Equivalence relations induced by extensional formulae: classification by means of a new fixed point property

by

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Abstract. Any formula $F(v)$ of PA induces an equivalence relation on ω defined as follows: $x \sim_F y$ iff $\vdash_{PA} F(\bar{x}) \leftrightarrow F(\bar{y})$. It is proved that if $F(v)$ is extensional then $\sim_F$ either is recursively isomorphic to the relation "provable equivalence" (defined in the set of sentences) or enjoys a "fixed point property". The subject is studied within the framework of numeration theory (in particular, we find a characterization of positive precomplete numerations); an algebraic translation is also discussed.

1. Introduction. The theory of numerations, due to Ju. Eršov [4], has been applied to logic first by A. Visser in [13] and then by the authors and A. Sorbi in [1], [10], [12]. We recall that a numeration γ is a pair $\langle \nu, S \rangle$, where S is a non-empty set and ν is a function from ω onto S; given a numeration γ, we can define an equivalence relation $\sim_\gamma$ as follows: $x \sim_\gamma y$ iff $\nu x = \nu y$. In fact, if we identify S with $\omega/\sim_\gamma$, we can speak indifferently of a numeration or an equivalence relation on ω.

A significant example is the numeration γ_{PA} of Peano Arithmetic: actually, Gödel numbers implicitly define a numeration, where S is the Lindenbaum algebra of PA (regarded as the set of equivalence classes of provably equivalent sentences). The corresponding equivalence relation $\sim_{\mathrm{PA}}$ is obviously defined as follows: $x \sim_{\mathrm{PA}} y$ iff the sentences of Gödel numbers x, y are provably equivalent in PA. As may be *expected*, the relation $\sim_{\mathrm{PA}}$ enjoys many *unexpected* properties (see [1] and [10]).

Now, our purpose is to carry this subject further, studying in particular ties between numerations and formulae of PA. (We always consider PA: in fact, we could consider any theory in which recursive functions are representable and which possesses partial truth predicates).

In the sequel, we mostly consider *positive* equivalence relations, i.e. those equivalence relations $\mathscr{R}$ for which the set $\{\langle x, y \rangle;\ x \mathscr{R} y\}$ is r.e. (and, when speaking about relations, we always refer to positive equivalence relations defined on ω, which are non total).

A formula $F(v)$, with only the variable v free, induces a relation $\sim_F$ defined as follows: $x \sim_F y$ iff $\vdash_{\mathrm{PA}} F(\bar{x}) \leftrightarrow F(\bar{y})$. A Theorem of [1] says that every positive equivalence relation coincides with $\sim_F$ for a suitable formula $F(v)$. Now, from

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a metatheoretic point of view, most significant formulae are *extensional*, i.e. preserve provable equivalence (in the sense that $\vdash_{PA} p \leftrightarrow q$ implies $\vdash_{PA} F(\bar{p}) \leftrightarrow F(\bar{q})$). So, we are mainly concerned in the study of the equivalence relations which are induced by extensional formulae. In this case we prove (Theorem 4) that either the induced relation is recursively isomorphic to $\sim_{PA}$ or the formula $F(v)$ enjoys the following fixed point property: for every total recursive function h there exists an n such that $\vdash_{PA} F(\bar{n}) \leftrightarrow F(\overline{hn})$.

But before examining the properties of the relations of the type $\sim_F$, we need some general lemmas, which are stated in Section 2: we provide new characterizations of $\sim_{PA}$ and of precomplete equivalence relations.

In Section 3, we prove the above mentioned result about extensional formulae, as well as a result which can be regarded as an inverse of it. Then we give several conditions for a formula $F(v)$ to induce a relation with the fixed point property and we consider many examples: in particular, we show that, if Theor(v) is the standard formula which numerates the set of theorems of PA, the relation $\sim_{\text{Theor}}$ is precomplete and hence enjoys the fixed point property.

In Section 4 we just discuss some consequences of this fact by means of algebraic methods. In fact, algebra can be very useful in studying fixed point properties, as is shown by C. Smorynski in [11] and by G. Boolos in [2]: we refer, in particular, to the concept of *diagonalizable algebra* introduced by R. Magari. (More generally, our opinion is that both recursive methods and algebraic ones can be fruitfully applied to logic). We find new fixed points related to the formula Theor and translate the situation in algebraic terms: in this way we come to defining an equational class (properly included in the equational class of diagonalizable algebras) of which the first properties are discussed.

2. Precomplete and u.f.p. positive equivalence relations. In order to simplify the notation, we express every notion related to numeration theory in terms of equivalence relations. So, for instance, if $\mathscr{R}$, $\mathscr{S}$ are relations, a *morphism* from $\mathscr{R}$ into $\mathscr{S}$ is regarded as a recursive function h such that, for every n, m, if $n\mathscr{R}m$ then $hn\mathscr{S}hm$. A morphism h is said to be *one-one* if it maps distinct equivalence classes modulo $\mathscr{R}$ in distinct classes modulo $\mathscr{S}$, that is: $n\mathscr{R}m$ iff $hn\mathscr{S}hm$; similarly, h is said to be *onto* if for every m there is an n such that $hn\mathscr{S}m$. Lastly, h is said to be an *isomorphism* if it is one-one and onto (1).

DEFINITION 1 (see [4]). An equivalence relation $\mathscr{R}$ is *precomplete* if for every partial recursive function ψ, there is a total recursive function f such that, for every n, if ψn converges, then $fn\mathscr{R}\psi n$. (We say that f *makes* ψ *total* modulo $\mathscr{R}$.)

PROPOSITION 1 (Eršov fixed point theorem; see [4]). *If $\mathscr{R}$ is a precomplete equivalence relation, then, for every total recursive function f, we can uniformly find a number n_0 such that $fn_0\mathscr{R}n_0$. (By "uniformly" we mean that there is a partial recursive function N such that if z is an index for a total recursive function f, N converges on z and $fNz\mathscr{R}Nz$)* (2).

In view of the application to logic, we are mainly concerned with relations induced by formulae: in such context, we shall see that the fixed point property (in the sense of the previous proposition) has remarkable consequences, whereas the notion of a precomplete relation seems to have minor direct consequences. Therefore, in the sequel we shall consider precompleteness essentially as a criterium for the fixed point property.

DEFINITION 2 (see [10]). An equivalence relation $\mathscr{R}$ is *uniformly finitely precomplete* (in short: *u.f.p.*) if, for every partial recursive function ψ such that range ψ is finite, there is a total recursive function f which makes ψ total modulo $\mathscr{R}$, and an index for such an f can be found in a uniform way starting from an index for ψ and a canonical index for a finite set containing range ψ.

As regards u.f.p. relations, Eršov proof of the fixed point theorem carries over; so we get

PROPOSITION 2 (see [10]). *If $\mathscr{R}$ is a u.f.p. equivalence relation, then, for every partial recursive function ψ having a finite range, there is an n_0 such that, if ψn_0 converges, then $\psi n_0\mathscr{R}n_0$; moreover such an n_0 can be found in a uniform way starting from an index for ψ and a canonical index for a finite set containing range ψ.*

Clearly, every precomplete relation is u.f.p.; the converse, however, does not hold: in fact, the relation $\sim_{PA}$ (provable equivalence in PA) is u.f.p. (see [10]), but it cannot be precomplete because the recursive function $\neg$ has no fixed point (if PA is consistent). In [10], the relation $\sim_{PA}$ is characterized as follows.

PROPOSITION 3. *An equivalence relation $\mathscr{R}$ is isomorphic to $\sim_{PA}$ iff it is positive, u.f.p. and there is a total recursive function Δ such that, for every $a_0, \ldots, a_m$, we have $\Delta\langle a_0, \ldots, a_m\rangle$ not $\mathscr{R}a_i$ for every i.*

Now, we intend to improve this characterization, in order to conclude (Corollary 1) that $\sim_{PA}$ is the unique (up to isomorphism) positive u.f.p. equivalence relation which does not enjoy the fixed point property.

THEOREM 1. *Every positive u.f.p. equivalence relation $\mathscr{R}$ admitting a total recursive diagonal function is isomorphic to $\sim_{PA}$.* (We recall that a function d is said *diagonal* if dx not $\mathscr{R}x$ for every x.)

Proof. Let $\mathscr{R}$ be a positive u.f.p. relation and let d be a total recursive diagonal function for $\mathscr{R}$. We get a Δ as in Proposition 3 as follows. Let $a_0, \ldots, a_m$ be given; define

$$\delta x = \begin{cases} da_i & \text{if } x\mathscr{R}a_i \ (i \leqslant m)\ (^3), \\ \text{divergent otherwise}. \end{cases}$$

(1) Since $\mathscr{R}$ and $\mathscr{S}$ are positive, any isomorphism h admits an inverse isomorphism k (in the sense that $khn\ \mathscr{R}n$ for every n). This property should be explicitly required in defining isomorphism between non positive relations.

(2) As shown in [4], the existence of a total recursive function N as above characterizes precomplete relations.

(3) A formal definition is, as usual, by dovetailing.

Clearly, range δ is finite; therefore, by Proposition 2, we can uniformly find (starting from $a_0, \ldots, a_m$) an n_0 such that, if δn_0 converges, then $\delta n_0 \mathscr{R} n_0$; now, δn_0 diverges, because, by definition, δx not $\mathscr{R} x$ for every $x \in \mathrm{Dom}\,\delta$. Since $\mathrm{Dom}\,\delta = \bigcup_{i \leq m} [a_i]_{\mathscr{R}}$, we can deduce n_0 not $\mathscr{R} a_i$ $(i \leq m)$: therefore it suffices to define $\Delta\langle a_0, \ldots, a_m\rangle = n_0$. $\mathscr{R}$ satisfies the hypothesis of Proposition 3 and hence is isomorphic to $\sim_{\mathrm{PA}}$.

COROLLARY 1. *Let $\mathscr{R}$ be a positive u.f.p. relation; if $\mathscr{R}$ is not isomorphic to $\sim_{\mathrm{PA}}$, then for every total recursive function f, there is an n_0 such that $fn_0 \mathscr{R} n_0$. In other words, $\mathscr{R}$ enjoys the fixed point property.*

Proof. Obvious.

Now, we introduce a simple notion in order to compare equivalence relations; this notion will provide characterizations both for positive u.f.p. relations and precomplete ones (Theorems 2 and 3). We recall (see [1]) that a positive equivalence relation $\mathscr{R}$ is said to be *m-complete* if, for every positive equivalence relation $\mathscr{S}$, there is a total recursive function h such that, for every n, m, we have $n \mathscr{S} m$ iff $hn \mathscr{R} hm$ (in other words, if for every $\mathscr{S}$ as above there is a one-one morphism from $\mathscr{S}$ into $\mathscr{R}$).

DEFINITION 3. Let $\mathscr{R}, \mathscr{S}$ be equivalence relations; we say that $\mathscr{R}$ is *less fine* than $\mathscr{S}$ if, for every n, m, if $n \mathscr{S} m$ then $n \mathscr{R} m$. Moreover, we write $\mathscr{R} \supseteq \mathscr{S}$ for $\mathscr{R}$, up to isomorphism, is less fine than $\mathscr{S}$.

LEMMA 1. (i) *Let $\mathscr{R}$, $\mathscr{S}$ be positive equivalence relations; then $\mathscr{R} \supseteq \mathscr{S}$ iff there is a morphism from $\mathscr{S}$ onto $\mathscr{R}$.*

(ii) (see [1]). *If $\mathscr{R}$ and $\mathscr{S}$ are equivalence relations and $\mathscr{R} \supseteq \mathscr{S}$, then if $\mathscr{S}$ is precomplete (u.f.p.) also $\mathscr{R}$ is.*

Proof. Straightforward.

THEOREM 2. *A positive equivalence relation $\mathscr{R}$ is u.f.p. iff $\mathscr{R} \supseteq \sim_{\mathrm{PA}}$.*

Proof. One direction is an obvious consequence of Lemma 1 (ii). Now, let $\mathscr{R}$ be a positive u.f.p. relation. We define a recursive function f by induction as follows.

Stage 0. We put $f0 = 0$.

Stage $2n+1$. Assume that, at the stages $0, \ldots, 2n$, we have defined f on the set $A_{2n} = \{a_0, \ldots, a_{2n}\}$ in such a way that $a_i \sim_{PA} a_j$ implies $fa_i \mathscr{R} fa_j$; let a_{2n+1} be the smallest natural number which does not belong to A_{2n}; define

$$\psi^{2n+1} x = \begin{cases} fa_i & \text{if } x \sim_{PA} a_i, \\ \text{divergent otherwise}. \end{cases}$$

Since range ψ^{2n+1} is finite and $\mathscr{R}$ is u.f.p., we can uniformly find a function $\bar{\psi}^{2n+1}$ which makes ψ^{2n+1} total modulo $\mathscr{R}$; then we put $fa_{2n+1} = \bar{\psi}^{2n+1} a_{2n+1}$.

Stage $2n+2$. Let u be the smallest natural number which does not belong to the set $\{fa_0, \ldots, fa_{2n+1}\}$. Let Δ be a recursive function such that $\Delta\langle a_0, \ldots, a_m\rangle$ not $\sim_{\mathrm{PA}} a_i$ for every i (see Proposition 3): define $a_{2n+2} = \Delta\langle a_0, \ldots, a_{2n+1}\rangle$ and $fa_{2n+2} = u$.

It is easily seen that f is a morphism from $\sim_{\mathrm{PA}}$ onto $\mathscr{R}$; by Lemma 1, $\mathscr{R} \supseteq \sim_{\mathrm{PA}}$.

THEOREM 3. *A positive equivalence relation $\mathscr{R}$ is precomplete iff $\mathscr{R} \supseteq \mathscr{S}$ for every m-complete relation $\mathscr{S}$.*

Proof ($\Rightarrow$). Let $\mathscr{S}$ be m-complete and let f be a one-one morphism from $\mathscr{R}$ into $\mathscr{S}$; we define a morphism h from $\mathscr{S}$ onto $\mathscr{R}$ by induction, as follows.

Stage 0. Define

$$\psi^0 x = \begin{cases} \text{the first element in the list of } f^{-1}\{y\} \text{ if } x \in [y]_{\mathscr{S}} \\ \quad \text{for some } y \in \text{range } f, \\ \text{divergent otherwise}. \end{cases}$$

Let $\bar{\psi}^0$ make ψ^0 total modulo $\mathscr{R}$; define $h0 = \bar{\psi}^0 0$.

Stage $n+1$. Define

$$\psi^{n+1} x = \begin{cases} \text{the first element in the list of } f^{-1}\{y\} \text{ if } x \text{ first} \\ \quad \text{appears in } [y]_{\mathscr{S}} \text{ for some } y \in \text{range} f, \\ hi \text{ if } x \text{ first appears in } [i]_{\mathscr{S}} \text{ for some } i \leq n, \\ \text{divergent otherwise}. \end{cases}$$

Now, let $\bar{\psi}^{n+1}$ make ψ^{n+1} total modulo $\mathscr{R}$; define $h(n+1) = \bar{\psi}^{n+1}(n+1)$.

By a simple inductive argument, we can show that h is a morphism from $\mathscr{S}$ onto $\mathscr{R}$; therefore, by Lemma 1(i) $\mathscr{R} \supseteq \mathscr{S}$.

($\Leftarrow$). Let $\mathscr{S}$ be any positive precomplete relation; by [1], $\mathscr{S}$ is m-complete, therefore $\mathscr{R} \supseteq \mathscr{S}$; then, by Lemma 1 (ii), $\mathscr{R}$ is in turn precomplete.

3. Positive equivalence relations induced by extensional formulae. The results of Section 2 allow us to characterize, up to isomorphism, the equivalence relations $\sim_F$ which are induced by extensional formulae $F(v)$ (see Introduction). First note that the total relation is induced, for example, by the extensional formula $v = v$. Of course, in the following, we restrict ourselves to the formulae which induce non-total relations.

THEOREM 4. *A positive equivalence relation is isomorphic to a relation induced by an extensional formula iff it is u.f.p. In particular, the equivalence relation induced by an extensional formula either enjoys the fixed point property or is isomorphic to $\sim_{\mathrm{PA}}$.*

Proof. Let $\mathscr{R}$ be isomorphic to $\sim_F$ for some extensional formula $F(v)$; since $F(v)$ is extensional, $\sim_F \supseteq \sim_{\mathrm{PA}}$; since $\sim_{\mathrm{PA}}$ is u.f.p., by Lemma 1 (ii) also $\mathscr{R}$ is.

Vice versa, let $\mathscr{R}$ be a positive u.f.p. relation. By Theorem 2, $\mathscr{R}$ is isomorphic to an equivalence relation $\mathscr{S}$ which is less fine than $\sim_{\mathrm{PA}}$; by [1], there is a formula $F(v)$ such that $\sim_F$ coincides with $\mathscr{S}$. Since $\mathscr{S}$ is less fine than $\sim_{\mathrm{PA}}$, $F(v)$ is extensional.

In the sequel we say that an extensional formula $F(v)$ is of the *first kind* if $\sim_F$ is isomorphic to $\sim_{\mathrm{PA}}$; otherwise, i.e. if $\sim_F$ enjoys the fixed point property, we say that $F(v)$ is of the *second kind*.

We are mainly interested in studying the formulae of the second kind; as regards them, let us point out that this fixed point property is not a direct consequence of Diagonalization Lemma and, in certain respects, is analogous to Recursion Theorem. Our aim is now to provide some useful criteria for deciding whether an extensional formula $F(v)$ is of the second kind; to do this, let us consider the following conditions.

(1) $F(v)$ induce a precomplete relation (and therefore is of the second kind).

(2) The partial recursive function φ defined as follows:

$$\varphi x = \begin{cases} \text{the first element in the list of the set } \{k\colon F(\bar{k}) \sim_{\mathrm{PA}} x\} \\ \qquad \text{if this set is non-empty,} \\ \text{divergent otherwise} \end{cases}$$

can be made total modulo $\sim_F$. (In a sense, we can regard φ as a function inverting the formula $F(v)$).

(3) The domain of $F(v)$ can be "bounded" by Σ_n (for a suitable $n \geqslant 1$) in the following sense: for every sentence p, there is a Σ_n-sentence q such that $\vdash_{\mathrm{PA}} F(\bar{p}) \leftrightarrow F(\bar{q})$ (in other words p and q have the same image under $F(v)$ up to provable equivalence).

(4) For every partial recursive function ψ, there is a formula $H(v)$ such that the function $\lambda x.\, H(\bar{x})$ makes ψ total modulo $\sim_F$.

(5) There is a formula $H(v)$ such that the function $\lambda x.\, H(\bar{x})$ makes the function φ defined in Condition (2) total modulo $\sim_F$.

(6) There is a formula $H(v)$ such that, for every n, $\vdash_{\mathrm{PA}} F(\bar{n}) \leftrightarrow F(\overline{H(\bar{n})})$.

THEOREM 5. *The following implications hold*:

$$(6) \Leftrightarrow (5) \Leftrightarrow (4) \Leftrightarrow (3) \Rightarrow (2) \Leftrightarrow (1).$$

Proof. The implications (4) $\Rightarrow$ (6), (4) $\Rightarrow$ (5), (4) $\Rightarrow$ (1), (6) $\Rightarrow$ (3) and (1) $\Rightarrow$ (2) are trivial; thus, we have only to prove (2) $\Rightarrow$ (1), (5) $\Rightarrow$ (4) and (3) $\Rightarrow$ (4).

(2) $\Rightarrow$ (1). Let h make the above defined function φ total modulo $\sim_F$ and let ψ be any partial recursive function. Define $\bar{\psi}x = F(\overline{\psi x})$ (here and in the following, $F(\overline{\psi x})$ is an abbreviation for $\exists y\,[\psi x = y \wedge F(y)]$; if ψx converges, say $\psi x = y$, then obviously $\vdash_{\mathrm{PA}} F(\overline{\psi x}) \leftrightarrow F(\bar{y})$.

We claim that $h\bar{\psi}$ makes ψ total modulo $\sim_F$. Indeed, suppose that $\psi x = z$; then $\bar{\psi}x \sim_{\mathrm{PA}} F(\bar{z})$; therefore, by the definition of φ, $\varphi\bar{\psi}x$ converges and $F(\overline{\varphi\bar{\psi}x}) \sim_{\mathrm{PA}} \bar{\psi}x \sim_{\mathrm{PA}} F(\bar{z})$; then $h\bar{\psi}x \sim_F \varphi\bar{\psi}x \sim_F z = \psi x$.

(5) $\Rightarrow$ (4). The proof is quite similar to the one of (2) $\Rightarrow$ (1).

(3) $\Rightarrow$ (4). Let ψ be any partial recursive function; define

$$\bar{\psi}x = \begin{cases} \text{the first element in the list of the set } \{k \in \Sigma_n\colon \vdash_{\mathrm{PA}} F(\bar{k}) \leftrightarrow F(\overline{\psi x})\} \\ \qquad \text{if } \psi x \text{ converges,} \\ \text{divergent otherwise.} \end{cases}$$

Now, define $H(v)$ to be the formula $\mathrm{Tr}_n(\overline{\bar{\psi}v})$, where Tr_n is a partial truth predicate for Σ_n-sentences. If ψx converges, then $\bar{\psi}x \sim_{\mathrm{PA}} H(\bar{x})$, whence $\bar{\psi}x \sim_F H(\bar{x})$, since $F(v)$ is extensional; then $\psi x \sim_F \bar{\psi}x \sim_F H(\bar{x})$.

Remark 2. The implication (2) $\Rightarrow$ (3) does not hold: in fact, in Example 1 below we construct a formula which induces a precomplete relation, but does not satisfy Condition (3); let us premise the following lemma.

LEMMA 2. *Let $F(v)$ be an extensional Σ_n-formula; assume that there is a Σ_n-sentence q such that* $\{[F(\bar{m})]_{\sim \mathrm{PA}}\colon m \in \omega\} = \{[p]_{\sim \mathrm{PA}}\colon p \in \Sigma_n$ *and* $\vdash_{\mathrm{PA}} q \to p\}$. (Referring to the Lindenbaum lattice of Σ_n-sentences, we can say that $F(v)$ is *onto* the filter generated by q.) *Then $F(v)$ is of the second kind. The analogous dual statement* ($\vdash_{\mathrm{PA}} p \to q$) *holds too.*

Proof. Let ψ be any partial recursive function and let φ be defined as in Condition (2). Define $\bar{\psi}x$ to be the formula $F(\overline{\psi x}) \vee q$ and put $hx = \varphi\bar{\psi}x$. By our hypothesis, for every x, there is a k such that $\bar{\psi}x \sim_{\mathrm{PA}} F(\bar{k})$; therefore, by the definition of φ, h is total; now, suppose that ψx converges, say $\psi x = y$. Then $\bar{\psi}x \sim_{\mathrm{PA}} F(\bar{y})$, whence $\vdash_{\mathrm{PA}} F(\overline{\varphi\bar{\psi}x}) \leftrightarrow F(\bar{y})$, that is, $hx = \varphi\bar{\psi}x \sim_F y = \psi x$. Then h makes ψ total modulo $\sim_F$.

EXAMPLE 1. Let $p_0, \ldots, p_n, \ldots$ be a primitive recursive list, without repetition, of all Σ_2-sentences; we define a recursive function h from ω onto $\{p_0, \ldots, p_n, \ldots\}$ by stages as follows.

Stage $2n$. Let r_i denote the input of h at the stage i, and let u be the smallest natural number which does not belong to the set $\{r_0, \ldots, r_{2n-1}\}$.

Define

$$\psi^{2n}x = \begin{cases} hr_j \text{ if, for some } j<2n, \text{ we have } x \sim_{\mathrm{PA}} r_j, \\ \text{divergent otherwise;} \end{cases}$$

$$hu = \mathrm{Tr}_2(\overline{\psi^{2n}u}).$$

Stage $2n+1$. Let t be the least v such that $r_i \in \Sigma_v$ for every $i \leqslant 2n$. We can find, uniformly from t, a sentence q provably equivalent to no Σ_t-sentence; then define $hq \doteq p_s$, where s is the smallest z such that $p_z \notin \{hr_0, \ldots, hr_{2n}\}$.

Now, let $A(v) = \mathrm{Tr}_2(\overline{hv})$; we claim that $A(v)$ *is of the second kind, but does not satisfy Condition* (3). First, for every n, $hn \sim_{\mathrm{PA}} A(\bar{n})$; moreover, by a simple inductive argument, we can show that $n \sim_{\mathrm{PA}} m$ implies $hn \sim_{\mathrm{PA}} hm$; therefore $A(v)$ is extensional. Since h is onto the set of all Σ_2-sentences, the hypothesis of Lemma 2 is satisfied: then $A(v)$ is of the second kind. Lastly, let us show that for every n there is a sentence p such that for no Σ_n-sentence q, $\vdash_{\mathrm{PA}} A(\bar{p}) \leftrightarrow A(\bar{q})$. We only sketch the proof. Let n be given and let i be an arbitrary natural number; by the definition of h, for no $l \in \Sigma_i$, $r_{2i+1} \sim_{\mathrm{PA}} l$; moreover, either $r_{2i} \sim_{\mathrm{PA}} r_j$ for some $j<2i$ or hr_{2i} is a false sentence. Now, let us consider a $k \geqslant 2n+1$ such that the following conditions hold:

a) hr_k is a true Σ_2-sentence;

b) for every $i<k$, hr_i not $\sim_{\mathrm{PA}} hr_k$.

It is a matter of routine to verify that for no Σ_n-sentence q we have $hr_k \sim_{PA} hq$; since, for every s, $A(\bar{s}) \sim_{PA} hs$, the formula $A(v)$ does not satisfy Condition (3).

We give four more examples of extensional formulae.

EXAMPLE 2. Theor(v), the usual Σ_1 formula expressing provability in PA, is of the second kind. This is a consequence of the following result, independently obtained by W. Goldfarb and H. Friedman [5].

PROPOSITION 4. *For every Σ_1-sentence p such that* $\vdash_{PA} \neg\mathrm{Con}_{PA} \to p$ *there is a Σ_1-sentence p' such that* $\vdash_{PA} p \leftrightarrow \mathrm{Theor}(\bar{p}')$.

Thus, the hypothesis of Lemma 2 is satisfied because $\vdash_{PA} \neg\mathrm{Con}_{PA} \to \mathrm{Theor}(\bar{p})$ for every p. In Section 4, we shall discuss the consequences of this fact.

EXAMPLE 3. We do not know whether the Rosser Predicate is extensional or not (see [7]); however, we can prove that a slight variant of it is. Indeed, let, for every n, gn compute the end formula of the nth proof in PA; clearly the Rosser predicate $R(v)$ is provably equivalent to the formula

$$\exists u[(gu = v) \wedge \forall w{<}u \neg(gw = \neg v)];$$

now, we define a total recursive function f, which, provably in PA, enumerates the set of theorems of PA and such that the formula

$$R^f(v) = \exists u[(fu = v) \wedge \forall w{<}u \neg(fw = \neg v)]$$

is extensional; the required f is defined by induction as follows.

Stage n. Define $D_{n-1} = \{x\colon fx$ has been defined at the previous stages$\}$; $R_{n-1} = \{p\colon \exists x \in D_{n-1}(fx = p) \wedge \forall y \in D_{n-1}(y<x) \to (fy \neq \neg p)\}$. Let p_n denote the conjunction of all formulae in R_{n-1} (if R_{n-1} is empty, we define p_n to be an arbitrary tautology) and let k be the least number of $\omega - D_{n-1}$; we distinguish three cases:

a) if $p_n \to \neg gn$ is not a tautology, we define $fk = gn$;

b) if $p_n \to \neg gn$ is a tautology and gn is not of the form $\neg q$, we define $fk = \neg gn$, $f(k+1) = gn$;

c) if $p_n \to \neg gn$ is a tautology and gn is of the form $\neg q$, we define $fk = \neg gn$, $f(k+1) = q$, $f(k+2) = gn$.

One can easily formalize in PA the following facts:

1) range f = range g;

2) the set $R_f = \bigcup_{n\in\omega} R_n = \{p\colon \exists x(fx = p) \wedge \forall y{<}x(fy \neq \neg p)\}$ is consistent;

3) R_f is closed under modus ponens, i.e., if p, $p \to q \in R_f$, then $q \in R_f$;

4) if PA is consistent, R_f coincides with the set of theorems of PA; otherwise, R_f is complete.

By 3), we obtain $\vdash_{PA} R^f(\bar{p}) \wedge R^f(\overline{p \to q}) \to R^f(\bar{q})$. Then R^f is extensional.

We claim that $R^f(v)$ is of the *first kind*; indeed, by formalizing the previous properties (2) and (4), we get, respectively, $\vdash_{PA} \neg R^f(\bar{p}) \vee \neg R^f(\overline{\neg p})$ and $\vdash_{PA} \neg\mathrm{Con}_{PA}$ $\to (R^f(\bar{p}) \vee R^f(\overline{\neg p}))$ for every p. Now, if R^f were of the second kind, there would be a sentence q such that $\vdash_{PA} R^f(\bar{q}) \leftrightarrow R^f(\overline{\neg q})$. So, it would follow $\vdash_{PA} \neg R^f(\bar{q})$ and, on the other hand, $\vdash_{PA} \neg\mathrm{Con}_{PA} \to R^f(\bar{q})$; we could conclude $\vdash_{PA} \mathrm{Con}_{PA}$, contradicting the Second Incompleteness Theorem.

EXAMPLE 4. In [1] it is shown that there is an extensional formula $F(v)$ which is *injective* in the sense that, for every two sentences p and q, $\vdash_{PA} p \leftrightarrow q$ iff $\vdash_{PA} F(\bar{p}) \leftrightarrow F(\bar{q})$. Clearly, any injective formula is of the first kind.

EXAMPLE 5. It is know that there are formulae $F(v)$ which induce *homomorphisms* from the Lindenbaum sentence algebra of PA into itself (see for instance [6]). Every such formula is of the first kind, otherwise there would be a p such that $\vdash_{PA} F(\bar{p}) \leftrightarrow F(\overline{\neg p})$; since F induces a homomorphism, we would also have $\vdash_{PA} F(\overline{\neg p}) \leftrightarrow \neg F(\bar{p})$, a contradiction.

Remark. Let $\mathfrak{P}$ be the Lindenbaum sentence algebra of PA; every extensional formula $F(v)$ induces a mapping $\tilde{F}$ from $\mathfrak{P}$ into $\mathfrak{P}$. It can be proven that $F(v)$ is of the first kind iff there is an injective formula $G(v)$ such that range $\tilde{G}$ = range $\tilde{F}$; in particular, any two extensional formulae $F(v)$, $G(v)$ such that range $\tilde{F}$ = range $\tilde{G}$ are of the *same kind*.

4. Hyperdiagonalizable algebras.

NOTATION. In this section, we denote Boolean operations by the symbols $+$, $\cdot$, v, 0, 1; the terms [4] $x \to y$ and $x - y$ are defined in the usual way. We shall not distinguish between an algebra and its base set.

R. Magari introduced the concept of a diagonalizable algebra in order to study the properties of the formula Theor(v) in an algebraic context. We recall that a *diagonalizable algebra* (in short, a DA) is a Boolean algebra enriched with a unary operation τ satisfying the following identities:

$$\tau 1 = 1; \quad \tau(x\cdot y) = \tau x \cdot \tau y; \quad \tau x \leqslant \tau\tau x; \quad \tau(\tau x \to x) = \tau x$$

(for the properties of DA's see [11]). In the sequel we shall use the symbol $\tau^n x$ (where $n \in \omega$) defined as follows:

$$\tau^0 x = x \cdot \tau x \quad \text{and} \quad \tau^{n+1} x = \tau\tau^n x.$$

For instance, considering the Lindenbaum algebra of PA and defining $\tau[p] = [\mathrm{Theor}(\bar{p})]$, we get the diagonalizable algebra of PA, which will be denoted by $\mathfrak{P}$.

Diagonalizable algebras have been deeply studied, both in an algebraic context and in the language of modal logic. Among other things, a fixed point theorem has been proved to hold for every diagonalizable algebra: for every term (or, more generally, for every polynomial) fx in which x occurs only within the scope of a τ, there exists in every diagonalizable algebra a unique a such that $fa = a$. Therefore,

(4) The meaning of the words *term* and *polynomial* is defined as in [3]: in particular, a polynomial is a term in which some variables have been substituted by elements of the considered algebra.

an algebraic translation of Diagonalization Lemma holds true in every diagonalizable algebra.

Now, since $\sim_{\text{Theor}}$ is a precomplete equivalence relation, from the Eršov fixed point theorem it follows that, in fact, many other fixed points do exist in the diagonalizable algebra $\mathfrak{P}$. Namely, for every function $f\colon \mathfrak{P}\to\mathfrak{P}$ which is induced by a total recursive function, there exists a sentence q such that $\tau f[q] = \tau[q]$.

From an algebraic point of view, this suggests the study of a subclass of the equational class of diagonalizable algebras: the subclass constituted by the diagonalizable algebras in which

(*) *for every term px there exists an a such that $\tau a = \tau pa$.*

This requirement is obviously satisfied in the so-called trivial diagonalizable algebras, i.e. in the diagonalizable algebras in which $\tau x = 1$ for every x, but it fails to be satisfied in every diagonalizable algebra. For instance, consider the equation $\tau x = \tau\nu x$ (a solution of which corresponds, by a logical point of view, to a Rosser sentence). If a is a solution, $\tau a = \tau\nu a = \tau(a\cdot\nu a) = \tau 0$; but on the other hand we have the following simple lemma.

Lemma 3. *In every non trivial diagonalizable algebra satisfying the identity*

(lin) $$\tau(x\cdot\tau x\to y)+\tau(y\cdot\tau y\to x) = 1$$

there is no element a such that $\tau a = \tau\nu a$.

(The identity (lin) has been studied in [9]; it holds in the free diagonalizable algebra on the empty set and in the Lindenbaum algebra of ZF, where τx translates the formula "x is true in any natural model of ZF").

Proof. By contradiction, let a satisfy $\tau a = \tau\nu a = \tau 0$. Substituting a and νa for x and y respectively in (lin), we can easily get

$$\tau(\tau\nu a\to\nu a)+\tau(\tau a\to a) = 1\,,\quad \text{i.e.}\quad \tau\nu a+\tau a = 1\,.$$

So, we conclude $\tau 0 = 1$ and the algebra is trivial.

Now, in view of condition (*), we give the following

Definition 7. A *hyperdiagonalizable algebra* (in short an HDA) is a structure $\mathfrak{B} = \langle B, +, \cdot, \nu, \tau, 0, 1, (c_n)_{n\in\omega}\rangle$ where the following conditions hold:

(a) $\langle B, +, \cdot, \nu, \tau, 0, 1\rangle$ is a DA;

(b) denoting the terms in just one variable by $p_0x, p_1x, p_2x, \ldots$, for every $n\in\omega$ we have $\tau p_n c_n = \tau c_n$ (where, of course, only the constants c_i with $i<n$ can occur in p_nx).

It is clear that HDA's constitute an equational class.

The first example of an HDA is of course the DA $\mathfrak{P}$ enriched with a suitable succession of constants (whose existence is granted by the precompleteness of $\sim_{\text{Theor}}$). Actually, we can find simpler examples and, in particular, finite non-trivial HDA's.

Let us start by considering the equations of the form $\tau(\nu x+\tau^n 0) = \tau x$ [5]. Let us assume that each of these equations admits a fixed point a_n.

Lemma 4. *If a satisfies the equation $\tau(\nu x+b) = \tau x$, then $\tau a = \tau b$. In particular, $\tau a_n = \tau^{n+1}0$.*

Proof. It is obvious that $\tau a = \tau(\nu a+b)\geqslant\tau b$. Conversely, $\tau a = \tau a\cdot\tau(\nu a+b) = \tau(a\cdot b)\leqslant\tau b$.

Lemma 5. (i) *We can assume that $a_n\leqslant\tau^{n+1}0$ for every $n\in\omega$.*

(ii) *For any m, the constants a_n can be chosen in such a way that $a_n\leqslant a_{n+1}$ for $n\leqslant m$;*

(iii) *If the* HDA, *as a Boolean algebra, is isomorphic to $\mathscr{P}(X)$ for a suitable X, the constants a_n can be chosen in such a way that $a_n\leqslant a_{n+1}$ for every n.*

Proof. (i) If s is an element such that $\tau(\nu s+\tau^n 0) = \tau s$, also $s\cdot\tau^{n+1}0$ satisfies the same equation: indeed

$$\tau\big(\nu(s\cdot\tau^{n+1}0)+\tau^n 0\big) = \tau\big(\nu\tau^{n+1}0+(\nu s+\tau^n 0)\big) = \tau\big(\tau(\nu s+\tau^n 0)\to(\nu s+\tau^n 0)\big)$$
$$= \tau(\nu s+\tau^n 0) = \tau^{n+1}0 = \tau(s\cdot\tau^{n+1}0)\,.$$

(ii) If s is as above, $s\cdot a_{n+1}$ can be taken as a_n: indeed

$$\tau\big(\nu(s\cdot a_{n+1})+\tau^n 0\big) = \tau\big(a_{n+1}\to(\nu s+\tau^n 0)\big)\leqslant\tau^{n+2}0\to\tau^{n+1}0\,.$$

We get

$$\tau\big(\nu(s\cdot a_{n+1})+\tau^n 0\big) = \tau\big(\nu(s\cdot a_{n+1})+\tau^n 0\big)\cdot\tau^2\big(\nu(s\cdot a_{n+1})+\tau^n 0\big) = \tau^{n+1}0\,.$$

(iii) (We assume some knowledge of representation theory for DA's — see [8], [11]). Let $K_n = \{\alpha\in X\,|\,\exists\beta\in\tau^{n+1}0 \text{ and } \beta<\alpha\}$. On the one hand, it is necessarily $K_n\subseteq a_n$ and, on the other, we can assume $a_n\subseteq K_{n+1}$ (the argument is similar to the previous ones).

In order to construct a simple example of an infinite non-trivial HDA we consider the Boolean algebra $\mathfrak{H} = \mathscr{P}(\omega)$ and define a binary relation $\dot{<}$ in ω as follows:

$n\dot{<}m$ iff n is greater than m and n is even.

Since the relation $\dot{<}$ is transitive and reversely well founded, if we put, for every $X\subseteq\mathscr{P}(\omega)$, $\tau X = \{n\,|\,\forall m\,(n\dot{<}m\Rightarrow m\in X)\}$, we get a DA (see [8]). It is easy to verify that, for instance, $\tau 0$ is the set $\{m\,|\,m = 0 \text{ or } m \text{ is odd}\}$ and $\tau^{n+1}0-\tau^n 0 = \{2n\}$.

In the DA $\mathfrak{H}$ there exist constants a_n satisfying the above considered equations: indeed, we can define $a_n = \{0, 1, \ldots, 2n\}$ and it is readily seen that each a_n satisfies the corresponding equation. (The representation of $\mathfrak{H}$ as well as the constants a_n are illustrated by Figure 1). More generally, we have

Theorem 6. *$\mathfrak{H}$ is an* HDA, *i.e. in $\mathfrak{H}$ the equation $\tau x = \tau px$ admits a solution, where px is any term or any polynomial.*

[5] The choice of these equations is quite natural: indeed, if other equations of the same degree of complexity are considered, a solution can be easily found.

Proof. If the constant 1 is not the desired solution, then $p1$ is different from 1: let n be the lowest number not belonging to $p1$. Let us distinguish two cases:

(i) n is even; we put $a = \{0, 1, \ldots, n\}$ (i.e., a is equal to the closure of n with respect to $\dot{<}$):

(ii) n is odd; we put $a = \{0, 1, \ldots, n-2, n\}$.

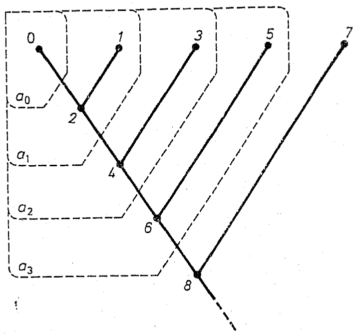

Fig 1

We claim that a is a solution of the given equation. We need the following

LEMMA 6. *For every polynomial px, if $x \cdot m = y \cdot m$, then $px \cdot \tau m \cdot m = py \cdot \tau m \cdot m$, where x, y, m are elements of any* DA.

Proof of the lemma. It suffices to note that x and y are congruent modulo the τ-filter generated by m, which is the Boolean filter generated by $m \cdot \tau m$.

End of the proof of Theorem 6. From the obvious equality $1 \cdot a = a \cdot a$, by Lemma 6 it follows $p1 \cdot \tau a \cdot a = pa \cdot \tau a \cdot a$. Since $\tau a \cdot a = a$, we have $pa \cdot a = p1 \cdot a = a - \{n\}$. Now, in both case (i) and case (ii), it is only a matter of routine to verify that, on the one side, $\tau pa = \{0, 1, \ldots, n\} + \tau 0$ and, on the other, $\tau a = \{0, 1, \ldots, n\} + \tau 0$, as required.

As a consequence of Theorem 6, it is now easy to construct finite (non-trivial) HDA's: it suffices to consider the quotients of $\mathfrak{H}$ modulo the filters generated by a_n (i.e. the Boolean algebras $\mathscr{P}(a_n)$, where the relation $\dot{<}$ between the elements of a_n is defined as above).

We conclude with a conjecture: the existence of the constants a_n is a sufficient condition for a DA to satisfy (*); in other words, if a DA possesses the constants a_n, then it can be made an HDA (the conjecture is suggested by the following fact: as is shown in Smorynski [11], if in the definition of DA's we assume only the first three identities concerning τ and an algebraic version of Diagonalization Lemma, the last identity can be proven. So this identity may be regarded as an instance of

Diagonalization Lemma, but this single instance is enough to redemonstrate Diagonalization Lemma in its full generality; analogously, maybe that from the existence of the constants a_n one can deduce that condition (*) holds in general) (6).

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(6) Added in proof. As shown in a forthcoming paper by the first author and M. Mirolli, the conjecture is true if the terms that are considered in (*) are built up by using only the operations of DA's while it is false if also the constans a_n are allowed to occur in the terms.