Hence the coefficients of $\sigma_{1/a}$ are obtained from 1 and the coefficients of σ_a by means of $+$, $\cdot$ and $^{-1}$. Similarly, the coefficients of $\sigma_{e(a)}$ are of the form $e(\sigma_a(1))c$ where c is obtained from 1 and the coefficients of σ_a by means of addition and multiplication. We observe that the coefficients of σ_a are independent of the concrete choice of the model C_1, the field K, and the ladder for K — they are completely determined by the diagram of the exponential field C.

By the order of a term t we mean the maximal number of iterations of the exponential function occurring in t. Thus we have proved

LEMMA 36. *Let $t(x, y_1, \ldots, y_n)$ be a term without parameters, let C_1 be a model of T and $c_1, \ldots, c_n \in C_1$. Then, for every $h \in H_\infty$, there exists a term $u(y_1, \ldots, y_n)$ of the same order as t such that*

$$\sigma_{t(x, c_1, \ldots, c_n)}(h) = u(c_1, \ldots, c_n).$$

LEMMA 37. *Suppose $a \neq 0$ and $h = \max\operatorname{supp}(\sigma_a)$. Then there is a positive $c \in C$ such that $|\sigma_a - \sigma_a(h)h| < \frac{c}{x}h$.*

Proof. The lemma is an easy consequence of the fact that x^{-1} is the largest element of H_∞ which is less than 1. ■

Since $\sigma\colon a \mapsto \sigma_a$ is an embedding of C_∞ into C^∞, Lemma 36 and Lemma 37 yield

THEOREM 38. *Let $t(x, y_1, \ldots, y_n)$ be a term without parameters, let C_1 be a model of T and $c_1, \ldots, c_n \in C_1$ such that $t(x, c_1, \ldots, c_n)$ is not identically zero in C_1. Then $C_1 \models$ "$\lim t(x, c_1, \ldots, c_n)$ exists" iff $\max\operatorname{supp}(\sigma_{t(x, c_1, \ldots, c_n)}) \leqslant 1$.*

If $c \in C_1$ is such that $C_1 \models$ "$\lim t(x, c_1, \ldots, c_n) = c$", then there is a term $u(y_1, \ldots, y_n)$ of the same order as t such that $c = u(c_1, \ldots, c_n)$.

COROLLARY 39. *Let C_1, C_2 be models of T containing an exponential field C and let t be a term with parameters from C. Then, for each $c \in C \cup \{\pm\infty\}$, $C_1 \models$ "$\lim t = c$" iff $C_2 \models$ "$\lim t = c$".*

Proof. Corollary 39 follows from Lemma 37 and Theorem 38 since the map $a \mapsto \sigma_a$ is the same for the models C_1 and C_2. ■

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The automorphism group of some semigroups

by

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Abstract. Let $F(Z)$ denote the collection of all finite non-empty subsets of the integers Z. $F(Z)$ can be considered as a semigroup with addition defined by $A+B = \{a+b \mid a \in A, b \in B\}$. The main result in this paper is the determination of the automorphism group of $F(Z)$. In order to determine this automorphism group some algebraic results for $F(G)$ where G is a group are obtained.

Introduction. Let $F(Z)$ denote the collection of all finite non-empty subsets of the integers Z. $F(Z)$ can be considered as a semigroup with addition defined by $A+B = \{a+b \mid a \in A, b \in B\}$. M. Deza and P. Erdös considered this set addition in [2] and G. A. Freĭman also uses this notion of set addition in his book [3]. The same idea is used, but mainly for infinite subsets, in the study of sequences, such as in H. Halberstam and K. Roth [4]. The main question that is considered in this paper is the determination of the automorphism group of this semigroup.

Since the answer to the main question can be obtained by considering the subsemigroup of $F(Z)$ composed of all subsets of the non-negative integers which contain 0, the first section is devoted to determining the automorphism group of this subsemigroup. It is necessary to introduce some algebraic results concerning retractions in order to answer the main question. Thus, the second section is devoted to providing the necessary facts about retractions in order to verify that the automorphism group of $F(Z)$ is a splitting extension of Z by the Klein four group.

Section I.

DEFINITION 1. For a group G let $F(G) = \{A \subset G \mid A \neq \emptyset \text{ and } |A| < \infty\}$. For the special case of $G = Z$ let $K = \{A \in F(Z) \mid 0 \in A,\ A \subset Z^+\}$.

The following lemma is due to Professor A. H. Clifford.

LEMMA 1. *If $\gamma \in \operatorname{Aut} K$ and n is a natural number, then $\{0, n\}\gamma = \{0, n\}$.*

Proof. Let $P_n = \{0, n\}\gamma$. Now $(n-1)\{0, 1\} + \{0, n\} = (2n-1)\{0, 1\}$ and thus $(n-1)P_1 + P_n = (2n-1)P_1$. Let $Q = \{0, \ldots, q\} = \{0, 1\}\gamma^{-1}$. Then $Q + q\{0, 1\}$

$= 2q\{0,1\}$ and so $\{0,1\}+qP_1 = 2qP_1$. Therefore, $1+q(\max P_1) = 2q(\max P_1)$ and so $q = \max P_1 = 1$. Hence $P_1 = \{0,1\}$. We now have $(n-1)P_1+P_n = (2n-1)P_1 = (2n-1)\{0,1\}$. Thus $(n-1)+\max P_n = 2n-1$ and so $\max P_n = n$. Suppose (by way of contradiction) that there is a $p\in P_n$ with $0<p<n$. Then

$$(n-2)\{0,1\}+P_n = (2n-2)\{0,1\},$$

and applying γ^{-1} to both sides, we obtain

$$(n-2)\{0,1\}+\{0,n\} = (2n-2)\{0,1\}.$$

But $n-1\in(2n-2)\{0,1\}$ while $n-1\notin(n-2)\{0,1\}+\{0,n\}$. Thus $P_n = \{0,n\}$.

LEMMA 2. *If* $\gamma\in \operatorname{Aut}K$ *and* $A\in K$, *then* $\max A = \max(A\gamma)$.

Proof. Let $m = \max A$. Then $m\{0,1\}+A = 2m\{0,1\}$. Thus

$$m\{0,1\}+A\gamma = 2m\{0,1\}.$$

It follows that $m+\max(A\gamma) = 2m$ and so $\max(A\gamma) = m$.

If $A\in K$ and the integers from k to m inclusive are in the set A, then we say that A is *solid* from k to m and denote A by

$$A = \{0,\ldots,\mathbf{k},\ldots,\mathbf{m},\ldots,n\}.$$

A *gap* in the set A is a sequence of consecutive natural numbers $x, x+1, \ldots, x+t$ which are not in the set A, while $x-1$ and $x+t+1$ are in A. In this case the *length* of the gap is $t+1$. A *maximum gap* in A is a gap of largest length.

LEMMA 3. *Let* $\gamma\in\operatorname{Aut}K$ *and* $A\in K$. *Then the length of a maximum gap of* A *equals the length of a maximum gap of* $A\gamma$.

Proof. Let t be the length of a maximum gap of A and s be the length of a maximum gap of $A\gamma$. Then

$$t\{0,1\}+A = (n+t)\{0,1\}$$

where $n = \max A$. By Lemma 1, we have that $t\{0,1\}+A\gamma = (n+t)\{0,1\}$. Thus we have that $s\leqslant t$. Applying γ^{-1} and using the same argument, we have that $t\leqslant s$ and so $t = s$.

LEMMA 4. *Let* $\gamma\in\operatorname{Aut}K$ *with* $\{0,2,3\}\gamma = \{0,2,3\}$. *Then for* $k\geqslant 2$ *and* $n\geqslant 2k-2$, $\{0,\mathbf{k},\ldots,\mathbf{n}\}\gamma = \{0,\mathbf{k},\ldots,\mathbf{n}\}$.

Proof. By Lemma 1, we have that $\{0,n\}\gamma = \{0,n\}$ for all natural numbers n. We first consider the case $k = 2$. $\{0,2\}\gamma = \{0,2\}$ and $\{0,2,3\}\gamma = \{0,2,3\}$. Applying Lemmas 2 and 3 we have that $\{0,2,3,4\}$ has only 4 possible images. By computation it can be verified that $\{0,2,3,4\}$ is fixed by γ. Now let $n\geqslant 5$ and assume that the result holds for all r such that $4\leqslant r<n$. Then

$$\{0,\mathbf{2},\ldots,\mathbf{n-2}\}\gamma = \{0,\mathbf{2},\ldots,\mathbf{n-2}\}.$$

Adding $\{0,2\}$ to each side of this equality and using the fact that $\{0,2\}\gamma = \{0,2\}$, we have that

$$\{0,\mathbf{2},\ldots,\mathbf{n}\}\gamma = \{0,\mathbf{2}\ldots,\mathbf{n}\}.$$

Thus the lemma is true for $k = 2$.

Before beginning the general case, we should note that

$$\{0,3,4,5,6,7\}\gamma \neq \{0,2,3,6,7\}.$$

To see this, we observe that $\{0,1\}+\{0,3\} = \{0,1,3,4\}$ is fixed by γ and

$$\{0,3,4,5,6,7\}+\{0,1,3,4\} = \{0,1,\mathbf{3},\ldots,\mathbf{11}\}$$

while

$$\{0,2,3,6,7\}+\{0,1,3,4\} = 11\{0,1\}.$$

Now suppose that $k\geqslant 3$, $n\geqslant 2k-2$, and the lemma holds for all l such that $2\leqslant l<k$. Let

$$A = \{0,\mathbf{k},\ldots,\mathbf{n}\}$$

and let $A\gamma = B = \{0,\ldots,n\}$. Now

$$A+\{0,k-1\} = \{0,\mathbf{k-1},\ldots,\mathbf{n+k-1}\}.$$

Thus

$$D = B+\{0,k-1\} = A\gamma+\{0,k-1\}\gamma = (A+\{0,k-1\})\gamma = \{0,\mathbf{k-1},\ldots,\mathbf{n+k-1}\}.$$

Note that $1,2,\ldots,k-2\notin B$ while $k\in B$. Now suppose that

$$2k-2\leqslant n\leqslant 3k-2$$

and suppose (by way of contradiction) that $k-1\in B$. Since A has a maximum gap of length $k-1$, there exists $x\in B$, $x>k$ with $x+1, x+2, \ldots, x+k-1\notin B$ while $x+k\in B$. Now

$$x+1, x+2, \ldots, x+k-1\in D.$$

Since $x+1\notin B$, there exists $y\in B$, $y\neq 0$ such that $y+k-1 = x+1$. Now $x+k-1 \leqslant n-1\leqslant 3k-3$ and so $x\leqslant 2k-2$. Thus $y+(k-1) = x+1\leqslant 2k-1$, and so $y\leqslant k$. Thus $y = k$ or $y = k-1$.

Case 1. $y = k-1$. Since $x+1 = y+k-1 = (k-1)+(k-1)$, we have that $x = 2k-3$. Now $x+k-1 = 3k-4\notin B$. Also $x+k\in B$ and $x+k = 3k-3$. Therefore, $3k-3\leqslant n\leqslant 3k-2$, and so $n = 3k-3$ or $n = 3k-2$.

Subcase 1a. $n = 3k-3$. Now $n+(k-1) = 4k-4$ and so $4k-5\in D$. If $4k-5\in B$, then $4k-5\leqslant n = 3k-3$, and $k\leqslant 2$, a contradiction. Thus $4k-5\notin B$. Therefore, there exists $z\in B$, $z\neq 0$, such that $z+k-1 = 4k-5$. But then $z = 3k-4\in B$ and this contradicts the fact that $3k-4 = x+k-1\notin B$.

Subcase 1b. $n = 3k-2$. Then we have that $n+k-1 = 4k-3$. Thus $4k-5\in D$. If $4k-5\in B$, then $4k-5\leqslant 3k-2$ and so $k\leqslant 3$. If $k = 3$, then $n = 7$ and $A = \{0,3,4,5,6,7\}$ while $B = \{0,2,3,6,7\}$ and this was taken care of in the above. Thus $4k-5 \notin B$. Therefore, there exists $z \in B$, $z \neq 0$, such that $z+k-1 = 4k-5$. Again we have that $z = 3k-4 = x+k-1 \notin B$.

It now follows that $y \neq k-1$.

Case 2. $y = k$. Then $x = 2k-2$. Now

$$x+k-1 = 2k-2+k-1 = 3k-3 \notin B.$$

Also, $x+k \leqslant n$ and so $n \geqslant 3k-2$. By assumption $n \leqslant 3k-2$ and therefore $n = 3k-2$. Thus $n+(k-1) = 4k-3$ and so $4k-4 \in D$. Now $4k-4 \notin B$ and hence there exists $z \in B$, $z \neq 0$, such that $z+k-1 = 4k-4$. But then $z = 3k-3 = x+k-1 \in B$, a contradiction.

Thus we have shown that $k-1 \notin B$. We now prove that $B = \{0, k, \ldots, n\}$. Assume that there exists $x \notin B$ with $k<x<n$. Then

$$2k-1<x+k-1<n+k-1$$

and so $x+k-1 \in D$. Since $x \notin B$, it follows that $x+(k-1) \in B$ and so $x+k-1 \leqslant n \leqslant 3k-2$. Therefore, $x \leqslant 2k-1$. If $x<2k-1$, then $x \notin D$, a contradiction. Thus, $x = 2k-1$ and it follows that $n = 3k-2$. Thus

$$A = \{0, k, \ldots, 3k-2\}$$

and $B = \{0, k, \ldots, 2k-2, 2k, \ldots, 3k-2\}$. Now $A+\{0, k\} = \{0, k, \ldots, 4k-2\}$ and $B+\{0, k\}$ has a gap at $2k-1$. By our argument we have that

$$\{0, k, \ldots, 3k-3\}\gamma = \{0, k, \ldots, 3k-3\}.$$

Adding $\{0, k+1\}$ to each side of this equality we have that

$$\{0, k, \ldots, 4k-2\}\gamma = \{0, k, \ldots, 4k-2\}$$

and this is a contradiction.

We claim now that $\{0, k, \ldots, n\}\gamma = \{0, k, \ldots, n\}$ if $(l-1)k-2 \leqslant n \leqslant lk-2$, where $l \geqslant 3$. We have just completed the case for $l = 3$. The induction step is clear.

LEMMA 5. *If* $\gamma \in \operatorname{Aut}K$ *and* $\{0, 2, 3\}\gamma = \{0, 2, 3\}$, *then* γ *is the identity automorphism of* K.

Proof. Let $M = \{A| \ A \in K$ and $A\gamma \neq A\}$ and assume that $M \neq \emptyset$. For $A \in M$, define $G(A) = \max A+1-|A|$. Let $A_0 \in M$ be such that

$$G(A_0)+G(A_0\gamma) = t_0$$

is minimal. Note that $t_0 \geqslant 2$. Let $n = \max A_0$ and y be the largest integer such that $y \leqslant n-1$ and $y \notin A_0$ or $y \notin A_0\gamma$. Then $A_0 = \{0, \ldots, y+1, \ldots, n\}$.

Since we can add to A_0 (and $A_0\gamma$) any set of the form $\{0, y+1, \ldots, m\}$, where $n+y+1 \leqslant m+1$, and preserve the structure of A_0 between 0 and $y+1$, we may assume that $n \geqslant 3y+1$. Next let

$$B_0 = A_0+\{0, y, \ldots, y+n\} = \{0, \ldots, y, \ldots, 2n+y\}.$$

Since $2y-2 \leqslant n+y$, we have by Lemma 4 that

$$\{0, y, \ldots, y+n\}\gamma = \{0, y, \ldots, y+n\}.$$

Thus $B_0\gamma = A_0\gamma+\{0, y, \ldots, y+n\}$ and hence

$$G(A_0+\{0, y, \ldots, y+n\})+G(A_0\gamma+\{0, y, \ldots, y+n\}) \leqslant t_0-1.$$

By assumption, we have that $A_0+\{0, y, \ldots, y+n\} \notin M$ and so

$$A_0+\{0, y, \ldots, y+n\} = A_0\gamma+\{0, y, \ldots, y+n\}.$$

Thus, $A_0\setminus\{y, \ldots, n\} = A_0\gamma\setminus\{y, \ldots, n\}$ and $A_0\setminus\{0, \ldots, y\} = A_0\gamma\setminus\{0, \ldots, y\}$. We may assume that $y \in A_0$ and $y \notin A_0\gamma$. Let

$$A_1 = \{0, \ldots, t\} = A_0\setminus\{y, \ldots, n\}.$$

Then $A_0 = A_1+\{0, y, \ldots, n-t\}$ and hence $A_0\gamma = A_1\gamma+\{0, y, \ldots, n-t\}$. Therefore $y \in A_0\gamma$ and this is a contradiction. Thus $M = \emptyset$ and so γ is the identity automorphism. This completes the proof of the lemma.

Define ν from K into K as follows: If $\{a_1, \ldots, a_n\} \in K$, where $a_1< \ldots <a_n$, then $\{a_1, \ldots, a_n\}\nu = \{a_n-a_i| \ i = 1, \ldots, n\}$. It is an easy matter to verify that ν is an automorphism of K.

THEOREM 1. *The only automorphisms of* K *are* ι *and* ν.

Proof. Suppose that $\gamma \in \operatorname{Aut}K$. Since maximal elements and lengths of maximum gaps are preserved by γ, we have that $\{0, 2, 3\}\gamma = \{0, 2, 3\}$ or $\{0, 2, 3\}\gamma = \{0, 1, 3\}$. Since $\{0, 2, 3\}\nu = \{0, 1, 3\}$, it follows that $\gamma = \iota$ or $\gamma = \nu$.

Section II.

DEFINITION 2. Given a group G, a *retraction* of G is a semigroup homomorphism σ, such that: (i) $\sigma\colon F(G) \to G$, (ii) $(\{g\})\sigma = g$.

For notation purposes $\operatorname{Aut}F(G)$ denotes the automorphisms of the semigroup $F(G)$ and $X\setminus Y$ denotes the elements of X which do not belong to Y.

Each automorphism of G induces in a natural way an automorphism of $F(G)$ and we call such an automorphism a *standard automorphism*. That is, for $f \in \operatorname{Aut}G$, denote the automorphism as σ_f, where $A\sigma_f = \{af| \ a \in A\}$. There are two fundamental questions concerning $\operatorname{Aut}F(G)$. First, does each element in $\operatorname{Aut}F(G)$ preserve set cardinality? In general the answer to this question has not been given but several special cases have been answered with a positive answer. Second, does each element in $\operatorname{Aut}F(G)$ preserve set inclusion? We prove that the only elements of $\operatorname{Aut}F(G)$ which preserve set inclusion are the standard automorphisms. Moreover, we show that for each torsion free group G, $F(G)$ admits nonstandard automorphisms.

In this section we investigate some relationships between group retractions and $\operatorname{Aut}F(G)$. A subset S of $F(G)$ is said to be *normal* if $g^{-1}Sg = S$ for every $g \in G$. A normal subsemigroup S of $F(G)$ is called a *G-complement* if, whenever $C \in S$, $g \in G$ and $Cg \in S$, then $g = \iota$; and whenever $A \in F(G)$, there exist $C \in S$ and $g \in G$

such that $A = Cg$. It is an easy matter to verify that if σ is a retraction of G, then $\ker\sigma = \{A |\ A \in F(G) \text{ and } A\sigma = \iota\}$ is a G-complement. Conversely, if S is a G-complement and $A \in F(G)$, then there exist unique $C \in S$ and unique $g \in G$ such that $A = Cg$. Thus if one defines $A\sigma = g$, then σ is a retraction of G. It was shown in [1, Corollary 2.11] that there is a one-to-one correspondence between retractions of G and G-complements of $F(G)$. If S and T are G-complements with $T \subseteq S$, then $T = S$ [1, Theorem 2.9].

The proof of the following theorem is straightforward and will be omitted.

THEOREM 2. *If $\theta \in \operatorname{Aut}F(G)$ and S is a G-complement of $F(G)$, then $S\theta$ is a G-complement of $F(G)$.*

If σ is a retraction of G and α is either an automorphisms or an anti-automorphism of G, then $\alpha\sigma\alpha^{-1}$ can be considered as a map from $F(G)$ into G and as such, is a retraction of G [1, Theorem 5.1]. In the case where α is the anti-automorphism of G given by $x\alpha = x^{-1}$ and $\sigma' = \alpha\sigma\alpha^{-1}$, then σ' is called the *dual of* σ. If S' is the kernel of σ' and S is the kernel of σ, then $S' = S^{-1} = \{A^{-1} |\ A \in S\}$ [1, Corollary 5.2]. In case $\sigma = \sigma'$, we say that σ is a *self dual retraction of* G. If G is a 2-divisible torsion-free abelian group, then G admits a self dual retraction.

Let $\leqslant$ be a total order for the abelian group G, $A \in F(G)$, and k be a fixed integer. If we define $A\sigma_k = (k+1)\max A - k\min A$, then it is easily verified that σ_k is a retraction of G. Moreover, σ_k is not self dual.

THEOREM 3. *Let σ be a retraction of the abelian group G and for each $A \in F(G)$ define*

$$A\varphi_\sigma = (A\sigma)A(A^{-1}\sigma).$$

Then φ_σ is an automorphism of $F(G)$ and if σ is not self dual, then φ_σ is an element of infinite order in $\operatorname{Aut}F(G)$.

Proof. It is easy to see that φ_σ is a homomorphism from $F(G)$ into $F(G)$. The map φ_σ^{-1} given by

$$A\varphi_\sigma^{-1} = (A\sigma)^{-1}A(A^{-1}\sigma)^{-1}$$

is a two-sided inverse for φ_σ and so $\varphi_\sigma \in \operatorname{Aut}F(G)$. It is not difficult to verify that

$$A\varphi_\sigma^n = (A\sigma)^n A(A^{-1}\sigma)^n$$

for each natural number n. Suppose that $\varphi_\sigma^n = \iota$ for some natural number n. Then if $A \in \ker\sigma = S$, we have that

$$A = A\varphi_\sigma^n = (A\sigma)^n A(A^{-1}\sigma)^n = A(A^{-1}\sigma)^n.$$

Since S is a G-complement, it follows that $(A^{-1}\sigma)^n = \iota$, and since G is torsion-free, we have that $A^{-1}\sigma = \iota$. Hence $A^{-1} \in S$ and so $S = S^{-1}$. Consequently, σ is self dual.

We remark in passing that if σ is self dual, then $\varphi_\sigma = \iota$.

COROLLARY 3.1. *If G is a torsion-free abelian group, then $\operatorname{Aut}F(G)$ is infinite.*

Note that if σ is a retraction of G, $\sigma \neq \sigma'$, and φ_σ is given in the statement of Theorem 3, then φ_σ is the identity automorphism on the group of units of $F(G)$ and so is not a standard automorphism of $F(G)$.

Let σ be a retraction of the abelian group G and $\alpha \in \operatorname{Aut}G$. For each $A \in F(G)$ define $A\psi_{\sigma,\alpha}$ as follows: $A = Cg$ for a unique $C \in \ker\sigma$ and unique $g \in G$. Then

$$A\psi_{\sigma,\alpha} = C(g\alpha).$$

It is a simple matter to verify that $\psi_{\sigma,\alpha} \in \operatorname{Aut}F(G)$ and $\psi_{\sigma,\alpha}^{-1} = \psi_{\sigma,\alpha^{-1}}$. We state some properties of $\psi_{\sigma,\alpha}$ in the following two theorems.

THEOREM 4. *If σ is a retraction of the abelian group G and $\alpha \in \operatorname{Aut}G$, then $\psi_{\sigma,\alpha}$ is an automorphism of $F(G)$ that leaves the elements of $\ker\sigma$ fixed. The map from $\operatorname{Aut}G$ into $\operatorname{Aut}F(G)$ that sends α to $\psi_{\sigma,\alpha}$ is an isomorphism of $\operatorname{Aut}G$ into $\operatorname{Aut}F(G)$. Moreover, $\langle\psi_{\sigma,\alpha}\rangle \cap \langle\varphi_\sigma\rangle = \iota$ and if $\alpha \neq \iota$, then $\psi_{\sigma,\alpha}$ is not a standard automorphism.*

THEOREM 5. *Let σ and τ be retractions of the abelian group G and let $\alpha \in \operatorname{Aut}G$ with $\alpha \neq \iota$. If $A\tau = \max A$ with respect to some total order of G, then $\varphi_\tau\psi_{\sigma,\alpha} \neq \psi_{\sigma,\alpha}\varphi_\tau$.*

An immediate consequence of Theorem 5 is

COROLLARY 5.1. *If G is a torsion-free abelian group, then $\operatorname{Aut}F(G)$ is non-abelian.*

The semigroup $F(G)$ is a partially ordered semigroup with respect to the relation of set containment. In general, an automorphism of $F(G)$ will not preserve this partial order. More specifically, we have

THEOREM 6. *The only order preserving automorphisms of $F(G)$ are the standard automorphisms.*

Proof. Suppose that θ is an order preserving automorphism of $F(G)$ and let η be the standard automorphism of $F(G)$ induced by $\theta|G$. We proceed by induction on the cardinality of the set $A \in F(G)$. If $|A| = 1$, then $A\theta = A\eta$. Assume that for all $A \in F(G)$ with $|A| \leqslant k$, $A\theta = A\eta$, and let $|B| = k+1$. If $D = B\eta$, then let $C \in F(G)$ be such that $C\theta = D$. Since θ is order preserving, if $b \in B$, then $b\theta = b\eta \in B\theta$. Thus $B\eta \subseteq B\theta$. If $x \in C$, then $x\eta = x\theta \in C\theta = D = B\eta$. Hence, $x\eta = b\eta$ for some $b \in B$ and so $x = b$. Therefore, $C \subseteq B$. Suppose that $C \neq B$. Then $|C| < |B| = k+1$ and, by induction $C\eta = C\theta = D = B\eta$. But then $C = B$. Thus θ is the standard automorphism η.

THEOREM 7. *Let G be a torsion-free abelian group and let σ, τ be retractions of G with kernels S and T respectively. There there exists $\theta \in \operatorname{Aut}F(G)$ with $S\theta = T$. Thus, any two G-complements of $F(G)$ are isomorphic semigroups.*

Proof. If $A \in F(G)$, then $A = Bg = Ch$ for unique $B \in S$, $C \in T$, and $g, h \in G$. Define $A\theta = Cg$. It is an easy matter to verify that $\theta \in \operatorname{Aut}F(G)$ and that $S\theta = T$.

THEOREM 8. *Let G be a torsion-free abelian group and σ, τ be distinct retractions of G with kernels S and T respectively. Then there exists $\theta \in \operatorname{Aut}F(G)$ such that $\theta|S = \iota$ and $\theta|T \neq \iota$.*

Proof. Let $\alpha \in \mathrm{Aut}\, G$ be given by $x\alpha = x^{-1}$ for every $x \in G$. Then by Theorem 4, $\psi_{\sigma,\alpha}|S = \iota$. If $A \in T \setminus S$, then $A = Cg$ with $C \in S$, $g \in G$ and $g \neq i$. Thus $A\psi_{\sigma,\alpha} = C(g\alpha) = Cg^{-1} \neq Cg = A$.

All retractions of Z are given by the formula

$$A\sigma_k = (k+1)\max A - k\min A$$

where k is an integer [1, Example 5.6]. The retraction induced by the natural order of Z is σ_0 and σ_{-1} is the retraction induced by the dual of this order. We shall use φ_k to denote the automorphism φ_{σ_K} of $F(G)$. Note that $K = \{A \in F(Z) |\, 0 \in A,\ A \subset Z^+\}$ is the kernel of σ_{-1}. The following lemmas are necessary for the determination of $\mathrm{Aut}\, F(Z)$.

Lemma 6. *For each* $k \in Z$,

$$(\varphi_{-1})^{-(2k+1)} = \varphi_k .$$

In particular, $\varphi_{-1}^{-1} = \varphi_0$.

Lemma 7. *For each* $k \in Z$,

$$(\ker\sigma_{-1})\varphi_{-1}^{k} = \ker\sigma_{k-1} .$$

Lemma 8. *Let G be a torsion-free abelian group, S and T be G-complements in $F(G)$, λ be an isomorphism of S onto T, and $\alpha \in \mathrm{Aut}\, G$. Then there is an automorphism θ of $F(G)$ such that $\theta|S = \lambda$ and $\theta|G = \alpha$.*

Proof. If $A \in F(G)$, then $A = Cg$ for unique $C \in S$ and $g \in G$. Define $A\theta = (C\lambda)(g\alpha)$. It is easy to verify that θ has the stated properties. We will denote θ by $\theta_{\lambda,\alpha}$.

Lemma 9. *Let α be the automorphism of Z given by $x\alpha = -x$ and define μ_1, μ_2, and μ_3 as follows:*

$$\mu_1 = \theta_{\iota,\alpha}, \quad \mu_2 = \theta_{\nu,\iota}, \quad \mu_3 = \theta_{\nu,\alpha} .$$

Then $\{\iota, \mu_1, \mu_2, \mu_3\}$ is a subgroup of $\mathrm{Aut}\, F(Z)$ which is isomorphic to the Klein four-group. Moreover, $\mu_1^{-1}\varphi_{-1}\mu_1 = \varphi_0$, $\mu_2^{-1}\varphi_{-1}\mu_2 = \varphi_{-1}$, and $\mu_3^{-1}\varphi_{-1}\mu_3 = \varphi_0$.

Theorem 9. *The automorphism group of $F(Z)$ is a splitting extension of Z by the Klein four-group.*

Proof. Let $\theta \in \mathrm{Aut}\, F(Z)$ and K be the kernel of σ_{-1}. By Theorem 2, $K\theta = \ker\sigma_j$ for $j \in Z$. By Lemma 7, we have that

$$(\ker\sigma_j)\varphi_{-1}^{-(j+1)} = K .$$

Thus $K\theta_{\varphi_{-1}}^{-(j+1)} = K$. Now

$$\theta\varphi_{\varphi_{-1}}^{-(j+1)}|K = \iota \text{ or } \nu \quad \text{and} \quad \theta\varphi_{-1}^{-(j+1)}|Z = \iota \text{ or } \alpha .$$

It follows that $\theta\varphi_{-1}^{-(j+1)} \in \{\iota, \mu_1, \mu_2, \mu_3\}$ and hence $\theta = \eta\varphi_{-1}^{j+1}$ for some $\eta \in \{\iota, \mu_1, \mu_2, \mu_3\}$.

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