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The limit behaviour of exponential terms

by

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Abstract. Let T be the theory of ordered exponential fields satisfying Rolle's schema and the intermediate value schema. It is shown that formulas of the form $\forall x\varphi$, where φ is quantifier free, persist under extensions of models of T. Asymptotic expansions of transfinite length are used to show that the limit of an exponential term in a model of T, if it exists, can be calculated from the coefficients of the term by means of another exponential term.

The main subject of this paper is the behaviour of exponential terms for large values of the argument, taken from some (possibly non-Archimedean) ordered exponential field. This will have consequences for the model theory, algebra and analysis of such fields.

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An exponential term is always a term which is built from the variable x and parameters from some specified set including 0, 1 and -1 by means of the unary function symbols $^{-1}$, e and the binary function symbols $+$ and $\cdot$. For every such term we can easily write a quantifier-free formula which is true for some value a of x iff the term is defined at a. T denotes the first order theory having as axioms

— the axioms of the theory of ordered fields,
— $e(x+y) = e(x)e(y)$,
— $e(x) \geqslant 1+x$,
— for every term $t(x, y_1, \ldots, y_n)$ an axiom saying that for all $c_1, \ldots, c_n, a, b$, if $a<b$ and $t(x, c_1, \ldots, c_n)$ is defined for all $x \in [a, b]$ and $t(a, c_1, \ldots, c_n) = t(b, c_1, \ldots, c_n) = 0$, then there is some $c \in (a, b)$ such that $t'(c, c_1, \ldots, c_n) = 0$ where t' is the formal derivative of t with respect to x (Rolle's schema) and
— for every term $t(x, y_1, \ldots, y_n)$ an axiom saying that for all $c_1, \ldots, c_n, a, b$, if $a<b$ and $t(x, c_1, \ldots, c_n)$ is defined for all $x \in [a, b]$ and $t(a, c_1, \ldots, c_n)<0<t(b, c_1, \ldots, c_n)$, then there is some $c \in (a, b)$ such that $t(c, c_1, \ldots, c_n) = 0$ (intermediate value schema).

In [DW] it has been proved that T is strong enough to prove that formal differentiation using the rule $(e(s))' = s'e(s)$ and differentiation applying the usual

ε-δ-definition yield the same function. The basic laws of calculus can be derived from T as usual. Especially, if $t(a)$ is defined for all a in some convex set J and $t'(a)>0$ for all such a, then t defines a monotonically increasing function on J. Similarly, if s and t define functions on the interval $[a,b]$ and $t'(x)\neq 0$ for all $x\in[a,b]$, then there is some $c\in(a,b)$ such that

$$\frac{s(a)-s(b)}{t(a)-t(b)}=\frac{s'(c)}{t'(c)}.$$

1. Identities of exponential terms. We fix, until stated otherwise, a model C_1 of T and an exponential subfield C of C_1. Terms will always be terms with parameters from C. For every term t we put

$$D_t=\{a\in C_1\colon\ t(a) \text{ is defined in } C_1\}$$

and

$$Z_t=\{a\in D_t\colon\ t(a)=0\}.$$

Though working with real numbers, Richardson [Ri] actually proved the following

LEMMA 1 (Richardson). *Let t be a term and let $J\subseteq D_t$ be an infinite convex subset of C_1. If $J\cap Z_t$ is infinite, then $J\subseteq Z_t$.*

By a neighbourhood system we understand a system N of infinite convex subsets of C_1 such that $J_1,J_2\in N$ implies $J_1\cap J_2\in N$ and $\bigcap N=\emptyset$.

LEMMA 2. *For every neighbourhood system N and every term t such that $J\subseteq D_t$ for some $J\in N$ there is a $J_1\in N$ such that $Z_t\cap J_1=\emptyset$ or $J_1\subseteq Z_t$.*

Proof. If $J\subseteq Z_t$ does not hold, then $J\cap Z_t$ is finite by Richardson's Lemma, say $J\cap Z_t=\{a_1,\ldots,a_n\}$. Since $\bigcap N=\emptyset$, there are $J_i\in N$ such that $a_i\notin J_i$ $(i=1,\ldots,n)$. Then $Z_t\cap J_1\cap\ldots\cap J_n=\emptyset$. ■

Let us fix some neighbourhood system N. C_N denotes the set of all terms t such that $J\subseteq D_t$ for some $J\in N$. Obviously C_N contains x and all $c\in C$, C_N is closed under addition, multiplication and exponentiation, and $t^{-1}\in C_N$ iff there is a $J\in N$ such that $Z_t\cap J=\emptyset$. $t\in C_N$ implies $t'\in C_N$. For $s,t\in C_N$ we say that s and t *are equal modulo N* ($s=t \bmod N$) if $J\subseteq Z_{s-t}$ for some $J\in N$. s_N denotes the class of s under this equivalence relation. As long as we work with a fixed neighbourhood system N we may write s instead of s_N. Analogously, we also use C_N to denote the set of equivalence classes of terms in C_N modulo N. Thus, we can consider C_N in a canonic way as a differential field. $C(x)_N$ denotes the set of equivalence classes of terms not containing exponentiation. Note that for different neighbourhood systems N_1,N_2, $C(x)_{N_1}=C(x)_{N_2}$ so that we may write $C(x)$ instead of $C(x)_N$.

For $s\in C_N$, Qs denotes the set of all rational multiples of s. For $F\subseteq C_N$, $e(F)$ denotes $\{e(f)\colon f\in F\}$. $F+G$, $F\cdot G$, $F<G$ are also defined in the canonic way.

A subfield K of C_N is called *F-normal* if there are $s_0,\ldots,s_{n-1}\in K$ such that

— $s_0\in C(x)$, $s_{i+1}\in C(x)\big(e(Qs_0+\ldots+Qs_i)\big)$ $(i<n-1)$,

— $F=Qs_0+\ldots+Qs_{n-1}$ and $K=C(x)\big(e(F)\big)$,

— if $f\in K$ is a constant, i.e., $f'=0 \bmod N$, then there is a $c\in C$ such that $f=c \bmod N$.

K is called *normal* if K is F-normal for some F.

Theorem 2.8 of [Wi] immediately yields the following

LEMMA 3 (Wilkie). *Supposed K is F-normal and $a,b\in K$ are such that $a'=b'a \bmod N$, then there exists a $c\in C$ and an $f\in F$ such that $a=ce(f) \bmod N$.*

We apply this to prove

LEMMA 4. *Let K be F-normal, $s\in K$ and let $e(s)\in C_N$ be algebraic over K. Then there are $f\in F$, $c\in C$ such that $s=f+c \bmod N$.*

Proof. Let $\sum_{i\leqslant n}a_iu^i$ be a non-trivial polynomial of minimal degree n such that $\sum_{i\leqslant n}a_ie(s)^i=0 \bmod N$, $a_0,\ldots,a_n\in K$, $a_n\neq 0 \bmod N$. We can assume without loss of generality that $a_0=1 \bmod N$. Then

$$\sum_{i=1}^{n}(a_i'+is'a_i)e(s)^{i-1}=0 \bmod N.$$

By the minimality of n this implies $a_n'+ns'a_n=0 \bmod N$. According to Wilkie's Lemma we can find some $c_1\in C$, $f\in F$ such that $a_n=c_1e(f) \bmod N$. $c_1\neq 0$ since $a_n\neq 0 \bmod N$.

Now

$$c_1(f+ns)'e(f+ns)=\big(ce(f+ns)\big)'=\big(a_ne(ns)\big)'=(a_n'+ns'a_n)e(ns)=0 \bmod N.$$

Therefore $f+ns$ is a constant of K and by the normality of K this implies $f+ns=c \bmod N$ for some $c\in C$. Now $s=-fn^{-1}+cn^{-1} \bmod N$ is a representation as required since the group F is divisible. ■

LEMMA 5. *Let K be F-normal, $s\in K$, and let $e(s)\in C_N$ be transcendental over K. Then $K\big(e(Qs)\big)$ is $F+Qs$-normal.*

Proof. The only non-trivial problem is to verify that the constants in $K\big(e(Qs)\big)$ are just the elements of C. It is obviously sufficient to show this for the field $K\big(e(s)\big)$. So let us consider some non-zero element

$$h=\frac{\sum_{i\leqslant n}a_ie(s)^i}{\sum_{j\leqslant m}b_je(s)^j},$$

from $K\big(e(s)\big)$ $(a_i,b_j\in K,\ a_n,b_m\neq 0)$.

We claim that

$$h'=0 \text{ implies } n=m \text{ and } \left(\frac{a_n}{b_m}\right)'=0.$$

(all equations are undrestood modulo N).

In fact $h' = 0$ implies

$$\sum_{k\leqslant n+m} e(s)^k \sum\{(a_i'+a_i is')b_j-(b_j'+b_j js')a_i:\ i+j=k,\ i\leqslant n,\ j\leqslant m\} = 0 .$$

Since $e(s)$ is transcendental over K, this polynomial in $e(s)$ has to be trivial, especially

$$(a_n'+a_n ns')b_m-(b_m'+b_m ms')a_n = 0 .$$

Dividing by b_m^2 yields

$$\left(\frac{a_n}{b_m}\right)' = ((m-n)s)'\frac{a_n}{b_m}.$$

By Wilkie's Lemma $a_n/b_m = ce(f)$ for some $c\in C$, $f\in F$, $c\neq 0$, hence there is some $d\in C$ such that $f+(n-m)s = d$. For $n\neq m$ this implies $e(s) = e\left(\frac{d}{n-m}\right)e\left(\frac{-f}{n-m}\right)\in K$ — a contradiction. Therefore $n = m$ and $f = d\in C$. Then $a_n/b_m = ce(d)\in C$ and also the second part of the claim is proved.

Hence $h' = 0$ implies that h can be written as

$$h = \frac{a_n}{b_m}+\frac{p_1(e(s))}{p_2(e(s))},$$

where p_1 and p_2 are polynomials with coefficients from K such that the degree of p_1 is less than the degree of p_2 if p_1 is non-trivial. In this case

$$\left(\frac{p_1(e(s))}{p_2(e(s))}\right)' \neq 0$$

by our claim, which contradicts $h' = (a_n/b_m)' = 0$. Therefore we see that $p_1 = 0$ and $h = a_n/b_m$ is already a constant of K. ■

From the Lemmas 4 and 5 it is easy to infer

PROPOSITION 6. *For every $a\in C_N$ there exists a normal subfield K of C_N such that $a\in K$.*

COROLLARY 7. *If $a\in C_N$ is a constant, then there is a $c\in C$ such that $a = c \bmod N$.*

LEMMA 8. *Let N_1, N_2 be two neighbourhood systems and let K_1 be an F_1-normal subfield of C_{N_1}. Then there is a field $K_2\subseteq C_{N_2}$, a group F_2 and a map $\varphi\colon K_1\to K_2$ such that*

(i) *K_2 is F_2-normal,*

(ii) *φ is an isomorphism of the differential fields K_1 and K_2,*

(iii) *φ induces an isomorphism between the groups F_1 and F_2,*

(iv) *φ is the identity on $C(x)$,*

(v) *for all $a\in K_1$, if $e(a)\in K_1$, then $e(\varphi(a))\in K_2$ and $e(\varphi(a)) = \varphi(e(a)) \bmod N_2$.*

Proof. If $K_1 = C(x)$, the assertion is obvious. Assume that it has been proved for K_1 and K_2. Choose $s\in K_1$. We want to extend φ to an isomorphism defined on $K_1(e(Qs))$. If $e(s)$ is algebraic over K_1, then Lemma 4 implies that $K_1(e(Qs))$ is K_1. Therefore we can assume that $e(s)$ is transcendental over K_1. If $e(\varphi(s))$ were algebraic over K_2, then, by Lemma 4, there would be an $f\in F_1$ such that $\varphi(s)-\varphi(f)\in C$. But then $e(s)\in K_1$, which contradicts the transcendency of $e(s)$ over K_1. Hence $e(s)$ and $e(\varphi(s))$ are transcendental over K_1 and K_2, respectively, and φ can be extended to an isomorphism of the differential fields $K_1(e(Qs))$ and $K_2(e(Q\varphi(s)))$ sending each $e(qs)$ to $e(q\varphi(s))$ $(q\in Q)$. If a, $e(a)\in K_1(e(Qs))$, then, by Lemma 4, there is a $c\in C$, $f\in F$, $q\in Q$ such that $a = f+qs+c \bmod N_1$. Now φ sends $e(a)$ to $e(c)\varphi(e(f))\varphi(e(qs)) = e(c+\varphi(f)+q\varphi(s)) \bmod N_2$ by our construction of φ. ■

PROPOSITION 9. *Let J_1 be an infinite convex subset of C_1 and let t be a term such that $J_1\subseteq D_t$. Then, to every infinite convex set $J_2\subseteq C_1$ there exists an infinite convex subset $J_2'\subseteq J_2$ such that $J_2'\subseteq D_t$. Moreover, $J_1\subseteq Z_t$ iff $J_2'\subseteq Z_t$.*

Proof. Let J_1 and J_2 be infinite convex subsets of C_1 such that $J_1\subseteq D_t$. We choose arbitrary $c_1, c_2\in J_1$ such that $c_1<c_2$, and define

$$N_1 = \{J_1\}\cup\{\{x\in J_1:\ c_1<x<c\}:\ c_1<c<c_2\}.$$

N_1 is a neighbourhood system containing J_1. Similarly we construct a neighbourhood system N_2 containing J_2.

By Proposition 6 and Lemma 8 we can find normal subfields K_1, K_2 of C_{N_1} respectively C_{N_2} and an isomorphism φ such that $t_{N_1}\in K_1$ and φ has the properties (i) ... (v) of Lemma 8. Now, since $J_1\subseteq D_t$, $K_1\models$ "t is defined at 'x'". Consequently, $K_2\models$ "t is defined at 'x'", which means that there is a $J_2'\in N_2$ such that $J_2'\subseteq D_t$.

Moreover $J_1\subseteq Z_t$ implies $K_1\models t(x) = 0$, $K_2\models t(x) = 0$ and hence $J_2''\subseteq Z_t$ for some $J_2''\in N_2$. But, by Richardson's Lemma, this is equivalent to $J_2'\subseteq Z_t$. The converse is proved similarly. ■

THEOREM 10. *For every term t the following is true.*

(i) *If $D_t\neq\varnothing$, then $C_1\setminus D_t$ is finite.*

(ii) *If $Z_t\neq D_t$, then Z_t is finite.*

Proof. We proceed by induction on the form of t. If t is x or a constant, there is nothing to prove. The induction step is trivial if t is the form t_1^{-1} or $e(t_1)$. So let t be of the form t_1+t_2 or t_1t_2 and assume that the assertion has been proved for t_1 and t_2. We have $D_t = D_{t_1}\cap D_{t_2}$ and (i) is obvious. In order to prove (ii) we assume that $D_t\neq\varnothing$ and that Z_t is infinite. Let

$$C_1\setminus D_t = \{a_0, \dots, a_{n-1}\},\quad \text{where}\quad a_0<\dots<a_{n-1}.$$

We put

$$J_0 = \{x\in C_1:\ x<a_0\},\quad J_{i+1} = \{x\in C_1:\ a_i<x<a_{i+1}\}\quad (i<n-1),$$

$$J_n = \{x\in C_1:\ a_{n-1}<x\}.$$

If $n = 0$, we put similarly $J_0 = C_1$. Then there is some i such that $Z_t \cap J_i$ is infinite. By Richardson's Lemma we have $J_i \subseteq Z_t$. Proposition 9 implies that for every $j \leqslant n$ there exists some infinite convex subset J_j' of J_j such that $J_j' \leqslant Z_t$. Again by Richardson's Lemma we conclude that $J_j \subseteq Z_t$ for all $j \leqslant n$, i.e., $D_t = Z_t$. ■

COROLLARY 11. *For every neighbourhood system N and all terms t and s*

(i) *$t \in C_N$ iff $C \setminus D_t$ is finite,*

(ii) *if $t, s \in C_N$, then $t = s \bmod N$ iff $D_{t-s} \cap C = Z_{t-s} \cap C$.*

Proof. If $t \in C_N$, then, by Theorem 10 (i), $C_1 \setminus D_t$ is finite and also $C \setminus D_t$ is finite. Conversely, if $C \setminus D_t$ is finite, then $D_t \neq \emptyset$, thus, again by Theorem 10 (i), $C_1 \setminus D_t$ is finite. Hence there is some $J \in N$ such that $J \cap (C_1 \setminus D_t) = \emptyset$, i.e., $J \subseteq D_t$.

In order to prove (ii) suppose $t = s \bmod N$. Then Z_{t-s} cannot be finite and hence $Z_{t-s} = D_{t-s}$ by Theorem 10 (ii).

Conversely, assume that $D_{t-s} \cap C = Z_{t-s} \cap C$. Since $t, s \in C_N$, by (i) $C \setminus D_{t-s}$ is finite. Therefore $C \cap Z_{t-s}$ is infinite and, by Theorem 10 (ii), $Z_{t-s} = D_{t-s}$. Hence every set $J \in N$ such that $J \subseteq D_{t-s}$ proves that $t = s \bmod N$. ■

Since C was assumed to be closed under the algebraic operations and exponentiation, it is now easy to show

COROLLARY 12. *The structure of C_N as a differential exponential field is uniquely determined by the diagram of C and the fact that it is of the form C_N for some neighbourhood system N in some T-model C_1.*

2. Dominance of exponential terms. Recall that we have a fixed model C_1 of T and an exponential subfield C of C_1 and that terms are understood to be terms having the variable x, parameters from C and being built by means of the operations $+, \cdot, ^{-1}$ and e only. The neighbourhood system of infinity of C_1 is defined to be $\{\{x \in C_1 : x > c_1\} : c_1 \in C_1\}$ and is denoted by ∞. Since T includes the intermediate value schema, Theorem 10 implies that C_∞ becomes an ordered field if we define

$$s < t \text{ iff there is some } J \in \infty \text{ such that } s(x) < t(x) \text{ for all } x \in J.$$

If $s < t$, then t is said to dominate s. The aim of this chapter is to show that the relation $<$ is uniquely determined by the structure of C as an ordered exponential field and the fact that it is the dominance relation on some T-model C_1 extending C. Since we know from the theory of ordered fields that the dominance relation for rational functions is independent of the choice of C_1, it suffices, by Proposition 6, to show that the order extends uniquely from $C(x)$ to each normal subfield of C_∞.

Though we work with the fixed model C_1, with its neighbourhood system ∞ and its dominance relation our arguments are independent of the particular choice of C_1.

If K is a subfield of C_∞ and $s, t \in C_\infty$, we define

$$s \ll t \bmod K \text{ iff } |s| \leqslant a|t| \text{ for all positive } a \in K.$$

An equivalence relation is defined by

$$s \sim t \bmod K \text{ iff } s - t \ll t \bmod K.$$

The following rules are easily verified.

If $s \ll t \bmod K$, $a \in K$, $a \neq 0$, then $s \ll at \bmod K$.

If $s_1 \ll t \bmod K$, $s_2 \ll t \bmod K$, then $s_1 + s_2 \ll t \bmod K$.

If $s_1 \ll t_1 \bmod K$ and $t_2 \ll s_2 \bmod K$, $s_2, t_2 \neq 0$, then $s_1 s_2^{-1} \ll t_1 t_2^{-1} \bmod K$.

$\sim \bmod K$ is a congruence relation with respect to $\ll$ and multiplication.

Unless stated otherwise, all fields $K, K_1, \ldots$ are normal subfields of C. All limits are understood for $x \to \infty$.

LEMMA 13. *For all terms s and t such that $|t| < x^{-1}$,*

$$s \ll t \bmod C \quad \textit{implies} \quad s' \ll t' \bmod C.$$

Proof. Supposed that the assertion is wrong, there is some positive $\varepsilon \in C$ such that $|s'(x)| > \varepsilon |t'(x)|$ for large values of x in C_1. If t is a constant of C_∞, then $t = 0$ since $|t| < x^{-1}$. But then $s \ll t \bmod C$ means $s = 0$ and hence $s' \ll t' \bmod C$. So we can assume that $t' \neq 0$. As we have mentioned in the introduction, for large $a, b \in C_1$ such that $a < b$ there has to be a $c \in C_1$ such that $a < c < b$ and

$$\left| \frac{s(a) - s(b)}{t(a) - t(b)} \right| = \left| \frac{s'(c)}{t'(c)} \right| > \varepsilon .$$

Our assumption yields $\lim t = \lim s = 0$. Therefore — if b tends to infinity — we see that $|s(a)/t(a)| \geqslant \varepsilon$ for large a, which contradicts the assumption that $s \ll t \bmod C$. ■

Let K be a subfield of C_∞ and $r \in C_\infty$. r is said to be *coinitial* in K if $r \in K$ and if for every positive $a \in K$ there is a natural number n such that $|r|^n < a$.

LEMMA 14. *Let K be F-normal and let r be coinitial in K. If there exists a nontrivial polynomial p with coefficients from K such that $p(e(r)) \ll 1 \bmod K$, then $e(r) \in K$.*

Proof. Let $p(u) = \sum_{i \leqslant n} a_i u^i$ be of minimal degree n such that $p(e(r)) \ll 1 \bmod K$. Since r is coinitial in K and $C(x) \subseteq K$, we have $|r| < q$ for all positive rational numbers q. Therefore $e(r) \geqslant 1 + r > \frac{1}{2}$. Now, if $a_0 = 0$, this suffices to claim that also

$$\sum_{i=1}^{n} a_i e(r)^{i-1} \ll 1 \bmod K ,$$

which contradicts the minimality of n. Thus we have $a_0 \neq 0$. Dividing p by a_0 if necessary, we see that we could have assumed that $a_0 = 1$. Now, for large m, $1 + \sum_{i=1}^{n} a_i e(r)^i \ll r^m \bmod C$ and $|r^m| < x^{-1}$. By Lemma 13 this implies

$$\sum_{i=1}^{n} (a_i' + i a_i r') e(r)^i \ll m r' r^{m-1} \bmod C .$$

Since r is coinitial in K and $r' \in K$, we have for each positive $a \in K$ for sufficiently large m $|mr'r^{m-1}| < a$ and therefore

$$\sum_{i=1}^{n} (a_i' + ia_i r') e(r)^{i-1} \ll 1 \bmod K$$

since $e(r) > \frac{1}{2}$. By our choice of n this is only possible if the polynomial on the left is trivial, especially $a_n' + na_n r' = 0$. By Wilkie's Lemma, $a_n = ce(f)$ for some $c \in C$, $f \in F$, and $(a_n e(nr))' = 0$ implies that $f + nr = d$ for some $d \in C$. Now $e(r) = e(d/n)e(-f/n) \in K$. ■

For every natural number k we define $e_k(x) = \sum_{i \leqslant k} \frac{x^i}{i!}$.

Proposition 15. *If $r \ll 1 \bmod Q$, then $|e(r) - e_k(r)| \leqslant |r|^k$.*

Proof. It is an easy exercise in calculus to prove the sentences

$$\forall x (x \geqslant 0 \to e(x) \leqslant e_k(x)),$$
$$\forall x (x \leqslant 0 \to e_k(x) \leqslant e(x) \leqslant e_{k+1}(x)) \quad (k \text{ odd}),$$
$$\forall x (x \geqslant 0 \wedge e(x) < 2 \to e(x) - e_k(x) \leqslant x^k)$$

in T by induction on k. These sentences imply

$$e(x) < 2 \to |e(x) - e_k(x)| \leqslant |x|^k.$$

Now, if $r \ll 1 \bmod Q$, then $e(r)(x) < 2$ for large values of x and hence $|e(r) - e_k(r)| \leqslant |r|^k$. ■

Lemma 16. *If r is coinitial in the normal field K, then*

(i) K is dense in $K(e(r))$.

(ii) for all polynomials $p_1(u)$, $p_2(u) \in K[u]$ such that $p_2 \neq 0$ $\frac{p_1(e(r))}{p_2(e(r))} > 0$ iff $\frac{p_1(e_k(r))}{p_2(e_k(r))} > 0$ for all sufficiently large k.

Proof. By Lemma 14 we can assume that for every non-trivial polynomial p with coefficients from K there is a positive $a \in K$ such that $|p(e(r))| > a$. Therefore K is cofinal in $K(e(r))$. The sequence $(e_k(r)\colon k < \omega)$ approximates $e(r)$ in $K(e(r))$ by Proposition 15. If $p_1(u), p_2(u) \in K[u]$, $p_2 \neq 0$, then the function $p_1(u)/p_2(u)$ is defined at $u = e(r)$ in the ordered field $K(e(r))$. By the continuity of rational functions, $(p_1(e_k(r))/p_2(e_k(r))\colon k < \omega)$ approximates $p_1(e(r))/p_2(e(r))$. Now (i) and (ii) follow immediately. ■

We call K_2 a *simple inner extension of* K_1 if $K_2 = K_1(e(Qr))$ for some r which is coinitial in K_1.

The field L is called an *inner extension of* K if it is obtained from K by a finite number of simple inner extensions.

From Corollary 7 it is clear that inner extensions of normal fields are again normal. Lemma 16 yields immediately

Proposition 17. *If K_2 is an inner extension of K_1, then the order extends uniquely from K_1 to K_2 and K_1 is dense in K_2.*

Recall that we are still working within C_∞ and note the obvious

Lemma 18. *Let K_2 be an inner extension of K_1, let L_1 be a subfield of C_∞ cofinally containing K_1 and let L_2 be the least subfield of C_∞ that contains L_1 and K_2. Then L_2 is an inner extension of L_1 which contains K_2 cofinally.*

Now we come to another important type of field extensions. Let G be a divisible ordered subgroup of $(K, +, 0, <)$. For the following definition we distinguish two cases.

a) $\{x^n\colon n < \omega\}$ is cofinal in K.

In this case $K(e(G))$ is said to be an *outer extension of K by means of G* if for each positive $g \in G$ there is a positive $c \in C$ such that $g > cx$.

b) $e(F)$ is a cofinal subset of K for some $F \subseteq K$.

In this case $K(e(G))$ is said to be an *outer* extension of K *by means of G* if for each positive $g \in G$ and for each $f \in F$ we have $f < g$.

Note that the order of K completely determines if $K(e(G))$ is an outer extension of K. Note also that $K(e(G))$ is normal if K is normal and if G has finite dimension as a vector space over the rationals. It is the crucial property of the outer extension that $e(g) > K$ for all positive $g \in G$. Let us look at some outer extension K_1 of K by means of some divisible group G. Every non-zero element of K_1 has a representation as

$$(+) \qquad f = \frac{\sum_{i \leqslant n} a_i e(g_i)}{\sum_{j \leqslant m} b_j e(h_j)},$$

where $a_i, b_j \in K$, $g_i, h_j \in G$, $g_0 > \ldots > g_n$, $h_0 > \ldots > h_m$, $a_0, b_0 \neq 0$. We define $\lim(f) = a_0/b_0$ and $\deg(f) = g_0 - h_0$ and have to show that these definitions do not depend on the concrete representation of f.

So let us assume that we also have the representation

$$f = \frac{\sum_{\iota \leqslant \nu} \alpha_\iota e(\gamma_\iota)}{\sum_{\lambda \leqslant \mu} \beta_\lambda e(\delta_\lambda)},$$

where $\alpha_\iota, \beta_\lambda \in K$, $\gamma_\iota, \delta_\lambda \in G$, $\gamma_0 > \ldots > \gamma_\nu$, $\delta_0 > \ldots > \delta_\mu$, $\alpha_0, \beta_0 \neq 0$. Then

$$\sum_{i \leqslant n} a_i e(g_i) \sum_{\lambda \leqslant \mu} \beta_\lambda e(\delta_\lambda) = \sum_{\iota \leqslant \nu} \alpha_\iota e(\gamma_\iota) \sum_{j \leqslant m} b_j e(h_j),$$

$$(*) \qquad a_0 \beta_0 e(g_0 + \delta_0) \Big(1 + \sum_{i=1}^{n} \frac{a_i}{a_0} e(g_i - g_0)\Big)\Big(1 + \sum_{\lambda=1}^{\mu} \frac{\beta_\lambda}{\beta_0} e(\delta_\lambda - \delta_0)\Big)$$
$$= \alpha_0 b_0 e(\gamma_0 + h_0) \Big(1 + \sum_{\iota=1}^{\nu} \frac{\alpha_\iota}{\alpha_0} e(\gamma_\iota - \gamma_0)\Big)\Big(1 + \sum_{j=1}^{m} \frac{b_j}{b_0} e(h_j - h_0)\Big).$$

Now, for $i\geqslant 1$, $g_0-g_i>0$. Our assumption that K_1 is an outer extension of K by means of G implies that $e(g_i-g_0)\ll 1 \bmod K$. Therefore $1+\sum_{i=1}^{n}\frac{a_i}{a_0}e(g_i-g_0)\sim 1 \bmod K$. Similar considerations for the other factors in the equation (*) lead to

$$a_0\beta_0 e(g_0+\delta_0)\sim \alpha_0 b_0 e(\gamma_0+h_0) \bmod K$$

and

$$\frac{a_0\beta_0}{b_0\alpha_0}\sim e(h_0-g_0+\gamma_0-\delta_0) \bmod K .$$

But this is only possible for $h_0-g_0+\gamma_0-\delta_0 = 0$, i.e., $g_0-h_0 = \gamma_0-\delta_0$ and $\frac{a_0\beta_0}{b_0\alpha_0}\sim 1 \bmod K$, hence $\frac{a_0}{b_0}=\frac{\alpha_0}{\beta_0}$. We note that $\lim(f)$ and $\deg(f)$ for $f\in K_1$ are already determined by the order of K.

PROPOSITION 19. *Let K_1 be an outer extension of K by means of G.*

(i) *For all non-zero $f\in K_1$*

$$f\sim \lim(f)e(\deg(f)) \bmod K .$$

(ii) *For all non-zero $f\in K_1$*

$$f>0 \text{ iff } \lim(f)>0 .$$

(iii) *$e(G)$ is cofinal in K_1 if G is non-trivial.*

Proof. Let f have the representation (+). Then

$$f-\frac{a_0}{b_0}e(g_0-h_0) = \frac{\sum_{i=1}^{n} a_i b_0 e(g_i+h_0)-\sum_{j=1}^{m} a_0 b_j e(g_0+h_j)}{b_0^2 e(2h_0)+\sum_{j=1}^{m} b_0 b_j e(h_0+h_j)} .$$

The denominator is $\bmod K$ equivalent to $b_0^2 e(2h_0)$ while for $i,j\geqslant 1$

$$e(g_i+h_0)\ll e(g_0+h_0) \bmod K \quad\text{and}\quad e(g_0+h_j)\ll e(g_0+h_0) \bmod K .$$

Therefore, $f-\frac{a_0}{b_0}e(g_0-h_0)\ll e(g_0-h_0) \bmod K$. This proves (i).

(ii) and (iii) are easy consequences of (i). ■

COROLLARY 20. *If K_1 is an outer extension of K, then the order extends uniquely from K to K_1.*

COROLLARY 21. *Let K_1 be an outer extension of K by means of G, let L be a subfield of C cofinably containing K and let $L_1 = L(e(G))$. Then L_1 is an outer extension of L by means of G, which contains K_1 cofinally.*

PROPOSITION 22. *Let K_1 be an outer extension of K by means of an Archimedean-ordered group G. Then, every $f\in K_1$ has a unique representation $f=p+r$ where p is a linear combination of elements of $\{e(g)\colon g\in G,\ g\geqslant 0\}$ with coefficients from K and r is coinitial in K_1.*

Proof. Let f have the representation (+). If $m = 0$, the assertion is obvious. So we can assume without loss of generality that $m>0$ and $b_1\neq 0$. We assume, moreover, that $g_n\geqslant h_0$. It is easily checked that

$$f=\sum_{i\leqslant n}\frac{a_i}{b_0}e(g_i-h_0)+f_1$$

where

$$f_1 = \frac{r_1}{\sum_{j\leqslant m} b_j e(h_j)} \quad\text{and}\quad r_1 = -\Big(\sum_{j=1}^{m} b_j e(h_j)\Big)\Big(\sum_{i\leqslant n}\frac{a_i}{b_0}e(g_i-h_0)\Big).$$

We have $\deg(f_1) = \deg(f)-(h_0-h_1)$. Now we can apply the same procedure to f_1. Iterating this k-times we find a representation $f=p_k+f_k$, where p_k is a linear combination as required, and $\deg(f_k) = \deg(f)-k(h_0-h_1)$. Since G is Archimedean-ordered we can find a least k such that $\deg(f_k)<0$. Then $p = p_k$ and $r = r_k$ define a representation as required.

In order to eliminate the assumption that $g_n\geqslant h_0$ let l be the greatest number such that $g_l\geqslant h_0$. Then

$$\frac{\sum_{i\leqslant l} a_i e(g_i)}{\sum_{j\leqslant m} b_j e(h_j)}$$

has a representation $p+r_0$ having the desired properties. Moreover, we have $\deg(f-p-r_0)<0$, which implies that $f-p$ is coinitial in K_1.

Now let us assume that $f = p_1+r_1 = p_2+r_2$ where

$$p_1 = \sum_{i\leqslant k}\alpha_i e(g_i),\qquad p_2 = \sum_{i\leqslant k}\beta_i e(g_i)$$

are linear combinations of the required form and r_1, r_2 are coinitial in K_1. If i is minimal such that $\alpha_i\neq\beta_i$, then $0 = p_1-p_2+r_1-r_2\sim(\alpha_i-\beta_i)e(g_i) \bmod K$, a contradiction. Therefore $p_1 = p_2$ and also $r_1 = r_2$. ■

COROLLARY 23. *Let K_1 be an outer extension of K by means of an Archimedean-ordered group G, and let K_1 be dense in the field K_2. Then every $f\in K_2$ has a unique representation $f=p+r$ where p is a linear combination of elements of $\{e(g)\colon g\in G,\ g\geqslant 0\}$ with coefficients from K and r is coinitial in K_2.*

Proof. Fix some $g\in G$, $g<0$ and choose $f_1\in K_1$ such that $|f-f_1|<e(g)$. f_1 has a representation $f_1 = p+r_1$ of the required form. Hence f has such a representation since r_1+f-f_1 is coinitial in K_2. ■

Similarly one can prove

PROPOSITION 24. *Let $C(x)$ be dense in K. Then every $f\in K$ has a unique representation $f=p+r$ where $p\in C[x]$ and r is coinitial in K.*

We now come to a central concept of our investigation of the structure of C_∞.

A sequence of fields $C(x) = K_0 \subseteq K_1 \subseteq \ldots \subseteq K_{2n+1} = K$ and ordered groups $G_1, \ldots, G_n$ is called a *ladder for the field K if*

(i) for every $i = 1, \ldots, n$ there is a $j<i$ such that

a) if $j>0$, then there is a non-trivial finite-dimensional Archimedean-ordered Q-vector space $V \subseteq (K_{2j-1}, +, 0, <)$ and a positive $g \in G_j$ such that $G_i = e(g)V \subseteq K_{2i-1}$;

b) if $j = 0$, then there is a non-trivial finite-dimensional Archimedean-ordered Q-vector space $V \subseteq (C, +, 0, <)$ and a positive $m \in \omega$ such that $G_i = x^m \subseteq K_{2i-1}$;

(ii) $K_{2i} = K_{2i-1}(e(G_i))$ is an outer extension of K_{2i-1} by means of G_i;

(iii) K_{2i+1} is an inner extension of K_{2i}.

The reader should satisfy himself that j, V, g and m occurring in (i) are unique because of (ii).

We say that a ladder $L_0, \ldots, L_{2m+1}, H_1, \ldots, H_m$ extends the ladder $K_0, \ldots, \ldots, K_{2n+1}, G_1, \ldots, G_n$ if for each $i = 0, \ldots, n$ there is a $j \in \{0, \ldots, m\}$ such that $G_i \subseteq H_j$, $K_{2i} \subseteq L_{2j}$, $K_{2i+1} \subseteq L_{2j+1}$. Proposition 17 and Corollary 20 yield immediately

PROPOSITION 25. *If there exists a ladder for K, then the order on K is unique.*

LEMMA 26. *If there is a ladder $K_0, \ldots, K_{2n+1}, G_1, \ldots, G_n$ for K and if r is coinitial in one of the fields K_{2i+1}, then there is a ladder for $K(e(Qr))$ extending $K_0, \ldots, \ldots, K_{2n+1}, G_1, \ldots, G_n$.*

Proof. We assume that r is coinitial in K_{2i+1}. $K_{2i+1}(e(Qr))$ is a simple inner extension of K_{2i+1}, hence it contains K_{2i+1} cofinally and is an inner extension of K_{2i}.

$K_{2i+2}(e(Qr)) = K_{2i+1}(e(Qr))(e(G_{i+1}))$ is an outer extension of $K_{2i+1}(e(Qr))$ by means of G_{i+1} by Corollary 21 and $e(G_{i+1})$ is cofinal in K_{2i+2} and in $K_{2i+2}(e(Qr))$. $K_{2i+3}(e(Qr))$ is an inner extension of $K_{2i+2}(e(Qr))$ by Lemma 18. By this Lemma we also know that K_{2i+3} is cofinal in $K_{2i+3}(e(Qr))$.

Similarly we proceed to show that $K_0, \ldots, K_{2i}, K_{2i+1}(e(Qr)), \ldots, K_{2n+1}(e(Qr))$, $G_1, \ldots, G_n$ is a ladder for $K(e(Qr))$. ■

LEMMA 27. *If there is a ladder $K_0, \ldots, K_{2n+1}, G_1, \ldots, G_n$ for K, if $g \in G_j$, $g>0$, $a \in K_{2j-1}$ for some $j>0$, then there is a ladder for $K(e(Qae(g)))$ extending $K_0, \ldots \ldots, K_{2n+1}, G_1, \ldots, G_n$.*

Proof. We can assume that a is positive. Let $i_1, \ldots, i_m$ be all $i = 1, \ldots, n$ in increasing order such that $G_{i_\nu} = e(g)V_\nu$ for some Archimedean-ordered Q-vector space $V_\nu \subseteq (K_{2j-1}, +, 0, <)$.

Case 1. There is a $\nu = 1, \ldots, m$ such that $V_\nu + Qa$ is Archimedean-ordered. Then, for $i = i_\nu$, $G_i + Qae(g)$ is Archimedean-ordered and

$K_0, \ldots, K_{2i-1}, K_{2i}(e(Qae(g))), \ldots, K_{2n+1}(e(Qae(g))), G_1, \ldots, G_{i-1}, G_i + Qae(g), G_{i+1}, \ldots G_n$ is a ladder for $K(e(Qae(g)))$ by Lemma 18 and Corollary 21.

Case 2. There is a $\nu = 1, \ldots, m$ and a $b \in V_1 + \ldots + V_m + Qa$ such that $V_\nu < b < V_{\nu+1}$. Observe that $b \in K_{2j-1}$ and $a \in V_1 + \ldots + V_m + Qb$. Then, for $i = i_\nu$ we can show as in case 1 that

$$K_0, \ldots, K_{2i+1}, K_{2i+1}(e(Qbe(g))), K_{2i+1}(e(Qbe(g))), K_{2i+2}(e(Qbe(g))), \ldots$$
$$\ldots, K_{2n+1}(e(Qbe(g))), G_1, \ldots, G_i, Qbe(g), G_{i+1}, \ldots, G_n$$

is a ladder for $K(e(Qae(g)))$.

Case 3. $m = 0$. In this case, for every $i = 1, \ldots, n$ either $G_i < ae(g)$ or $ae(g) < h$ for every positive $h \in G_i$. Let $i \leqslant n$ be maximal such that $G_i < ae(g)$ or $i = 0$ if there is no such $i \geqslant 1$. Then

$$K_0, \ldots, K_{2i+1}, K_{2i+1}(e(Qae(g))), K_{2i+1}(e(Qae(g))), K_{2i+2}(e(Qae(g))), \ldots$$
$$\ldots, K_{2n+1}(e(Qae(g))), G_1, \ldots, G_i, Qae(g), G_{i+1}, \ldots, G_n$$

is a ladder for $K(e(Qae(g)))$. ■

COROLLARY 28. *If there exists a ladder $K_0, \ldots, K_{2n+1}, G_1, \ldots, G_n$ for K and if p is a linear combination of elements of the form $e(g)$ for positive $g \in G_j$, with coefficients from K_{2j-1} for some $j>0$, then there is a ladder for a field $K_1 \supseteq K(e(Qp))$ which extends $K_0, \ldots, K_{2n+1}, G_1, \ldots, G_n$.*

The following Lemma is proved in a similar way.

LEMMA 29. *If there is a ladder for K and if $p \in C[x]$ is such that $p(0) = 0$, then there is a ladder for a field $K_1 \supseteq K(e(Qp))$.*

PROPOSITION 30. *If there is a ladder for K and $f \in K$, then there is a ladder for a field $K_1 \supseteq K(e(Qf))$.*

Proof. Let $K_0, \ldots, K_{2n+1}, G_1, \ldots, G_n$ be a ladder for K and let $f \in K$. By Corollary 23, $f = p_n + f_n + r_n$, where p_n is a linear combination of elements $e(g)$ with positive $g \in G_n$ and coefficients from K_{2n-1}, $f_n \in K_{2n-1}$ and r_n is coinitial in K. Similarly, f_n decomposes into a linear combination p_{n-1} of $e(g)$, $g \in G_{n+1}$, $g>0$, with coefficients from K_{2n-3}, an $f_{n-1} \in K_{2n-3}$, and an element r_{n-1} which is coinitial in K_{2n-1}. Continuing this process and using Proposition 24 we obtain a representation $f = p_n + \ldots + p_0 + f_0 + r_0 + \ldots + r_n$ where each p_i ($i>0$) is a linear combination of $e(g)$ with positive $g \in G_i$ and coefficients from K_{2i-1}, $p_0 \in C[x]$, $p_0(0) = 0$, $f_0 \in C$ and where each r_i is coinitial in K_{2i+1}.

Now, by Lemma 26 and Corollary 28, we can extend the ladder for K to a ladder for some field K_1 containing $e(Qp_i)$, $e(Qr_i)$ $(i = 0, \ldots, n)$ and hence containing $e(Qf)$. ■

THEOREM 31. *The order extends uniquely from C to C_∞.*

Proof. For every $f \in C_\infty$ we can find, by Proposition 30, a field K which has a ladder and contains f. Then Theorem 31 follows by Proposition 25. ■

COROLLARY 32. *If C_1 is a model of T containing the exponential field C and if s, t are terms with parameters from C, then $C_1 \models$ "s dominates t" iff* Diagram$(C) \cup \cup T \vdash$ "*s dominates t*".

3. Applications.

THEOREM 33. *Let C_1, C_2 be models of T such that $C_1 \subseteq C_2$. Let t be a term with parameters from C_1 such that $t(c) \neq 0$ for some $c \in C_1$. Then there is a $c_1 \in C_1$ such that*

(i) *$t(x)$ is defined for every $x \in C_2$ such that $x > c_1$ and*

(ii) *$C_2 \models \forall x(x > c_1 \to t(x) > 0)$ or $C_2 \models \forall x(x > c_1 \to t(x) < 0)$.*

Proof. Clearly, the assertion is true for every term without exponentiation. Let t be a term with a minimal number of iterations of the exponential function such that the assertion is false for t. It suffices to obtain a contradiction for $t = \sum_{i \leqslant n} s_i e(t_i)$, where all the s_i and t_i contain only a smaller number of iterations of the exponential function. Let n be chosen minimal too. If s_n were identically zero in C_1, it would be zero in C_2 by Theorem 10 (ii) and we could omit the summand $s_n e(t_n)$. Hence, there is a $c \in C_1$ such that e.g. $C_2 \models \forall x(x > c \to s_n(x) > 0)$. We put $r = t/s_n$. If r is not constant, then our assumption on t implies that there exists a $d \in C_1$ such that either

$$C_2 \models \forall x(x > d \to r'(x) > 0) \quad \text{or} \quad C_2 \models \forall x(x > d \to r'(x) < 0).$$

So we can assume without loss of generality that $r(x)$ is monotonically decreasing in C_2 for $x > d$. If $t(d_1) < 0$ for some $d_1 \in C_1$, $d_1 > d$, then we are done. If, however $t(d_1) > 0$ for every $d_1 \in C_1$, $d_1 > d$, then, by Corollary 32,

$$T \cup \mathrm{Diagram}(C_1) \vdash \exists y \forall x(x > y \to r(x) > 0).$$

Hence $r(x)$ is positive for large $x \in C_2$, too. But, since r is monotonically decreasing, this implies that $r(x) > 0$ for all $x > d$, $x \in C_2$ and hence the same holds for t, which contradicts our assumption. ■

COROLLARY 34. *Let C_1, C_2 be models of T such that $C_1 \subseteq C_2$ and let t be a term with parameters from C_1 which is not identically zero in C_2. If $c \in C_2$ is such that $t(c) = 0$, then $c \in C_1$.*

Proof. We choose $t = \sum_{i \leqslant n} s_i e(t_i)$ minimal such that the assertion is false, i.e. $t(c) = 0$, $c \in C_2 \setminus C_1$. Theorem 10 (ii) implies that t is not identically zero in C_1. By Theorem 33, c cannot be greater than C_1. Similarly, $c < C_1$ cannot hold. Therefore, there are $a, b \in C_1$ such that $a < c < b$.

By our assumption $s_n(c) \neq 0$. So we can consider the term $t/s_n = r$. If r is constant, t must be identically zero. Otherwise, by the induction hypothesis, all the finitely many zeros of r' are in C_1 and we can assume without loss of generality that $r'(x) \neq 0$ if $a < x < b$, $x \in C_2$. Hence r is strictly monotonic in $[a, b]$. Then $t(c) = 0$ implies $r(a)r(b) < 0$ and since the intermediate value schema holds in C_1, there must be a $c_1 \in C_1$ such that $a < c_1 < b$ and $r(c) = 0$. But $c_1 = c$ since r is strictly monotonic in $[a, b]$. ■

THEOREM 35. *Let C_1, C_2 be models of T such that $C_1 \subseteq C_2$. If $\varphi(x)$ is a quantifier free formula with parameters from C_1 and one free variable x, then $C_1 \models \exists x \varphi(x)$ if and only if $C_2 \models \exists x \varphi(x)$.*

Proof. We can assume that $\varphi(x)$ is the formula

$$\bigwedge_{i=1}^{n} t_i(x) = 0 \wedge \bigwedge_{j=1}^{m} s_j(x) > 0.$$

Let $C_2 \models \varphi(c)$. If one of the t_i's is not identically zero in C_2, then $c \in C_1$ by Corollary 34. So we can assume that $n = 0$. If $c > C_1$, then $s_j(x) > 0$ for all large $x \in C_1$ ($j = 1, \ldots, m$) by Theorem 33. Similarly, $C_1 \models \exists x \varphi(x)$ if $c < C_1$. If $a < c < b$ for some $a, b \in C_1$, then we can choose a, b such that $s_j(x) \neq 0$ for all $x \in [a, b]$, $x \in C_2$, $j = 1, \ldots, m$. Now, by the intermediate value schema, $s_j(x) > 0$ for all such x, j. Hence $C_1 \models \varphi(a)$. ■

Our study of the structure of C_∞ provides us with a new representation of exponential terms. In order to describe it we require some general algebraic concepts.

Let K be an ordered field and let H be a multiplicatively written ordered Abelean group. A function σ: $H \to K$ is called a *transfinite series on H with coefficients from K* if its support, i.e., $\mathrm{supp}(\sigma) = \{h \in H: \sigma(h) \neq 0\}$, has no infinite ascending chain with respect to the order of H. The set of all such series is denoted by $K[[H]]$. For given series σ, τ we define $\sigma + \tau$ and $\sigma \cdot \tau$ by

$$(\sigma + \tau)(h) = \sigma(h) + \tau(h),$$

$$(\sigma \cdot \tau)(h) = \sum \{\sigma(h_1)\tau(h_2): h_1 h_2 = h\}.$$

Moreover, we define $\sigma > 0$ iff $\mathrm{supp}(\sigma) \neq \emptyset$ and $\sigma(\max \mathrm{supp}(\sigma)) > 0$. We quote the following two lemmas from [Fu], pp. 134ff.

LEMMA 36 (Hahn, Neumann). *$K[[H]]$ is an ordered field.*

LEMMA 37. *If $\sigma \in K[[H]]$ is such that $\sigma \neq 0$ and $\mathrm{supp}(\sigma) < 1$, then for any sequence a_n of elements from K, $\sum_{n=0}^{\infty} a_n \sigma^n$ is meaningful.*

Now let C be any ordered exponential field. Let $H_{-1} = \{1\}$ and $H_0 = \{x^m: m \in Z\}$. H_{-1} and H_0 are multiplicatively written ordered Abelean groups. We put $C^0 = C[[H_0]]$. Now suppose H_n to be defined and $C^n = C[[H_n]]$. Let

$$G^{n+1} = \{\sigma \in C^n: \mathrm{supp}(\sigma) > H_{n-1}\} \cup \{0\}$$

and let E be an isomorphism of the additively written ordered group $G^{n+1} \subseteq (C^n, +, 0, <)$ onto some new multiplicatively written ordered group $E(G^{n+1})$. We define $H_{n+1} = H_n E(G^{n+1})$ and order this group lexicographically by

$$hE(\sigma) > 1 \quad \text{iff} \quad \sigma > 0 \text{ or } \sigma = 0 \text{ and } h > 1.$$

Then $C^{n+1} = C[[H_{n+1}]]$ is defined and is isomorphic with $C^n[[E(G^{n+1})]]$. The groups H_n and the fields C^n form increasing chains. Let H_∞ and C^∞ denote the limits of these chains. Then $C^\infty \subseteq C[[H]]$ and the isomorphisms E defined above can be combined to an embedding

$$E\colon \{\sigma \in C^\infty\colon \operatorname{supp}(\sigma) > 1\} \to H_\infty .$$

C^∞ becomes an exponential field as follows. For every $\sigma \in C^\infty$ there are unique σ_∞, $\sigma_0 \in C^\infty$, $c \in C$ such that $\sigma = \sigma_\infty + c + \sigma_0$, $\operatorname{supp}(\sigma_\infty) > 1 > \operatorname{supp}(\sigma_0)$. We put

$$e(\sigma) = E(\sigma_\infty)e(c)\sum_{i=0}^{\infty}\frac{1}{i!}\sigma_0^i .$$

Now let us again consider C as a subfield of some T-model C_1 and let C_∞ be the field of functions definable by exponential terms with parameters from C and ordered according to their limit behaviour in C_1. Until stated otherwise we fix a normal subfield K of C_∞ and a ladder $K_0, \ldots, K_{2n+1}, G_1, \ldots, G_n$ for K. We are going to define an embedding $\sigma\colon a \mapsto \sigma_a$ of the ordered field K into C^∞ such that

a) $\sigma_a = a$ if $a \in C$ or $a \in \{x^m\colon m \in \mathbf{Z}\}$,

b) if $a \in G_j$, then $\operatorname{supp}(\sigma_a) > 1$ and $\sigma_{e(a)} = E(\sigma_a)$ $(j = 1, \ldots, n)$,

c) if $\Gamma_0 = \{x^m\colon m \in \mathbf{Z}\}$ and for each $i = 1, \ldots, n$ $\Gamma_i = \{E(\sigma_g)\colon g \in G_i\}$ and if $a \in K_{2j+1}$, then $\operatorname{supp}(\sigma_a) \subseteq \Gamma_0 \cdot \ldots \cdot \Gamma_j$.

We follow the convention that $\sigma_a(h) = 0$ unless stated otherwise.

For every $a \in K_1$ and for every integer m, by Proposition 24 there is a unique representation $ax^{-m} = p + c + r$ such that $p \in C[x]$, $p(0) = 0$ and r is coinitial in K_1. We put $\sigma_a(x^m) = c$. Then for every m

$$a - \sum_{\nu \geq m} \sigma_a(x^\nu)x^\nu \ll x^m \bmod C .$$

It is routine to verify that $\sigma\colon a \mapsto \sigma_a$ is an embedding of the ordered field K_1 into $C[[\Gamma_0]]$.

Now suppose that σ has been defined on K_{2i+1} such that a), b), c) above are satisfied for all $j \leq i$ and that $i < n$. Observe that $E\sigma\colon g \mapsto E(\sigma_g)$ is an isomorphism of the additively written ordered group $G_1 + \ldots + G_i$ onto the multiplicatively written ordered group $\Gamma_1 \cdot \ldots \cdot \Gamma_i$.

If $G_{i+1} = x^m V$ for some $m \in \omega$, $m > 0$, $V \subseteq C$, then obviously $1 < \operatorname{supp}(\sigma_g)$ for each $g \in G_{i+1}$. Therefore we assume that there is some $j > 0$ such that $G_{i+1} = e(g)V$ for some positive $g \in G_j$, $V \subseteq K_{2j-1}$. Since by b) $\operatorname{supp}(\sigma_{e(g)}) = \{E(\sigma_g)\} \subseteq \Gamma_j$ and for every $v \in V$ $\operatorname{supp}(\sigma_v) \subseteq \Gamma_0 \cdot \ldots \cdot \Gamma_{j-1}$ by c), we have $\operatorname{supp}(\sigma_{e(g)v}) \subseteq \Gamma_0 \cdot \ldots \cdot \Gamma_{j-1} E(\sigma_g)$. But $g > 0$ implies $E(\sigma_g) > \Gamma_0 \cdot \ldots \cdot \Gamma_{j-1}$, hence $\operatorname{supp}(\sigma_{e(g)v}) > 1$. Therefore we can define $\Gamma_{i+1} = \{E(\sigma_g)\colon g \in G_{i+1}\}$. Now let $a \in K_{2i+3}$. For every $g_0 \in G_{i+1}$, by Corollary 23, there is a unique representation $ae(-g_0) = p + c + r$ such that p is a linear combination of elements of $\{e(g)\colon g \in G_{i+1}, g > 0\}$ with coefficients from K_{2i+1}, $c \in K_{2i+1}$ and r coinitial in K_{2i+3}. We put $\tau_a(g_0) = c$. Then for every $g_0 \in G_{i+1}$

$$a - \sum_{g \geq g_0} \tau_a(g)e(g) \ll e(g_0) \bmod K_{2i+1} .$$

It is routine to verify that $\tau\colon a \mapsto \tau_a$ is an embedding of the ordered field K_{2i+3} into the ordered field $K_{2i+1}[[e(G_{i+1})]]$ being isomorphic with $K_{2i+1}[[\Gamma_{i+1}]]$. Hence σ extends from K_{2i+1} to an embedding of K_{2i+3} into $C[[\Gamma_0 \cdot \ldots \cdot \Gamma_{i+1}]]$ satisfying a), b), c) for $j \leq i+1$.

In order to show that for all a $\sigma_{e(a)} = e(\sigma_a)$ let $a \in K$ be such that $e(a) \in K$. If F is such that K is F-normal, then, by Lemma 4, $a - f \in C$ for some $f \in F$. By the definition of a ladder a has to be of the form $a = g_n + \ldots + g_1 + c + r_0 + \ldots + r_n$, where $g_i \in G_i$, $c \in C$ and r_i is coinitial in K_{2i+1}. σ has been constructed in such a way that $\sigma_{e(g_i)} = E(\sigma_{g_i})$ and $\operatorname{supp}(\sigma_{r_i}) \subseteq \Gamma_0 \cdot \ldots \cdot \Gamma_i$. Proposition 15 yields that $|e(r_i) - e_k(r_i)| \leq |r_i^k|$ for every natural number k. If we put

$$\tau_i = \sigma_{e(r_i)} - \sum_{\nu=0}^{\infty}\frac{1}{\nu!}(\sigma_{r_i})^\nu ,$$

we see that $|\tau_i| \leq |\sigma_{r_i}|^k$ for each k and $\operatorname{supp}(\tau_i) \subseteq \Gamma_0 \cdot \ldots \cdot \Gamma_i$. But r_i is coinitial in K_{2i+1} and, if $i > 0$, $e(G_i) \subseteq K_{2i+1}$. Hence for every $g \in G_i$ $|\sigma_{r_i}|^k < E(\sigma_g)$ for large k. Therefore $\operatorname{supp}(\tau_i) < \Gamma_i$. But this is possible only if $\operatorname{supp}(\tau_i) = \varnothing$, i.e.,

$$\sigma_{e(r_i)} = \sum_{\nu=0}^{\infty}\frac{1}{\nu!}(\sigma_{r_i})^\nu .$$

If $i = 0$, a similar argument can be applied.

Now $e(a) = \prod_{i=1}^{n} e(g_i)e(c)\prod_{i=0}^{n} e(r_i)$, hence

$$\sigma_{e(a)} = \prod_{i=1}^{n} E(\sigma_{g_i})\cdot e(c)\cdot \prod_{i=0}^{n}\sum_{\nu=0}^{\infty}\frac{1}{\nu!}(\sigma_{r_i})^\nu$$

$$= E\Big(\sum_{i=1}^{n}\sigma_{g_i}\Big)\cdot e(c)\sum_{\nu=0}^{\infty}\frac{1}{\nu!}\Big(\sum_{i=0}^{n}\sigma_{r_i}\Big)^\nu = e(\sigma_a) .$$

We observe that the coefficients of σ_{a+b} and σ_{ab} can be calculated from the coefficients of σ_a and σ_b by means of addition and multiplication. If $\sigma_a \neq 0$, $h = \max\operatorname{supp}(\sigma_a)$ and $\tau = 1 - \dfrac{\sigma_a}{\sigma_a(h)h}$ then $\operatorname{supp}(\tau) < 1$. Hence $\sum_{n=0}^{\infty}\tau^n$ is meaningful. It is easy to check that

$$\frac{1}{\sigma_a} = \frac{1}{\sigma_a(h)h}\sum_{n=0}^{\infty}\tau^n .$$

Hence the coefficients of $\sigma_{1/a}$ are obtained from 1 and the coefficients of σ_a by means of $+$, $\cdot$ and $^{-1}$. Similarly, the coefficients of $\sigma_{e(a)}$ are of the form $e(\sigma_a(1))c$ where c is obtained from 1 and the coefficients of σ_a by means of addition and multiplication. We observe that the coefficients of σ_a are independent of the concrete choice of the model C_1, the field K, and the ladder for K — they are completely determined by the diagram of the exponential field C.

By the order of a term t we mean the maximal number of iterations of the exponential function occurring in t. Thus we have proved

LEMMA 36. *Let $t(x, y_1, \ldots, y_n)$ be a term without parameters, let C_1 be a model of T and $c_1, \ldots, c_n \in C_1$. Then, for every $h \in H_\infty$, there exists a term $u(y_1, \ldots, y_n)$ of the same order as t such that*

$$\sigma_{t(x, c_1, \ldots, c_n)}(h) = u(c_1, \ldots, c_n).$$

LEMMA 37. *Suppose $a \neq 0$ and $h = \max\operatorname{supp}(\sigma_a)$. Then there is a positive $c \in C$ such that $|\sigma_a - \sigma_a(h)h| < \frac{c}{x}h$.*

Proof. The lemma is an easy consequence of the fact that x^{-1} is the largest element of H_∞ which is less than 1. ■

Since σ: $a \mapsto \sigma_a$ is an embedding of C_∞ into C^∞, Lemma 36 and Lemma 37 yield

THEOREM 38. *Let $t(x, y_1, \ldots, y_n)$ be a term without parameters, let C_1 be a model of T and $c_1, \ldots, c_n \in C_1$ such that $t(x, c_1, \ldots, c_n)$ is not identically zero in C_1. Then $C_1 \models$ "$\lim t(x, c_1, \ldots, c_n)$ exists" iff $\max\operatorname{supp}(\sigma_{t(x, c_1, \ldots, c_n)}) \leqslant 1$.*

If $c \in C_1$ is such that $C_1 \models$ "$\lim t(x, c_1, \ldots, c_n) = c$", then there is a term $u(y_1, \ldots, y_n)$ of the same order as t such that $c = u(c_1, \ldots, c_n)$.

COROLLARY 39. *Let C_1, C_2 be models of T containing an exponential field C and let t be a term with parameters from C. Then, for each $c \in C \cup \{\pm\infty\}$, $C_1 \models$ "$\lim t = c$" iff $C_2 \models$ "$\lim t = c$".*

Proof. Corollary 39 follows from Lemma 37 and Theorem 38 since the map $a \mapsto \sigma_a$ is the same for the models C_1 and C_2. ■

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The automorphism group of some semigroups

by

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Abstract. Let $F(Z)$ denote the collection of all finite non-empty subsets of the integers Z. $F(Z)$ can be considered as a semigroup with addition defined by $A+B = \{a+b \mid a \in A, b \in B\}$. The main result in this paper is the determination of the automorphism group of $F(Z)$. In order to determine this automorphism group some algebraic results for $F(G)$ where G is a group are obtained.

Introduction. Let $F(Z)$ denote the collection of all finite non-empty subsets of the integers Z. $F(Z)$ can be considered as a semigroup with addition defined by $A+B = \{a+b \mid a \in A, b \in B\}$. M. Deza and P. Erdös considered this set addition in [2] and G. A. Freĭman also uses this notion of set addition in his book [3]. The same idea is used, but mainly for infinite subsets, in the study of sequences, such as in H. Halberstam and K. Roth [4]. The main question that is considered in this paper is the determination of the automorphism group of this semigroup.

Since the answer to the main question can be obtained by considering the subsemigroup of $F(Z)$ composed of all subsets of the non-negative integers which contain 0, the first section is devoted to determining the automorphism group of this subsemigroup. It is necessary to introduce some algebraic results concerning retractions in order to answer the main question. Thus, the second section is devoted to providing the necessary facts about retractions in order to verify that the automorphism group of $F(Z)$ is a splitting extension of Z by the Klein four group.

Section I.

DEFINITION 1. For a group G let $F(G) = \{A \subset G \mid A \neq \emptyset \text{ and } |A| < \infty\}$. For the special case of $G = Z$ let $K = \{A \in F(Z) \mid 0 \in A,\ A \subset Z^+\}$.

The following lemma is due to Professor A. H. Clifford.

LEMMA 1. *If $\gamma \in \operatorname{Aut}K$ and n is a natural number, then $\{0, n\}\gamma = \{0, n\}$.*

Proof. Let $P_n = \{0, n\}\gamma$. Now $(n-1)\{0, 1\} + \{0, n\} = (2n-1)\{0, 1\}$ and thus $(n-1)P_1 + P_n = (2n-1)P_1$. Let $Q = \{0, \ldots, q\} = \{0, 1\}\gamma^{-1}$. Then $Q + q\{0, 1\}$