An index and a Nielsen number for n-valued multifunction

by

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Abstract. A multifunction is called *n-valued* if all its point images consist of exactly n points. The fact that the fixed point set of an n-valued continuous multifunction $\varphi\colon |K| \to |K|$ from a compact polyhedron $|K|$ to itself is generically finite is used to introduce a fixed point index for such multifunctions. Fixed point classes of φ are defined with the help of the Splitting Lemma which shows that any restriction of φ to a contractible subset is equivalent to n single-valued functions. Hence a Nielsen number $N(\varphi)$ is obtained which is a lower bound for the number of fixed points of φ, and the homotopy invariance of $N(\varphi)$ is proved.

1. Introduction. A multifunction $\varphi\colon X \to Y$ is called *n-valued* if $\varphi(x)$ consists, for all $x \in X$, of exactly n points. It is known that n-valued continuous multifunctions inherit from single-valued functions some of those properties which are basic in fixed point theory. A multi-valued analogue of the simplicial approximation theorem [7], Theorem 4 can be used to show that the fixed point set Fix φ $= \{x \in X \mid x \in \varphi(x)\}$ of an n-valued continuous multifunction $\varphi\colon X \to X$ is generically finite if X is a compact polyhedron ([7], Theorem 6; see also Theorem 2.2 below). Lefschetz numbers $L(\varphi)$ with the usual property that $L(\varphi) \neq 0$ implies that φ has at least one fixed point have been obtained by B. O'Neill [4] for a class of multifunctions which includes the n-valued continuous ones as a special case.

Here we are interested in the behaviour of such multifunctions with respect to Nielsen fixed point theory. If $f\colon X \to X$ is a map (i.e. a single-valued continuous function) on a compact ANR, then a Nielsen number $N(f)$ can be defined which is a lower bound for the number of fixed points for all maps in the homotopy class of f. Hence $N(f) \neq 0$ implies that every map homotopic to f has not only one, but at least $N(f)$ fixed points. To obtain $N(f)$ the fixed points of f are first divided into finitely many equivalence classes, the so-called fixed point classes, and an index is associated with each fixed point class. The Nielsen number $N(f)$ is the number of fixed point classes which have a non-zero index. (See e.g. [2], Chapter VI, or [9].)

In order to introduce a Nielsen number for n-valued continuous multifunctions it is therefore necessary to develop first a fixed point index for such functions. This will be done in §§ 3 and 4. We start with the definition of the fixed point index of an isolated fixed point with a Euclidean neighbourhood. It follows from the Splitting Lemma 2.1 (see also [7], Lemma 1) that an n-valued continuous multifunction is locally (but not globally) equivalent to n single-valued continuous functions, and thus the fixed point index of an isolated fixed point can simply be defined as the fixed point index of an isolated fixed point of a map. We then define the index of a multifunction with a finite fixed point set additively, and finally use the Fix-Finite Approximation Theorem 2.2 (see also [7], Theorem 6) to define the fixed point index in general. This fixed point index has the properties of localization (Theorem 4.3), additivity (Theorem 4.4) and homotopy invariance (Theorem 4.5).

Fixed point classes and the Nielsen number are introduced in § 5, and the Splitting Lemma 2.1 is again a crucial tool in these definitions. The proof of the homotopy invariance of the Nielsen number in § 6 uses the Splitting Lemma as well as the work by U. K. Scholz [9]. Some examples of Nielsen numbers are discussed in the final paragraph.

Many problems remain. If $n = 1$, then our results reduce to the corresponding ones for maps, and these exist for maps on compact ANR's [9], while we consider multifunctions on compact polyhedra only. We deal neither with the computation of the Nielsen number $N(\varphi)$ of a multifunction, nor with its realization, i.e. with the possible existence of an n-valued continuous multifunction in the (multi-) homotopy class of φ which has precisely $N(\varphi)$ fixed points.

The multifunctions considered here are very special ones, and the only other multifunctions for which a Nielsen number exists, the so-called small ones [5], are quite different in nature. But fixed points of symmetric product maps can be interpreted as fixed points of certain finite-valued multifunctions [8] § 6, and a Nielsen number for symmetric product maps has been obtained by S. Masih [3]. It may also be possible to extend some of the results of this paper to finite-valued continuous multifunctions for which all point images consist of either one or exactly n points, especially as Lefschetz numbers for such multifunctions have been defined by O'Neill [4]. But the definition of a meaningful Nielsen number for acyclic upper semi-continuous multifunctions, which behave so well with respect to Lefschetz fixed point theory, seems to be more difficult.

2. Background. We shall start with some definitions and results concerning n-valued multifunctions. Of these only Theorem 2.3 is new, the rest of the material can be found in [7]. A basic reference for multifunctions is e.g. [1], Chapter VI.

A multifunction $\varphi\colon X\to Y$ from a topological space X to a topological space Y is a correspondence which assigns to each point $x\in X$ a non-empty subset $\varphi(x)$ of Y. The multifunction φ is called *upper semi-continuous* (usc) if $\varphi(x)$ is closed for all $x\in X$ and if for each open set V with $\varphi(x)\subset V\subset Y$ there exists an open set U with $x\in U\subset X$ and $\varphi(U)\subset V$. It is called *lower semi-continuous* (lsc) if for each $x\in X$ and open set $V\subset Y$ with $\varphi(x)\cap V\neq\emptyset$ there exists an open set U with $x\in U\subset X$ and $\varphi(x')\cap V\neq\emptyset$ for all $x'\in U$. If φ is both usc and lsc, then it is called *continuous*. By a map we always mean a single-valued continuous function.

A multifunction $\varphi\colon X\to Y$ *splits into n distinct maps* if

$$\varphi(x) = \{f_1(x), f_2(x), \ldots, f_n(x)\}$$

for all $x\in X$, where $f_j\colon X\to Y$ are maps with $f_j(x)\neq f_k(x)$ for every $j, k = 1, 2, \ldots, n$ and $j\neq k$. We shall write $\varphi = \{f_1, f_2, \ldots, f_n\}$, and simply call this a *splitting* of φ, as only splittings into distinct maps are considered here. The following lemma ([7], Lemma 1) is a basic tool in [7] as well as in this paper.

SPLITTING LEMMA 2.1. *Let X and Y be compact Hausdorff. If X is path connected and simply connected and $\varphi\colon X\to Y$ is n-valued and continuous, then φ splits into n distinct maps.*

Now denote by $|K|$ a compact polyhedron which is the realization of a finite simplicial complex K, and define the distance $\bar{d}(\varphi, \psi)$ between two multifunctions $\varphi, \psi\colon X\to |K|$ by

$$\bar{d}(\varphi, \psi) = \sup\{\varrho(\varphi(x), \psi(x))\mid x\in X\},$$

where ϱ is the Hausdorff metric on $|K|$ induced by the barycentric metric d. A multifunction $\varphi\colon X\to X$ is called *fix-finite* if its fixed point set $\mathrm{Fix}\,\varphi$ is finite. The next theorem will be used in the definition of the fixed point index. It follows from [7], Theorem 6 and its proof.

THEOREM 2.2 (Fix-finite approximation of n-valued multifunctions). *Let $|K|$ be a compact polyhedron and $\varphi\colon |K|\to|K|$ an n-valued continuous multifunction. Given $\varepsilon>0$, there exists an n-valued continuous multifunction $\varphi'\colon |K|\to|K|$ with the following properties*:

(i) *φ' is fix-finite,*

(ii) *the fixed points of φ' are contained in maximal simplexes of $|K|$,*

(iii) $\bar{d}(\varphi, \varphi')<\varepsilon$.

To prove the homotopy invariance of the fixed point index, we shall need an extension of Theorem 2.2 to homotopies. A (multivalued) homotopy is a multifunction $\Phi\colon X\times I\to Y$ where $I = [0, 1]$, and if $X = Y$, then the *fixed point set of the homotopy Φ* is defined as

$$\mathrm{Fix}\,\Phi = \{(x, t)\in X\times I\mid x\in\Phi(x, t)\}.$$

Φ is called a fix-finite homotopy if the multifunction $\varphi_t\colon X\to X$ given by $\varphi_t(x) = \Phi(x, t)$ is fix-finite for all $t\in I$. A *hyperface* of the polyhedron $|K|$ is an open simplex σ so that $\bar{\sigma} = \bar{\sigma}'\cap\bar{\sigma}''$, where σ' and σ'' are maximal simplexes of $|K|$ and $\bar{\sigma}$ denotes the closed simplex corresponding to σ. (The definition in [6], p. 532 is wrong, and should be corrected.)

THEOREM 2.3 (Fix-finite approximation of n-valued homotopies). *Let $|K|$ be a compact polyhedron, let $|K_1|\subseteq|K|$ be a subpolyhedron and let $\Phi\colon |K_1|\times I\to|K|$ be an n-valued continuous homotopy such that φ_0 and φ_1 are fix-finite and have all their fixed points located in maximal simplexes of $|K_1|$. Given $\varepsilon>0$, there exists an* *n-valued continuous homotopy* Φ': $|K_1|\times I\to|K|$ *from* $\varphi_0'=\varphi_0$ *to* $\varphi_1'=\varphi_1$ *with the following properties*:

(i) Φ' *is fix-finite*,

(ii) *the fixed points of each* φ_t' *are located in maximal simplexes or hyperfaces of* $|K_1|$,

(iii) *Fix* Φ' *is a one-dimensional finite polyhedron in* $|K_1|\times I$ *so that no edge lies is a section* $|K_1|\times\{t\}$ *of* $|K_1|\times I$,

(iv) $\bar{d}(\Phi,\Phi')<\varepsilon$.

Theorem 2.3 reduces to [6], Theorem 2 if $n=1$ and $K_1=K$. The proof of Theorem 2.3 is fairly long and technical, but requires nothing more than a combination of the techniques used in the proofs of [6], Theorem 2 and [7], Theorem 6. It is therefore omitted.

3. A fixed point index for n-valued multifunctions: Fix-finite case. Our aim is to define a fixed point index for all n-valued continuous multifunctions on compact polyhedra. For this purpose we first define the index of an isolated fixed point with a Euclidean neighbourhood, and then use additivity and the Fix-Finite Approximation Theorem 2.2 to extend the scope of the definition.

So let φ: $|K|\to|K|$ be an n-valued continuous multifunction on a compact polyhedron $|K|$ and x an isolated fixed point of φ which lies in a maximal simplex σ. We use the Splitting Lemma 2.1 to obtain a splitting $\varphi|\bar{\sigma}=\{f_1,f_2,\ldots,f_n\}$, where $f_j(x)=x$ and hence $f_k(x)\neq x$ if $k\neq j$. We define the *fixed point index of* φ *at* x as $\operatorname{ind}(\varphi,x)=\operatorname{ind}(f_j,x)$, where $\operatorname{ind}(f_j,x)$ is the ordinary fixed point index of the map f_j at x ([2], p. 122). If φ is fix-finite on the open set $U\subset|K|$ and if all points of $\operatorname{Fix}\varphi\cap U\neq\varnothing$ lie in maximal simplexes, we define the *fixed point index of* φ *on* U as

$$\operatorname{ind}(\varphi,U)=\sum(\operatorname{ind}(\varphi,x)|\ x\in\operatorname{Fix}\varphi\cap U).\tag{3.1}$$

Finally we put $\operatorname{ind}(\varphi,U)=0$ if $\operatorname{Fix}\varphi\cap U=\varnothing$. Hence $\operatorname{ind}(\varphi,U)$ is the ordinary fixed point index if $n=1$, i.e. if φ is single-valued.

The next lemma implies that $\operatorname{ind}(\varphi,U)$ is homotopy invariant. Other properties of the index will be derived in the more general setting of the next paragraph, but the definitions and proofs of § 4 will make use of Lemma 3.2. The inequality

$$d(x,A)\leqslant d(x,B)+\varrho(A,B)$$

for every point $x\in|K|$ and subsets $A,B\subset|K|$, which is an easy consequence of the definition of d and ϱ, will occur in the next and in several later proofs. We write $\operatorname{Cl}A$ and $\operatorname{Bd}A$ to denote the closure and the boundary of the set A.

LEMMA 3.2. *Let* $U\subset|K|$ *be an open subset of* $|K|$ *and* φ,ψ: $|K|\to|K|$ *be two n-valued continuous multifunctions which are fix-finite on U and have all their fixed points on U located in maximal simplexes. Let* Φ: $\operatorname{Cl}U\times I\to|K|$ *be an n-valued continuous homotopy from* $\varphi_0=\varphi|\operatorname{Cl}U$ *to* $\varphi_1=\psi|\operatorname{Cl}U$ *so that* $\operatorname{Fix}\Phi\cap(\operatorname{Bd}U\times I)=\varnothing$. *Then* $\operatorname{ind}(\varphi,U)=\operatorname{ind}(\psi,U)$.

Proof. Let pr: $\operatorname{Cl}U\times I\to\operatorname{Cl}U$ be the projection onto the first factor. As $\operatorname{Fix}\Phi\cap(\operatorname{Bd}U\times I)=\varnothing$ and $\operatorname{Fix}\Phi$ is compact, there exists $\xi>0$ so that

$$\operatorname{pr}(\operatorname{Fix}\Phi)\subset N(\operatorname{pr}(\operatorname{Fix}\Phi),\xi)\subset U,$$

where $N(\operatorname{pr}(\operatorname{Fix}\Phi),\xi)=\{x\in|K|\,|\,d(x,\operatorname{pr}(\operatorname{Fix}\Phi))<\xi\}$. Now let K' be a subdivision of K with mesh $\mu(K')<\xi/2$, and denote by $\operatorname{st}_{K'}v$ the (open) star of the vertex v in $|K'|$. If

$$V=\bigcup[\operatorname{st}_{K'}v|\ \operatorname{st}_{K'}v\cap\operatorname{pr}(\operatorname{Fix}\Phi)\neq\varnothing],$$

then V is open in $|K|$, $\operatorname{Cl}V$ is a subpolyhedron of $|K'|$ and

$$\operatorname{pr}(\operatorname{Fix}\Phi)\subset V\subset N(\operatorname{pr}(\operatorname{Fix}\Phi),\xi)\subset U.$$

As $\operatorname{pr}(\operatorname{Fix}\Phi)\cap\operatorname{Bd}V=\varnothing$, there exists $\eta\geqslant0$ so that $d(x,\varphi_t(x))>\eta$ for all $(x,t)\in\operatorname{Bd}V\times I$. We now replace $\Phi|\operatorname{Cl}V\times I$ by a fix-finite homotopy Φ': $\operatorname{Cl}V\times I\to|K|$ with the properties (i) to (iii) of Theorem 2.3 and $\bar{d}(\Phi|\operatorname{Cl}V\times I,\Phi')<\eta/2$. Then for all $(x,t)\in\operatorname{Bd}V\times I$

$$d(x,\varphi_t'(x))\geqslant d(x,\varphi_t(x))-\varrho(\varphi_t(x),\varphi_t'(x))>0,$$

and therefore it follows easily from the Splitting Lemma 2.1 and from [6], Propositions 2 and 3 that

$$\operatorname{ind}(\varphi_0,V)=\operatorname{ind}(\varphi_1,V).$$

But φ and ψ are fixed point free on $\operatorname{Cl}U-V$, so it follows from the additive definition (3.1) of the fixed point index that

$$\operatorname{ind}(\varphi,U)=\operatorname{ind}(\varphi_0,V)=\operatorname{ind}(\varphi_1,V)=\operatorname{ind}(\psi,U).$$

4. A fixed point index for n-valued multifunctions: general case. We start the extension of the definition of the fixed point index with a lemma. It will be stated for compact polyhedra, but is true for compact ANR's, and generalizes a well-known result for maps [2], p. 40, Corollary 4. We say that two n-valued continuous multifunctions φ_0,φ_1: $|K|\to|K|$ are *ε-homotopic* if there exists an n-valued continuous homotopy Φ: $|K|\times I\to|K|$ from φ_0 to φ_1 so that $\bar{d}(\varphi_s,\varphi_t)<\varepsilon$ for all $s,t\in I$. The *gap* $\gamma(\varphi)$ of an n-valued continuous multifunction φ: $X\to|K|$ was defined in [7], § 3 as

$$\gamma(\varphi)=\inf\{d(y_j,y_k)|\ y_j,y_k\in\varphi(x),\ x\in X,\ y_j\neq y_k\},$$

and if X is compact, then $\gamma(\varphi)>0$.

LEMMA 4.1. *Let* $|K|$ *be a compact polyhedron,* $A\subset|K|$ *a closed subset of* $|K|$ *and* φ: $A\to|K|$ *an n-valued continuous multifunction. Given* $0<\varepsilon<\gamma(\varphi)/2$, *there exists* $0<\delta\leqslant\varepsilon$ *such that every n-valued continuous multifunction* ψ: $A\to|K|$ *with* $\bar{d}(\varphi,\psi)<\delta$ *is ε-homotopic to* φ.

Proof. Choose $0<\varepsilon<\gamma(\varphi)/2$. It follows from [2], p. 39 that $\delta>0$ can be determined so that there exists a map $w\colon W\times I\to|K|$ on the space

$$W=\{(x,x')\in|K|\times|K|\mid d(x,x')<\delta\}$$

with $w(x,x',0)=x$, $w(x,x',1)=x'$ and $d(w(x,x',s),w(x,x',t))<\varepsilon$ for all $x,x'\in|K|$ and $s,t\in I$. Clearly, $\delta\leqslant\varepsilon$.

For any $x\in A$ let $\varphi(x)=\{y_1,y_2,\dots,y_n\}$. As $\bar d(\varphi,\psi)<\delta<\gamma(\varphi)/2$, there exists a unique indexing of the image points in $\psi(x)=\{z_1,z_2,\dots,z_n\}$ so that $d(y_j,z_k)<\delta$ if and only if $j=k$. Define $\varphi_t\colon A\to|K|$ by

$$\varphi_t(x)=\{w(y_1,z_1,t),w(y_2,z_2,t),\dots,w(y_n,z_n,t)\}$$

for all $t\in I$. Then

$$d(w(y_j,z_j,s),w(y_j,z_j,t))<\varepsilon$$

and

$$d(w(y_j,z_j,t),w(y_k,z_k,t))\geqslant\gamma(\varphi)-2\varepsilon>0\ \text{for all}\ j\neq k$$

imply that the multifunction $\Phi\colon A\times I\to|K|$ given by $\Phi(x,t)=\varphi_t(x)$ is the desired ε-homotopy.

We now call the triple $(|K|,\varphi,U)$ *admissible* if $\varphi\colon|K|\to|K|$ is an n-valued continuous multifunction on a compact polyhedron $|K|$ and U an open subset of $|K|$ so that $\operatorname{Fix}\varphi\cap\operatorname{Bd}U=\emptyset$. We shall define the fixed point index $\operatorname{ind}(\varphi,U)$ generically.

If $U\neq\emptyset$, let $\varepsilon'=\inf(d(x,\varphi(x))\mid x\in\operatorname{Bd}U)$ and $\varepsilon=\min(\varepsilon',\gamma(\varphi))>0$, where $\gamma(\varphi)$ is the gap of φ. We now use Lemma 4.1 to select $0<\delta\leqslant\varepsilon/2$ so that any n-valued continuous multifunction $\psi\colon|K|\to|K|$ with $\bar d(\varphi,\psi)<\delta$ is $\varepsilon/2$-homotopic to φ. Then we use the Fix-Finite Approximation Theorem 2.2 to obtain a fix-finite n-valued continuous multifunction $\varphi'\colon|K|\to|K|$ so that all fixed points of φ' are situated in maximal simplexes of $|K|$ and $\bar d(\varphi,\varphi')<\delta/2$. For all $x\in\operatorname{Bd}U$

$$d(x,\varphi'(x))\geqslant d(x,\varphi(x))-\varrho(\varphi(x),\varphi'(x))\geqslant\varepsilon-\delta/2>0\,,$$

so $\operatorname{ind}(\varphi',U)$ exists. Therefore we define for any admissible triple $(|K|,\varphi,U)$, the *fixed point index of* φ *on* U by

$$\operatorname{ind}(\varphi,U)=\operatorname{ind}(\varphi',U)\,. \tag{4.2}$$

It is necessary to show that this definition is independent of the choice of φ'.

Hence let φ'_i, where $i=1,2$, be two such choices. As $\bar d(\varphi,\varphi'_i)<\delta/2$, we have $\bar d(\varphi'_1,\varphi'_2)<\delta$, so φ'_1 and φ'_2 are $\varepsilon/2$-homotopic. We call this $\varepsilon/2$-homotopy Φ'. Now for all $(x,t)\in|K|\times I$

$$\begin{aligned}\varrho(\varphi(x),\varphi'_t(x))&\leqslant\varrho(\varphi(x),\varphi'_1(x))+\varrho(\varphi'_1(x),\varphi'_t(x))\\&\leqslant\delta/2+\varepsilon/2\leqslant3\varepsilon/4\,.\end{aligned}$$

So if $(x,t)\in\operatorname{Bd}U\times I$, then

$$d(x,\varphi'_t(x))\geqslant d(x,\varphi(x))-\varrho(\varphi(x),\varphi'_t(x))>0\,,$$

which shows that Φ' is fixed point free on $\operatorname{Bd}U\times I$, and therefore $\operatorname{ind}(\varphi'_1,U)=\operatorname{ind}(\varphi'_2,U)$ is a consequence of Lemma 3.2.

We finally define $\operatorname{ind}(\varphi,\emptyset)=0$.

We now show that $\operatorname{ind}(\varphi,U)$ shares some of the properties of the fixed point index of maps. There is no commutativity, as the composite of two n-valued multifunctions need not be n-valued, but this property is not needed in the definition of the Nielsen number and in the proof of its homotopy invariance in § 5 and § 6. (Compare [9], p. 84, Remarks.)

THEOREM 4.3. (Localization). *Let $(|K|,\varphi,U)$ and $(|K|,\psi,U)$ be admissible and $\varphi(x)=\psi(x)$ for all $x\in\operatorname{Cl}U$. Then* $\operatorname{ind}(\varphi,U)=\operatorname{ind}(\psi,U)$.

Proof. We may assume that $U\neq\emptyset$. Let $\varepsilon'=\inf(d(x,\varphi(x))\mid x\in\operatorname{Bd}U)=\inf(d(x,\psi(x))\mid x\in\operatorname{Bd}U)$ and $0<\varepsilon''=\min(\varepsilon',\gamma(\varphi),\gamma(\psi))$, and use Lemma 4.1 to determine $0<\delta\leqslant\varepsilon''/2$ so that any two n-valued continuous multifunctions from $\operatorname{Cl}U$ to $|K|$ with distance $<\delta$ are $\varepsilon''/2$-homotopic. We then use the Fix-Finite Approximation Theorem 2.2 to obtain fix-finite approximations $\varphi',\psi'\colon|K|\to|K|$ of φ and ψ so that $\bar d(\varphi,\varphi')<\delta/2$ and $\bar d(\psi,\psi')<\delta/2$. Hence the restrictions of φ' and ψ' to $\operatorname{Cl}U$ are $\varepsilon''/2$-homotopic. If $\Phi'\colon\operatorname{Cl}U\times I\to|K|$ is this homotopy, then for all $(x,t)\in\operatorname{Bd}U\times I$

$$\begin{aligned}d(x,\varphi_t(x))&\geqslant d(x,\varphi(x))-\varrho(\varphi(x),\varphi'(x))-\varrho(\varphi'(x),\varphi'_t(x))\\&\geqslant\varepsilon''-\delta/2-\varepsilon''/2>0\,,\end{aligned}$$

so Φ' is fixed point free on $\operatorname{Bd}U\times I$, and thus the result follows from Lemma 3.2.

THEOREM 4.4 (Additivity). *Let $(|K|,\varphi,U)$ be admissible and $U_1,U_2,\dots,U_s$ mutually disjoint open subsets of U so that φ has no fixed points on $\operatorname{Cl}U-\bigcup_{j=1}^{s}U_j$. Then*

$$\operatorname{ind}(\varphi,U)=\sum_{j=1}^{s}\operatorname{ind}(\varphi,U_j)\,.$$

Proof. We select $0<\eta\leqslant\gamma(\varphi)$ so that $d(x,\varphi(x))\geqslant\eta$ for all $x\in\operatorname{Cl}U-\bigcup_{j=1}^{s}U_j$, and choose the fix-finite approximation φ' of φ in the definition (4.2) of $\operatorname{ind}(\varphi,U)$ so that $\bar d(\varphi,\varphi')<\eta$. Then $x\in\operatorname{Cl}U-\bigcup_{j=1}^{s}U_j$ implies that

$$d(x,\varphi'(x))\geqslant d(x,\varphi(x))-\varrho(\varphi(x),\varphi'(x))>0\,,$$

and it follows from the additive definition (3.1) of $\operatorname{ind}(\varphi',U)$ that

$$\operatorname{ind}(\varphi,U)=\operatorname{ind}(\varphi',U)=\sum_{j=1}^{s}\operatorname{ind}(\varphi',U_j)=\sum_{j=1}^{s}\operatorname{ind}(\varphi,U_j)\,.$$

THEOREM 4.5 (Homotopy). *Let $\Phi\colon|K|\times I\to|K|$ be an n-valued continuous homotopy such that $(|K|,\varphi_t,U)$ is admissible for all $t\in I$. Then* $\operatorname{ind}(\varphi_0,U)=\operatorname{ind}(\varphi_1,U)$.

Proof. It is sufficient to show that for every admissible triple $(|K|, \varphi, U)$ there exists a $\delta>0$ so that if $(|K|, \varphi, U)$ is admissible and $\bar{d}(\varphi, \psi)<\delta$, then $\operatorname{ind}(\varphi, U) = \operatorname{ind}(\psi, U)$, as compactness of I then implies Theorem 4.5.

So let $(|K|, \varphi, U)$ be admissible, let $\varepsilon>0$ be determined as in the definition (4.2) of $\operatorname{ind}(\varphi, U)$, and let $\delta>0$ be selected with the help of Lemma 4.1 so that any two n-valued continuous multifunctions on $|K|$ with distance $<\delta$ are $\varepsilon/4$-homotopic. If the fix-finite approximations φ' of φ and ψ' of ψ in the definitions (4.2) of $\operatorname{ind}(\varphi, U)$ and $\operatorname{ind}(\psi, U)$ are chosen so that $\bar{d}(\varphi, \varphi')<\delta$ and $\bar{d}(\psi, \psi')<\delta$, then $\bar{d}(\varphi, \psi)<\delta$ implies that there exists a $3\varepsilon/4$-homotopy $\Phi'\colon |K|\times I \to |K|$ from φ' to ψ'. Hence we have for all $(x, t) \in \operatorname{Bd} U\times I$

$$d(x, \varphi_t(x)) \geqslant d(x, \varphi'(x)) - \varrho(\varphi'(x), \varphi_t'(x)) \geqslant \varepsilon - 3\varepsilon/4 > 0,$$

so we can use Lemma 3.2 to obtain

$$\operatorname{ind}(\varphi, U) = \operatorname{ind}(\varphi', U) = \operatorname{ind}(\psi', U) = \operatorname{ind}(\psi, U).$$

We conclude this paragraph with two corollaries, which follow from Theorems 4.4 and 4.5 in the standard way. (See e.g. [2], p. 53, Corollaries 1 and 2.)

Corollary 4.6. *If $(|K|, \varphi, U)$ is admissible and* $\operatorname{ind}(\varphi, U) \neq 0$, *then φ has a fixed point on U.*

Corollary 4.7. *If $\varphi\colon |K| \to |K|$ is n-valued and continuous and if* $\operatorname{ind}(\varphi, |K|) \neq 0$, *then every $\psi\colon |K| \to |K|$ related to φ by an n-valued continuous homotopy has a fixed point.*

5. Fixed point classes and the Nielsen number.

We now define fixed point classes and the Nielsen number of an n-valued continuous multifunction $\varphi\colon |K| \to |K|$ on a compact polyhedron. The restriction to polyhedra is necessary as we do not have an index in a more general setting.

So let $x, x' \in \operatorname{Fix}\varphi$ and let $p\colon I \to |K|$ be a path from $p(0) = x$ to $p(1) = x'$. According to the Splitting Lemma 2.1 the multifunction $\varphi \circ p = \{g_1, g_2, \ldots, g_n\}$ splits into n distinct maps. We say that x and x' are *φ-equivalent* if there exist such a path p and splitting of $\varphi\circ p$ so that one g_j is a path from $g_j(0) = x$ to $g_j(1) = x'$ which is homotopic to p by a fixed end-point (single-valued) homotopy. It is clear that φ-equivalence is an equivalence relation on $\operatorname{Fix}\varphi$, and we call these equivalence classes the *fixed point classes* of φ. If $n = 1$, then φ-equivalence reduces to the usual f-equivalence on $\operatorname{Fix} f$ induced by a map [2], p. 86.

The proof of the next theorem needs a rather technical lemma, which follows from the proof of Theorem 1, [2], p. 86.

Lemma 5.1. *Let U be an open subset of the compact polyhedron $|K|$ and $f\colon \operatorname{Cl} U \to |K|$ a map. Given $\delta>0$, there exists $0<\eta<\delta$ so that $x, x' \in \operatorname{Fix} f$ and $d(x, x')<\eta$ imply that x and x' are f-equivalent, and that a path $p\colon I \to |K|$ from $p(0) = x$ to $s(1) = x'$ which is homotopic to $f\circ p$ by a fixed end-point homotopy can be chosen do that $d(p(s), p(t))<\delta$ for all $s, t\in I$.*

Theorem 5.2. *Any n-valued continuous multifunction $\varphi\colon |K| \to |K|$ on a compact polyhedron has finitely many fixed point classes.*

Proof. Let $\gamma(\varphi)$ be the gap of φ and $x \in \operatorname{Fix}\varphi$. We choose $0<\xi<\gamma(\varphi)/2$ so that $\operatorname{Cl} N(x, \xi)$ is contractible, and use Lemma 2.1 to obtain a splitting $\varphi|\operatorname{Cl} N(x, \xi) = \{f_1, f_2, \ldots, f_n\}$, indexed so that $f_1(x) = x$ and hence $f_j(x) \neq x$ if $j \neq 1$. By uniform continuity we can find $0<\delta<\xi$ so that $d(x, x')<\delta$ implies $d(f_j(x), f_j(x'))<\gamma(\varphi)/2$ for $j = 1, 2, \ldots, n$. We now select $0<\eta<\delta$ according to Lemma 5.1, where f_1 and $N(x, \xi)$ replace f and U. If $x' \in \operatorname{Cl} N(x, \eta) \cap \operatorname{Fix}\varphi$ and $j = 2, 3, \ldots, n$, then

$$\begin{aligned} d(x', f_j(x')) &\geqslant d(f_1(x), f_j(x)) - d(x, x') - d(f_j(x), f_j(x')) \\ &\geqslant \gamma(\varphi) - \eta - \gamma(\varphi)/2 > 0, \end{aligned}$$

so $x' \in \operatorname{Fix} f_1$. Hence it follows from Lemma 5.1 that x and x' are f_1-equivalent, and that the path p from x to x' which is end-point homotopic to $f_1\circ p$ can be chosen so that $p(I) \subset N(x, \delta)$. Therefore $\varphi\circ p = \{f_1\circ p, f_2\circ p, \ldots, f_n\circ p\}$, and so x and x' are φ-equivalent. The proof of Theorem 5.2 can now be completed in the same way as the proof of the corresponding theorem for maps [2], Theorem 1, p. 86 by using the fact that each fixed point class of φ is open and $\operatorname{Fix}\varphi$ is compact.

Using Theorem 5.2 we can find for each fixed point class F of $\varphi\colon |K| \to |K|$ an open set U containing F so that $\operatorname{Fix}\varphi \cap \operatorname{Cl} U = F$. We define the *index of the fixed point class F* by

$$\operatorname{ind}(F) = \operatorname{ind}(\varphi, U). \tag{5.3}$$

The independence of $\operatorname{ind}(F)$ from the choice of U follows from Theorem 4.4, i.e. from the additivity of the index, in the same way as in the single-valued case. (See [2], Theorem 1, p. 87.) If $\operatorname{ind}(F) \neq 0$, then we call F an *essential fixed point class*. The *Nielsen number* $N(\varphi)$ of the n-valued continuous multifunction $\varphi\colon |K| \to |K|$ is defined as the number of essential fixed point classes of φ. The following theorem is an immediate consequence of the definition.

Theorem 5.4. *Any n-valued continuous multifunction $\varphi\colon |K| \to |K|$ has at least $N(\varphi)$ fixed points.*

6. Homotopy invariance of the Nielsen number.

We shall prove here that $N(\varphi_0) = N(\varphi_1)$ if there exists an n-valued continuous homotopy $\varphi\colon |K|\times I \to |K|$ from φ_0 to φ_1. The proof follows the pattern in [9], pp. 83–84, but also depends heavily on the Splitting Lemma 2.1. The next lemma, which concerns splittings, is new to our situation. We shall write $p\simeq p'$ if $p, p'\colon I \to |K|$ are fixed end-point homotopic paths in $|K|$.

Lemma 6.1. *Let $\varphi\colon |K| \to |K|$ be n-valued and continuous, let $x, x' \in \operatorname{Fix}\varphi$ and let $p, p'\colon I \to |K|$ be two paths from $p(0) = p'(0) = x$ to $p(1) = p'(1) = x'$ with $p\simeq p'$. If $\varphi\circ p = \{g_1, g_2, \ldots, g_n\}$ and $g_1 \simeq p$, then there exists a splitting $\varphi\circ p' = \{g_1', g_2', \ldots, g_n'\}$ so that $g_1' \simeq p'$.*

Proof. Let $H\colon I\times I\to |K|$ be a homotopy with

$$\begin{array}{ll} H(s,0)=p(s), & H(s,1)=p'(s) \\ H(0,t)=x, & H(1,t)=x' \end{array} \quad \text{for all } (s,t)\in I\times I.$$

We index the splitting $\varphi\circ H=\{F_1,F_2,\ldots,F_n\}\colon I\times I\to|K|$ so that $F_j(s,0)=g_j(s)$ for all $s\in I$ and $j=1,2,\ldots,n$, and define $g_j'\colon I\to|K|$ by $g_j'(s)=F_j(s,1)$. As

$$\begin{array}{l} F_1(0,1)=F_1(0,0)=g_1(0)=x \\ F_1(1,t)=F_1(1,0)=g_1(1)=x' \end{array} \quad \text{for all } t\in I,$$

we see that F_1 induces $g_1\simeq g_1'$, and hence we have $g_1'\simeq p'$.

We now use $\boldsymbol{\Phi}\colon |K|\times I\to|K|\times I$ to denote the n-valued continuous multifunction given by $\boldsymbol{\Phi}(x,t)=(\Phi(x,t),t)$, where $\Phi\colon|K|\times I\to|K|$ is an n-valued continuous homotopy. The t-slice of a set $A\subset|K|\times I$ is the set $A_t\subset|K|$ defined by $A_t=\{x|\ (x,t)\in A\}$.

Lemma 6.2. *Let $\Phi\colon|K|\times I\to|K|$ be an n-valued continuous homotopy and F a fixed point class of $\boldsymbol{\Phi}$. Then for each $t\in I$ either $F_t=\emptyset$ or F_t is a single fixed point class of φ_t.*

Proof. Let $t\in I$. It is clearly sufficient to show that two points (x,t) and (y,t) are $\boldsymbol{\Phi}$-equivalent if and only if they are φ_t-equivalent.

If (x,t) and (y,t) are $\boldsymbol{\Phi}$-equivalent, then there exist a path $\boldsymbol{p}\colon I\to|K|\times I$ and a splitting $\boldsymbol{\Phi}\circ\boldsymbol{p}=\{\boldsymbol{G}_1,\boldsymbol{G}_2,\ldots,\boldsymbol{G}_n\}$ so that $\boldsymbol{G}_1(0)=\boldsymbol{p}(0)=(x,t)$, $\boldsymbol{G}_1(x,1)=\boldsymbol{p}(1)=(y,t)$ and $\boldsymbol{G}_1\simeq\boldsymbol{p}$. We write $\boldsymbol{p}(s)=(p'(s),p''(s))$ and define the path $\boldsymbol{p}'\colon I\to|K|\times I$ by $\boldsymbol{p}'(s)=(p'(s),t)$. Clearly $\boldsymbol{p}\simeq\boldsymbol{p}'$, and so it follows from Lemma 6.1 that there exists a splitting $\boldsymbol{\Phi}\circ\boldsymbol{p}'=\{\boldsymbol{G}_1',\boldsymbol{G}_2',\ldots,\boldsymbol{G}_n'\}$ with $\boldsymbol{G}_1'\simeq\boldsymbol{p}'$. If $\mathrm{pr}\colon|K|\times I\to|K|$ is the projection onto the first factor, then $\mathrm{pr}(x,t)=x$, $\mathrm{pr}(y,t)=y$ and $\mathrm{pr}\circ\boldsymbol{p}'=p'$. Hence if we define $G_j'\colon I\to|K|$ by $G_j'=\mathrm{pr}\circ\boldsymbol{G}_j'$ for $j=1,2,\ldots,n$ then

$$\varphi_t\circ p'=\mathrm{pr}\circ\boldsymbol{\Phi}\circ\boldsymbol{p}'=\{G_1',G_2',\ldots,G_n'\}.$$

As $\boldsymbol{G}_1'\simeq\boldsymbol{p}'$ implies $G_1'\simeq p'$, we see that x and y are φ_t-equivalent.

If, on the other hand, x and y are φ_t-equivalent, then there exist a path $q\colon I\to|K|$ and a splitting $\varphi_t\circ q=\{h_1,h_2,\ldots,h_n\}$ so that $h_1(1)=q(0)=x$, $h_1(1)=q(1)=y$ and $h_1\simeq q$. If we define the path $\boldsymbol{q}\colon I\to|K|\times I$ by $\boldsymbol{q}(s)=(q(s),t)$ and maps $\boldsymbol{h}_j\colon I\to|K|\times I$ by $\boldsymbol{h}_j(s)=(h_j(s),t)$ for $j=1,2,\ldots,n$, then $\boldsymbol{\Phi}\circ\boldsymbol{q}=\{\boldsymbol{h}_1,\boldsymbol{h}_2,\ldots,\boldsymbol{h}_n\}$ and $\boldsymbol{h}_1\simeq\boldsymbol{q}$, so (x,t) and (y,t) are $\boldsymbol{\Phi}$-equivalent.

Lemma 6.3. *Let $\Phi\colon|K|\times I\to|K|$ be an n-valued continuous homotopy and let F', for a given $t\in I$, be a fixed point class of φ_t. Then there exists a unique fixed point class F of $\boldsymbol{\Phi}$ so that $F'=F_t$.*

Proof. If $x'\in F'$, then $(x,t)\in F$. Hence if F is the unique fixed point class of $\boldsymbol{\Phi}$ with $(x,t)\in F$, then $F'=F_t$ according to Lemma 6.2.

Lemma 6.4. *Let $\Phi\colon|K|\times I\to|K|$ be an n-valued continuous homotopy and let F be a fixed point class of $\boldsymbol{\Phi}$. Then* $\mathrm{ind}(F_0)=\mathrm{ind}(F_1)$.

Proof. As I is compact, it is sufficient to show that for every $r\in I$ there exists an $\eta>0$ so that $s\in I$ and $|r-s|<\eta$ imply $\mathrm{ind}(F_r)=\mathrm{ind}(F_s)$.

We choose an open set $U\subset|K|\times I$ so that $F\subset U$ and $\mathrm{Cl}\,U\cap\mathrm{Fix}\,\boldsymbol{\Phi}=F$. If $r\in I$, then $F_r\subset U_r$ and $\mathrm{Cl}\,U_r\cap\mathrm{Fix}\,\varphi_r=F_r$, hence $\mathrm{ind}(F_r)=\mathrm{ind}(\varphi_r,U_r)$ according to definition (5.3). The set $A=\mathrm{pr}(F)-U_r$ is compact and $A\cap\mathrm{Fix}\,\varphi_r=\emptyset$, and so we can choose $\xi>0$ with $d(x,\varphi_r(x))>\xi$ for all $x\in A$. Now Φ is uniformly continuous on $A\times I$ (see [1], p. 127, Corollary), therefore there exists an $\eta>0$ so that $|r-s|<\eta$ implies that $\varrho(\varphi_r(x),\varphi_s(x))<\xi$ for all $x\in A$, and hence

$$d(x,\varphi_s(x))\geqslant d(x,\varphi_r(x))-\varrho(\varphi_r(x),\varphi_s(x))>0.$$

Thus $A\cap\mathrm{Fix}\,\varphi_s=\emptyset$. But $F_s\subset\mathrm{pr}(F)$ and $F_s\cap A=\emptyset$, so $F_s\subset U_r$. As $\mathrm{Cl}\,U\cap\mathrm{Fix}\,\boldsymbol{\Phi}=F$, it follows that $\mathrm{Cl}\,U_r\cap\mathrm{Fix}\,\varphi_s=F_s$, and so $\mathrm{ind}(F_s)=\mathrm{ind}(\varphi_s,U_r)$ by definition (5.3).

We define the n-valued continuous homotopy $\Phi^{r,s}\colon|K|\times I\to|K|$ from φ_r to φ_s by $\Phi^{r,s}(x,t)=\Phi(x,(1-t)r+ts)$. Then $\mathrm{Fix}\,\varphi_t^{r,s}\cap\mathrm{Bd}\,U_r=\emptyset$ for all $t\in I$, and so

$$\mathrm{ind}(F_r)=\mathrm{ind}(\varphi_r,U_r)=\mathrm{ind}(\varphi_s,U_r)=\mathrm{ind}(F_s)$$

follows from the definition (5.3) and the homotopy invariance of the index (Theorem 4.4).

Theorem 6.5. *Let $\Phi\colon|K|\times I\to|K|$ be an n-valued continuous homotopy. Then $N(\varphi_0)=N(\varphi_1)$.*

Proof. Any fixed point class of φ_0 is the 0-slice of a fixed point class of $\boldsymbol{\Phi}$ by Lemma 6.3. So it follows from Lemma 6.4 that $\mathrm{ind}(F_0)=\mathrm{ind}(F_1)$, hence F_1 is essential if F_0 is, and $N(\varphi_0)\leqslant N(\varphi_1)$. Similarly $N(\varphi_1)\leqslant N(\varphi_0)$, and so Theorem 6.5 is true.

7. Some examples. In order to illustrate our definitions we compute the Nielsen number of some simple multifunctions. The first examples are obtained as special cases from Theorem 7.1. A *selection* $f\colon X\to Y$ of a multifunction $\varphi\colon X\to Y$ is a map with $f(x)\in\varphi(x)$ for all $x\in X$.

Theorem 7.1. *Let $|K|$ be a compact polyhedron and $\varphi\colon|K|\to|K|$ be n-valued and continuous. If φ has a selection f, then $N(\varphi)=N(f)+N(\psi)$, where $\psi\colon|K|\to|K|$ is the $(n-1)$-valued continuous multifunction given by $\psi(x)=\varphi(x)-\{f(x)\}$ for all $x\in|K|$.*

Proof. It is clear that ψ is $(n-1)$-valued, and easy to check that it is continuous and that $\mathrm{Fix}\,\varphi$ is the disjoint union of $\mathrm{Fix}\,f$ and $\mathrm{Fix}\,\psi$. We now clarify the relation between φ-, f- and ψ-equivalence.

Let $x,x'\in\mathrm{Fix}\,\varphi$ and $p\colon I\to|K|$ be a path from $p(0)=x$ to $p(1)=x'$. We can use the Splitting Lemma 2.1 to write

$$\varphi\circ p=\{g_1,g_2,\ldots,g_{n-1},f\circ p\},\quad \text{where}\quad \psi\circ p=\{g_1,g_2,\ldots,g_{n-1}\}.$$

If $x, x' \in \mathrm{Fix} f$ and x, x' are φ-equivalent, then $f \circ p(0) = x$ implies $g_j(0) \neq x$ for $j = 1, 2, \ldots, n-1$, so $g_j \simeq p$ is impossible and therefore we must have $f \circ p \simeq p$, i.e. x, x' must be f-equivalent. As f-equivalence of x and x' clearly implies φ-equivalence, we see that $x, x' \in \mathrm{Fix} f$ are φ-equivalent if and only if they are f-equivalent. The proof that $x, x' \in \mathrm{Fix}\psi$ are φ-equivalent if and only if they are ψ-equivalent is similar. If $x \in \mathrm{Fix} f$ and $x' \in \mathrm{Fix}\psi$, then $f \circ p(1) \neq x'$ and $g_j(0) \neq x$ for $j = 1, 2, \ldots, n-1$, and therefore x and x' cannot be φ-equivalent. Hence the set of fixed point classes of φ is the disjoint union of the sets of fixed point classes of f and fixed point classes of ψ.

It remains to show that a fixed point class of φ is essential if and only if it is essential as a fixed point class of f resp. ψ. So let us assume that F is a fixed point class of φ and $F \subset \mathrm{Fix} f$, and let U be an open set with $F \subset U$ and $\mathrm{Fix}\varphi \cap \mathrm{Cl}\, U = F$. Then $\mathrm{Fix}\psi \cap \mathrm{Cl}\, U = \emptyset$, and hence $d(x, \psi(x)) \geqslant \xi$ for all $x \in \mathrm{Cl}\, U$ and some $\xi > 0$. Let $\delta > 0$ be determined as in the definition (4.2) of $\mathrm{ind}(\varphi, U)$, and let f' and ψ' be fix-finite approximations of f and ψ obtained with the help of Theorem 2.2 so that $\bar{d}(f, f') < \eta$ and $\bar{d}(\psi, \psi') < \eta$, where $\eta = \min(\delta/2, \xi, \gamma(\varphi)/2)$. Then the multifunction φ' given by $\varphi'(x) = \{f'(x)\} \cup \psi'(x)$ for all $x \in |K|$ is n-valued and continuous, and is a fix-finite approximation of φ. As $\mathrm{Fix}\psi' \cap \mathrm{Cl}\, U = \emptyset$, it follows from the definitions (3.1), (4.2) and (5.3) that

$$\mathrm{ind}(F) = \mathrm{ind}(\varphi, U) = \mathrm{ind}(\varphi', U) = \mathrm{ind}(f', U) = \mathrm{ind}(f, U),$$

so F is an essential fixed point class of φ if and only if it is an essential fixed point class of f. The proof for $F \subset \mathrm{Fix}\psi$ is similar, and thus Theorem 7.1 is true.

Remark. Theorem 7.1 easily extends to the case where φ splits into two distinct multifunctions ψ_j which are n_j-valued $(j = 1, 2)$ and continuous, with $n_1 + n_2 = n$.

We now state some corollaries of Theorem 7.1. The first one, which follows by induction, is also an immediate consequence of the definition of splitting.

COROLLARY 7.2. *If $|K|$ is a compact polyhedron and $\varphi = \{f_1, f_2, \ldots, f_n\}$: $|K| \to |K|$ a splitting of the n-valued continuous multifunction φ, then $N(\varphi) = \sum_{j=1}^{n} N(f_j)$.*

COROLLARY 7.3. *If $|K|$ is a compact polyhedron and $\varphi\colon |K| \to |K|$ an n-valued continuous multifunction which is constant, then $N(\varphi) = n$.*

Proof. If $\varphi(x) = \{c_1, c_2, \ldots, c_n\}$ for all $x \in |K|$, then φ splits into n constant maps $f_j\colon |K| \to |K|$ given by $f_j(x) = c_j$. But the Nielsen number of a constant maps is one.

Next we apply Theorem 7.1 to *multivalued identities*, i.e. to multifunctions which have the identity map as a selection. The Euler characteristic of $|K|$ is denoted by $\chi(K)$.

COROLLARY 7.4. *If $|K|$ is a compact polyhedron and $\varphi\colon |K| \to |K|$ an n-valued continuous identity, then $N(\varphi) = 0$ if $\chi(K) = 0$ and $N(\varphi) = 1$ if $\chi(K) \neq 0$.*

Proof. The identity map id: $|K| \to |K|$ is a selection of φ, and if $\psi(x) = \varphi(x) - \{\mathrm{id}(x)\}$, then $x \notin \psi(x)$ for all $x \in |K|$. So $N(\varphi) = N(\mathrm{id}) + N(\psi) = N(\mathrm{id})$. (Compare [2], p. 96, Exercise 4.)

The reader may start to wonder by now whether all n-valued continuous multifunctions split. But they need not do so if $|K|$ is not simply connected, and this is precisely the situation in which the Nielsen number is more powerful than the Lefschetz number. We finish by discussing a simple example.

Let $S^1 = \{\exp(ti) \mid 0 \leqslant t < 2\pi\}$ be the unit circle in the complex plane, and let $\varphi\colon S^1 \to S^1$ be given by

$$\varphi(\exp(ti)) = \{\exp(-\tfrac{1}{2}ti), \exp((\pi - \tfrac{1}{2}t)i)\}$$

for all $0 \leqslant t < 2\pi$. Then φ is 2-valued, continuous, and does not split. Its three fixed points occur at $t = 0$, $t = 2\pi/3$ anf $t = 4\pi/3$, and it is easy to check that each fixed point has index one and that no two fixed points are φ-equivalent. Hence $N(\varphi) = 3$ and any 2-valued continuous multifunction in the homotopy class of φ has at least three fixed points.

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