrm{LC}$, then for $a < a_n$ $\underline{K}(f_a) < \underline{K}(f_{a_n})$ and for $a > a_n$ $\underline{K}(f_a) > K(f_{a_n})$.*

Proof. Let $f_{a_n}^n(\frac{1}{2}) = \frac{1}{2}$ and $\underline{K}(f_{a_n}) = \mathrm{RLL} \dots \mathrm{LC}$. We shall show that

$$\frac{\partial}{\partial a} f_a^n(\tfrac{1}{2})|_{a=a_n} < 0 . \tag{1}$$

Set $b_i = |f_{a_n}'(f_{a_n}^i(\frac{1}{2}))|$ for $i = 1, 2, \dots, n-1$. We have

$$a_n \frac{\partial}{\partial a}(f_a^n(\tfrac{1}{2}))|_{a=a_n} = \sum_{i=1}^{n} f_{a_n}^i(\tfrac{1}{2})(f_{a_n}^{n-i})'(f_{a_n}^i(\tfrac{1}{2}))$$
$$= -(f_{a_n}(\tfrac{1}{2}) b_1 b_2 \dots b_{n-1} - f_{a_n}^2(\tfrac{1}{2}) b_2 \dots b_{n-1} - \dots - f_{a_n}^{n-1}(\tfrac{1}{2}) b_{n-1} - \tfrac{1}{2})$$
$$= -f_{a_n}(\tfrac{1}{2})\left(b_1 b_2 \dots b_{n-1} - \frac{f_{a_n}^2(\frac{1}{2})}{f_{a_n}(\frac{1}{2})} b_2 \dots b_{n-1} - \dots - \frac{f_{a_n}^n(\frac{1}{2})}{f_{a_n}(\frac{1}{2})}\right).$$

Thus we have to show that

$$b_1 b_2 \dots b_{n-1} - \frac{f_{a_n}^2(\frac{1}{2})}{f_{a_n}(\frac{1}{2})} b_2 \dots b_{n-1} - \dots - \frac{f_{a_n}^n(\frac{1}{2})}{f_{a_n}(\frac{1}{2})} > 0 . \tag{2}$$

Since $f_{a_n}^i(\frac{1}{2}) < f_{a_n}(\frac{1}{2})$, $i = 2, \dots, n$, it is sufficient to prove

$$b_1 b_2 \dots b_{n-1} - b_2 \dots b_{n-1} - \dots - b_{n-2} b_{n-1} - b_{n-1} - 1 > 0 . \tag{3}$$

Set $b = (b_1 b_2 \dots b_{n-1})^{1/(n-1)}$. Because of the concavity of f we have

$$b_1 \geqslant b_2 \geqslant \dots \geqslant b_{n-1} \quad \text{and} \quad (b_i b_{i+1} \dots b_{n-1})^{1/(n-1)} \leqslant b .$$

Hence the following inequalities hold:

$$b_i b_{i+1} \dots b_{n-1} \leqslant b^{n-i}, \quad b_1 b_2 \dots b_{n-1} = b^{n-1}. \tag{4}$$

From the kneading theory we know that f has the same entropy as the subshift of finite type with the matrix:

$$\begin{bmatrix} 0 & 0 & & \dots & 0 & 1 \\ 1 & 0 & & \dots & 0 & 1 \\ 0 & 1 & 0 & \dots & 0 & 1 \\ \dots & & & & & \\ 0 & 0 & \dots & 0 & 1 & 1 \end{bmatrix}.$$

The characteristic polynomial of this matrix is equal to

$$q(x) = x^{n-1} - x^{n-2} - \dots - x - 1 .$$

Since $\exp h(f_{a_n})$ is the largest zero of q, it is sufficient to show that $b \geqslant \exp h(f_{a_n})$. We shall use the formula

$$h(f) = \lim_{k\to\infty} \frac{1}{k} \log \operatorname{Var} f^k = \lim_{k\to\infty} \frac{1}{k} \log \int_{f^2_{a_n}(1/2)}^{f_{a_n}(1/2)} |(f^k)'|$$

(see [6]).

The points $f^i_{a_n}(\frac{1}{2})$ $(i = 0, 1, \ldots, n-1)$ and $(f^i_{a_n}(\frac{1}{2}))^* = 1 - f^i_{a_n}(\frac{1}{2})$ divide the interval $[f^2_{a_n}(\frac{1}{2}), f_{a_n}(\frac{1}{2})]$ into $2n-3$ parts. Write

$$A_1 = [(f^2_{a_n}(\tfrac{1}{2}))^*, f_{a_n}(\tfrac{1}{2})], \quad A^*_i = [(f^{i+1}_{a_n}(\tfrac{1}{2}))^*, (f^i_{a_n}(\tfrac{1}{2}))^*],$$
$$A_i = [f^i_{a_n}(\tfrac{1}{2}), f^{i+1}_{a_n}(\tfrac{1}{2})] \quad (i = 2, \ldots, n-1).$$

For $x \in A_i$ (or $x \in A^*_i$), $|f'_{a_n}(x)| \leqslant b_i$.

The transformation f_{a_n} maps every subinterval A_i (and A^*_i) on the union of some others, namely

$$f_{a_n}(A_1) = A_2,\ f_{a_n}(A_2) = f_{a_n}(A^*_2) = A_3,\ \ldots,\ f_{a_n}(A_{n-2}) = f_{a_n}(A^*_{n-2}) = A_{n-1},$$
$$f_{a_n}(A_{n-1}) = f_{a_n}(A^*_{n-1}) = A_1 \cup A^*_2 \cup \ldots \cup A^*_{n-1}.$$

Hence for $x \in [f^2_{a_n}(\frac{1}{2}), f_{a_n}(\frac{1}{2})]$ $|(f^k_{a_n})'|(x)$ is not larger than an ordered product of numbers b_i with b_i followed by b_{i+1} if $i \leqslant n-2$.

$$|(f^k_{a_n})'(x)| \leqslant (b_{i_1} b_{i_1+1} \ldots b_{n-1})(b_{i_2} b_{i_2+1} + \ldots b_{n-1}) \ldots (b_l b_{l+1} \ldots b_{l+t})$$

where $t < n$. Using (4), we obtain:

$$|(f^k_{a_n})'(x)| \leqslant b^{k-t} g^k \leqslant b^k g^{n-1} \quad \text{where} \quad g = \sup_{y\in[0,1]} |f'_{a_n}(y)|.$$

Thus

$$\frac{1}{k} \log(\operatorname{Var} f^k_{a_n}) = \frac{1}{k} \log \int_{f^2_{a_n}(1/2)}^{f_{a_n}(1/2)} |(f^k_{a_n})'|' \leqslant \frac{1}{k} \log b^k g^{n-1} = \log b + \frac{1}{k} \log g^{n-1}.$$

Hence $h(f_{a_n}) \leqslant \log b$. This inequality ends the proof

Remark. For some maps the proof is easier. Write $\bar{a} = 2f_{a_n}(\frac{1}{2})$ and $g_a(x) = \frac{\bar{a}}{2} - \bar{a}|x - \frac{1}{2}|$. Let p be the closest point to 0 for which $g^{n-2}_{\bar{a}} p = \frac{1}{2}$ (p has an invariant coordinate LL...LC). Since $f_{a_n}(x) \geqslant g_{\bar{a}}(x)$ for $x \in [0, 1]$, p has to be larger than $f^2_{a_n}(\frac{1}{2})$. Moreover, $g^2_{\bar{a}}(\frac{1}{2}) < f^2_{a_n}(\frac{1}{2})$.

Hence, for $(K_i) = \underline{K}(g_{\bar{a}})$ we have

$$K_1 = R,\ K_2 = L,\ \ldots,\ K_i = L \quad \text{and} \quad i \geqslant n.$$

It follows that $\underline{K}(f_{a_n}) < \underline{K}(g_{\bar{a}})$ and $h(f_{a_n}) < h(g_{\bar{a}}) = \log \bar{a}$. Thus $\bar{a} > \exp h(f_{a_n})$. If, moreover,

$$(5) \qquad b_{n-1} \geqslant \bar{a},$$

then $b \geqslant \exp h(f)$, which gives the statement of the theorem.

It is easy to verify that condition (5) holds in the family $a \mapsto ax(1-x)$ for all f_a such that $\underline{K}(f_a) = $ RLL...LC.

I would like to thank M. Misiurewicz for his encouragement and helpful remarks.

References

[1] P. Collet and J. P. Eckmann, *Iterated maps on the interval as dynamical systems*, Bosto 1980.

[2] A. Douadý, *Systèmes dynamiques holomorphes*, Seminaire Bourbaki 599 (1982).

[3] J. Guckenheimer, *Sensitive dependence to initial conditions for one dimensional maps*, Comm. Math. Phys. 70 (1979), pp. 133–160.

[4] F. Hofbauer, *Topological entropy of the family $ax(1-x)$*, preprint.

[5] S. Matsumoto, *On the bifurcation of periodic points of one dimensional dynamical systems of a certain kind*, preprint.

[6] M. Misiurewicz and W. Szlenk, *Entropy of piecewise monotonic mappings*, Asterisque 50 (1977), pp. 299–310.

[7] D. Singer, *Stable orbits and bifurcations of maps of the interval*, SIAM J. Appl. Math 35 (1978), pp. 260–267.

INSTITUTE OF MATHEMATICS
WARSAW UNIVERSITY
00-901 Warszawa, Poland
PKiN, IX p.

Received 23 May 1983