Cantor sets in Prohorov spaces

by

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Abstract. Let X be a Suslin set in a Prohorov (e.g. complete) metric space and $\mathscr{E}$ a partition of X to F_σ-sets. It is proved that either $\mathscr{E}$ is σ-discretely decomposable, or X contains a compact set homeomorphic to the Cantor set which meets uncountably many members of $\mathscr{E}$.

Introduction. A classical theorem of Suslin states that every analytic subset of a Polish (separable complete metric) space is either countable, or contains a copy of the Cantor set. A generalization to the non-separable case has been obtained by El'kin [1]: every absolutely analytic space (i.e. homeomorphic to a Suslin subset of some complete metric space) is either σ-discrete, or contains a copy of the Cantor set. This theorem had previously been proved by Stone [14] for absolutely Borel spaces.

In this paper we show that completeness can be replaced by the Prohorov property, a measure-theoretic property enjoyed by complete metric spaces. Moreover, a stronger form is obtained, which involves a partition of the space to F_σ-sets (see the abstract). This result is proved in Theorem 2 for a class of spaces which includes Suslin subsets of Prohorov metric spaces (Proposition 4). It is also proved that F_σ-sets cannot be replaced by G_δ-sets.

Preliminaries. All topological spaces considered in this paper are assumed to be at least Hausdorff. A non-negative finite Borel measure μ on a space X is called *tight* if $\mu(B) = \sup\{\mu(K)\colon K\subset B,\ K \text{ is compact}\}$ for all Borel sets B in X. We denote by $M^+(X)$ the space of non-negative tight measures on X, endowed with the weak topology. That is, for a net $\{\mu_i\}$ in $M^+(X)$, $\mu_i \to \mu$ if and only if $\int f d\mu_i \to \int f d\mu$ for all bounded continuous real-valued functions f on X. We say that X is a *Prohorov space* if every compact set H in $M^+(X)$ is uniformly tight, that is, for every $\varepsilon>0$ there exists a compact set K in X such that $\mu(X\setminus K)<\varepsilon$ for all $\mu\in H$.

It is well-known that complete metric spaces are Prohorov, the first result being that Polish spaces are Prohorov [11]. Here all Prohorov spaces will be assumed to be metrizable. One of our main tools will be the deep result of Preiss [10, Theorem 5] that every Prohorov metric space is a Baire space. We shall also use the fact that the Prohorov property is preserved by countable products and G_δ-subspaces (see [7]), as well as the following lemma.

Lemma 1 ([15, Lemmas 5.1 and 5.3]). *Let H be a compact subset of $M^+(X)$ and F a closed subset of X. Then*:

(i) *The set $\{\mu|_F: \mu\in H\}$ is relatively compact in $M^+(F)$, where $\mu|_F$ denotes the restriction of μ to the Borel sets in F.*

(ii) *If $\mu(F)=0$ for all $\mu\in H$, then for every $\varepsilon>0$ there is an open set V in X such that $F\subset V$ and $\mu(V)<\varepsilon$ for all $\mu\in H$.*

We now present some notions from Hansell ([4] and [5]), needed for the statement of Theorem 2 below.

Let $\mathscr{E}$ be a family of subsets of a space X. We say that $\mathscr{E}$ is *σ-discretely decomposable* [4], abbreviated σ-dd, if there is a family $\{A_{E,n}: E\in\mathscr{E},\ n\in N\}$ such that $E=\bigcup\{A_{E,n}: n\in N\}$ for every $E\in\mathscr{E}$ and $\{A_{E,n}: E\in\mathscr{E}\}$ is a discrete family for every $n\in N$. A *base* for $\mathscr{E}$ is a family $\mathscr{B}$ of subsets of X such that for every $E\in\mathscr{E}$ there corresponds $\mathscr{B}_E\subset\mathscr{B}$ with $E=\bigcup\mathscr{B}_E$. A *σ-discrete base* is a base which is also a σ-discrete family.

Clearly, every σ-dd family of sets has a σ-discrete base; the converse holds when the family is disjoint.

Finally, a function $f: X\to Y$ is called *co-σ-discrete* [5], if $f(\mathscr{E})$ has a σ-discrete base whenever $\mathscr{E}$ is discrete (or, equivalently, has a σ-discrete base).

The main theorem. In this section we prove

Theorem 2. *Let X be a continuous co-σ-discrete image of a Prohorov metric space and $\mathscr{E}$ a partition of X to F_σ-sets. Then exactly one of the following holds: either* (i) *$\mathscr{E}$ is σ-dd, or* (ii) *X contains a compact set C which meets uncountably many members of $\mathscr{E}$. In the latter case, C can be chosen to be homeomorphic to the Cantor set.*

For the proof of this theorem we shall use the following notation and Lemma 3. Given a family $\mathscr{E}$ of subsets of a space X we set

$$\mathscr{E}_Z=\{E\in\mathscr{E}: E\cap Z\neq\emptyset\}$$

and

$$\mathscr{E}|_Z=\{E\cap Z: E\in\mathscr{E}\}$$

for every subset Z of X. As in [9] we say that Z is *$\mathscr{E}$-discrete* if for every $x\in Z$ there exists $E\in\mathscr{E}$ with $E\cap Z=\{x\}$. We shall also denote by $K(\mathscr{E},X)$ the largest subset Z of X with the property that no nonempty open set in Z is contained in any member of $\mathscr{E}$; $K(\mathscr{E},X)$ is closed in X and is called the "non-locally-$\mathscr{E}$ kernel of X" (see [13, Theorem 1] for the existence of this kernel).

Every continuous function $f: X\to Y$ induces a continuous function $f_*: M^+(X)\to M^+(Y)$ defined by $f_*(\mu)(B)=\mu(f^{-1}(B))$ for all Borel sets B in Y. In the particular case where X is a subspace of Y and f is the inclusion map, we write $\bar\mu$ instead of $f_*(\mu)$. Thus, if H is a compact set in $M^+(X)$, then $\{\bar\mu: \mu\in H\}$ is compact in $M^+(Y)$.

The following lemma is based on an idea of [3] and [8]; namely, in a separable metric space we can obtain a 0-dimensional subspace by removing a set of arbitrarily small measure.

Lemma 3. *Let Y be a separable metric space, D a countable subset of Y and H a relatively compact subset of $M^+(D)$. Then for every $\varepsilon>0$ there exists a closed 0-dimensional subset B of Y such that $\bar\mu(Y\setminus B)<\varepsilon$ for all $\mu\in H$.*

Proof. Clearly we can assume that H is compact. If d denotes the metric of Y, then for every open set U in Y and every $x\in U$ there is $r>0$ such that

$$\{y\in Y: d(y,x)<r\}\subset U$$

and

$$\{y\in Y: d(y,x)=r\}\cap D=\emptyset.$$

(Such $r>0$ exists because the first relation holds for uncountably many $r>0$ and the corresponding sets in the second are disjoint). It follows that there is a base $\mathscr{B}$ for the topology of Y such that $\vartheta U\cap D=\emptyset$ for all $U\in\mathscr{B}$. Since Y is second countable, we can assume that $\mathscr{B}$ is countable, say $\mathscr{B}=\{U_n: n\in N\}$. By Lemma 1 (ii), for every $n\in N$ there is an open set V_n with $V_n\supset\vartheta U_n$ and $\bar\mu(V_n)<\varepsilon/2^n$ for all $\mu\in H$. Setting $B=Y\setminus\bigcup_n V_n$, we have that B is closed, 0-dimensional (since $B\subset Y\setminus\bigcup_n\vartheta U_n$) and $\bar\mu(Y\setminus B)<\varepsilon$ for all $\mu\in H$.

Proof of Theorem 2. First we notice that (i) $\Rightarrow\sim$ (ii). Indeed, if $\mathscr{E}$ is σ-dd and C is a compact set in X, then $\mathscr{E}|_C$ is a σ-dd partition of C and [2, Lemma 2] implies that $\mathscr{E}_C$ is countable.

Next we prove that $\sim$(i) $\Rightarrow$ (ii). Thus, we assume that $\mathscr{E}$ is not σ-dd. Let f be a continuous co-σ-discrete function from a Prohorov metric space Y onto X and set $\mathscr{D}=\{f^{-1}(E): E\in\mathscr{E}\}$. Since f is continuous, $\mathscr{D}$ is a partition of Y to F_σ-sets. Since f is co-σ-discrete, $\mathscr{D}$ is not σ-dd. (Otherwise, $\mathscr{E}$ would be a disjoint family with a σ-discrete base and so σ-dd).

Let $Z=K(\mathscr{D},Y)$. If Z is empty, then by [13, Theorem 4′] $Y=\bigcup_{n\in N}F_n$, where each F_n is closed in Y and locally $\mathscr{D}$ (that is, every $x\in F_n$ has a neighborhood in F_n contained in some member of $\mathscr{D}$). But now for every $n\in N$, $\{D\cap F_n: D\in\mathscr{D}\}$ is a discrete family and for every $D\in\mathscr{D}$, $D=\bigcup_n D\cap F_n$, which contradicts the fact that $\mathscr{D}$ is not σ-dd. Therefore Z is nonempty. Moreover, it is easy to see that $\mathscr{D}|_Z$ is a partition of Z to F_σ-sets with empty interior in Z. Note also that Z, as a closed subspace of Y, is by Preiss' theorem a Baire space. It follows that every countable union of members of $\mathscr{D}|_Z$ has empty interior in Z. Using this fact we can easily construct by induction points $y(s)\in Z$ for every finite sequence s of natural numbers such that the set Q of all $y(s)$ is $\mathscr{D}|_Z$-discrete and $0<d(y(s),y(s,n))<1/n$, where d denotes the metric of Y and $(s,n)=(s_1,s_2,\ldots,s_m,n)$ if $s=(s_1,s_2,\ldots,s_m)$. It is clear that Q is countable and dense in itself; so by a well-known theorem of Sierpiński (see [6, p. 287]), Q is homeomorphic to the space of rational numbers.

By Preiss' theorem, Q is not Prohorov and so there is a compact set H of probability measures on Q, which is not uniformly tight. Since Y is a Prohorov space, for every $\varepsilon>0$ there is a compact set $K\subset Y$ such that

$$(*)\qquad \mu(K\cap Q)>1-\varepsilon \quad \text{for all } \mu\in H.$$

CLAIM. *There is some $\varepsilon>0$ such that for any compact set K satisfying (∗), $\mathscr{D}_K$ is uncountable.*

Suppose that the claim is false and let $\varepsilon>0$. Choose a compact set $K\subset Y$ such that $\mu(K\cap Q)>1-\varepsilon/2$ for all $\mu\in H$ and $\mathscr{D}_K$ is countable. Then

$$K\setminus Q = \bigcup\{K\cap D\colon D\in\mathscr{D}_K\}\setminus Q = \bigcup\{K\cap D\setminus D\cap Q\colon D\in\mathscr{D}_K\},$$

where $D\cap Q$ is either the empty set or a singleton (since Q is $\mathscr{D}$-discrete). This shows that $K\setminus Q$ is F_σ in K, hence $K\cap Q$, as a G_δ-set in K, is a Prohorov space. Applying the Prohorov property for the relatively compact set $H_0=\{\mu|_{K\cap Q}\colon \mu\in H\}$ (see Lemma 1(i)), there is a compact set $L\subset K\cap Q$ such that $\mu(K\cap Q\setminus L)<\varepsilon/2$ for all $\mu\in H$. Then we have $\mu(Q\setminus L)<\varepsilon$ for all $\mu\in H$, which contradicts the fact that H is not uniformly tight.

Now we fix an $\varepsilon>0$ as in the claim. If K is any compact set satisfying (∗) and we set $C=f(K)$, then $\mathscr{E}_C$ is uncountable and the proof of $\sim$(i) ⇒ (ii) is complete.

It remains to show that C can be chosen to be a Cantor set. Let K be a compact set in Y with $\mu(K\cap Q)>1-\varepsilon/2$ for all $\mu\in H$. We apply Lemma 3 for the compact metric space $f(K)$, the countable subset $f(K\cap Q)$ and the relatively compact set

$$H_1=\{f_*(\mu|_{K\cap Q})\colon \mu\in H\}=f_*(H_0)$$

of measures on $f(K\cap Q)$. So there is a closed 0-dimensional subset B of $f(K)$ such that $\bar{\nu}(f(K)\setminus B)<\varepsilon/2$ for all $\nu\in H_1$. Setting $K_1=f^{-1}(B)\cap K$ we have that K_1 is compact and for every $\mu\in H$

$$\mu(K\cap Q\setminus K_1)=\mu(K\cap Q\setminus f^{-1}(B))=\overline{f_*(\mu|_{K\cap Q})}(f(K)\setminus B)<\varepsilon/2.$$

Therefore $\mu(K_1\cap Q)>1-\varepsilon$ for all $\mu\in H$. By the claim, $\mathscr{D}_{K_1}$ is uncountable, so $\mathscr{E}_{f(K_1)}=\mathscr{E}_B$ is uncountable. Finally, by the Cantor–Bendixson theorem [6, p. 253], $B=A\cup C$ where A is countable and C is compact perfect. Since C is also 0-dimensional, C is homeomorphic to the Cantor set. Since $\mathscr{E}_A$ is countable and $\mathscr{E}_B\subset\mathscr{E}_A\cup\mathscr{E}_C$, $\mathscr{E}_C$ is uncountable.

Suslin sets in Prohorov spaces. In this section we show that the class of spaces for which Theorem 2 holds includes Suslin sets in Prohorov metric spaces. Recall that the Suslin sets in a space X are obtained from the family of closed sets in X by the Suslin operation (or $\mathscr{A}$-operation [6]).

As in [2] we say that a function $f\colon X\to Y$ is *σ-dd-preserving* if $f(\mathscr{E})$ is σ-dd whenever $\mathscr{E}$ is a discrete (or, equivalently, σ-dd) family of subsets of X. It is clear that every σ-dd-preserving function is co-σ-discrete.

The following facts about σ-dd-preserving functions may be proved exactly as the analogous properties of co-σ-discrete functions (see Corollary 3.9 and Propositions 3.11 and 3.3 in [5]): Let $f\colon X\to Y$ be a function between metrizable spaces. Then we have (a) f is σ-dd-preserving if and only if $f(\mathscr{B})$ is σ-dd for some σ-discrete base $\mathscr{B}$ for the topology of X; (b) if f is open, onto and $f^{-1}(\{y\})$ is separable for all $y\in Y$, then f is σ-dd-preserving; and (c) if f is σ-dd-preserving, then so is the restriction $f|_E$ of f to any $E\subset X$.

We are now ready to prove

PROPOSITION 4. *Every Suslin set in a Prohorov metric space is a continuous σ-dd-preserving (hence also co-σ-discrete) image of some Prohorov metric space.*

Proof. Let X be a Prohorov metric space and let $\mathscr{S}(X)$ denote the family of all subsets of X which are continuous σ-dd-preserving images of Prohorov metric spaces. We first prove that $\mathscr{S}(X)$ is closed under countable unions and countable intersections.

Let $\{Y_n\colon n=1,2,\ldots\}$ be a sequence of elements of $\mathscr{S}(X)$. Then there are Prohorov metric spaces X_n and functions $f_n\colon X_n\to Y_n$ for $n=1,2,\ldots$, such that each f_n is continuous, σ-dd-preserving and onto. The topological sum $\sum_n X_n$ of the sequence $\{X_n\colon n=1,2,\ldots\}$ is a Prohorov metric space (see e.g. [15, Theorem 5.5]) and the function $F\colon \sum_n X_n\to\bigcup_n Y_n$ with $F(x)=f_n(x)$, if $x\in X_n$, is continuous, σ-dd-preserving and onto. Therefore $\bigcup_n Y_n\in\mathscr{S}(X)$.

To show that $\bigcap_n Y_n\in\mathscr{S}(X)$ we consider the function $f\colon\prod_n X_n\to\prod_n Y_n$ defined by $f((x_n)_{n\in N})=(f_n(x_n))_{n\in N}$, which is continuous and onto. Moreover, it is not hard to see, using (a) above, that f is σ-dd-preserving. Note also that $\prod_n X_n$ is a Prohorov metric space and that the set

$$\Delta=\{(y_n)_{n\in N}\in\prod_n Y_n\colon y_1=y_2=\ldots\}$$

is closed in $\prod_n Y_n$. Thus, $f^{-1}(\Delta)$ is a Prohorov metric space and $f|_{f^{-1}(\Delta)}\colon f^{-1}(\Delta)\to\Delta$ is σ-dd-preserving. Since Δ is homeomorphic to $\bigcap_n Y_n$, it follows that $\bigcap_n Y_n\in\mathscr{S}(X)$.

Finally, we prove that $\mathscr{S}(X)$ is closed under the Suslin operation. This will complete the proof because X is assumed to be a Prohorov space and so every closed subset of X is in $\mathscr{S}(X)$. Let Y be a subset of X which is obtained form $\mathscr{S}(X)$ by the Suslin operation. By [12, Theorem 2. 6.2] there is $B\subset X\times N^N$ of the form

$$B=\bigcap_{n\in N}\bigcup_{m\in N}(S_{(m,n)}\times T_{(m,n)}),$$

where $S_{(n,m)}\in\mathscr{S}(X)$ and $T_{(m,n)}$ is closed in N^N, such that $Y=\mathrm{pr}_X(B)$. Since each $S_{(n,m)}\times T_{(n,m)}$ clearly belongs to $\mathscr{S}(X\times N^N)$, it follows by the above that $B\in\mathscr{S}(X\times N^N)$. Notice also that the projection $\mathrm{pr}_X\colon X\times N^N\to X$ is σ-dd-preserving by (b) above. Thus, using the fact that the composition of σ-dd-preserving functions is σ-dd-preserving, it follows that $Y\in\mathscr{S}(X)$.

Remarks. Since every function on a separable metric space is co-σ-discrete, Theorem 2 holds when X is a continuous image of a separable Prohorov metric space. Note also that in this case $\mathscr{E}$ is σ-dd if and only if $\mathscr{E}$ is countable.

By Proposition 4, Theorem 2 holds, in particular, when X is absolutely analytic. If, moreover, $\mathscr{E}$ is the partition of X to singletons, Theorem 2 reduces to the case of El'kin's theorem mentioned in the introduction.

It is worth noting that a proof of $\sim$(i) $\Rightarrow$ (ii) in Theorem 2 for absolutely analytic spaces is possible without using the Prohorov property. To do this, observe as in Proposition 4 that every absolutely analytic space is a continuous co-σ-discrete image of a complete metric space. (This actually characterizes absolutely analytic spaces; see Hansell [5]). Now we can proceed as in the proof of Theorem 2, where Y is a complete metric space, and find a $\mathscr{D}$-discrete, dense in itself, countable subset Q of Y. However, the elements $y(s)$ of Q are now chosen so that:

$$0<d(y(s),y(s,n))<1/2^{\|s\|+1},$$

where $\|s\| = \sum_{j=1}^{m} s_j$ if $s = (s_1, s_2, \ldots, s_m)$, and d is a complete metric inducing the topology of Y. Then we have that Q is totally d-bounded, so its closure $K = \overline{Q}$ is compact. We claim that $K \cap D$ has empty interior in K for every $D \in \mathscr{D}$. This will complete the proof because by the Baire Category Theorem $\mathscr{D}_K$, hence also $\mathscr{E}_{f(K)}$, must be uncountable. To prove the claim assume, if possible, that there is a non-empty open set V in K with $V \subset K \cap D$. Then $y(s) \in V$ for some $y(s) \in Q$. Since $y(s,n) \to y(s)$, it follows that $y(s,n) \in V$ for some n, which contradicts the fact that Q is $\mathscr{D}$-discrete. This paragraph was communicated to me by Professor D. H. Fremlin to whom I express my thanks.

Finally, we show by an example that Theorem 2 fails for partitions to Borel sets of higher classes (even for partitions to G_δ-sets). Let Ω denote the space of countable ordinals with the discrete topology and set $S = \Omega^{\mathbf{N}}$. Then S is a complete metric space and the family $\mathscr{E}$ of the sets $S_\alpha = \{x \in S\colon \sup_n x(n) = \alpha\}$, $\alpha \in \Omega$, is easily seen to be a partition of S to G_δ-sets. By [14, Lemma 5] there is an $\mathscr{E}$-discrete set which is not σ-discrete and so $\mathscr{E}$ is not σ-dd. On the other hand, it is easy to see that every compact subset of S meets only countably many of the S_α's.

We also mention that Theorem 2 fails in the case of non-metrizable Prohorov spaces. However, a weaker form holds, which in particular solves Problem 12.15 in [16]. Namely, the Sorgenfrey line is not Prohorov. We hope to publish these results elsewhere.

References

[1] A. G. El'kin, *A-sets in complete metric spaces*, Dokl. Akad. Nauk SSSR 175 (1967), pp. 517–520 = Soviet Math. Dokl. 8 (1967), pp. 874–877.

[2] Z. Frolík and P. Holický, *Decomposability of completely Suslin-additive families*, Proc. Amer. Math. Soc. 82 (1981), pp. 359–365.

[3] B. R. Gelbaum, *Cantor sets in metric measure spaces*, Proc. Amer. Math. Soc. 24 (1970), pp. 341–343.

[4] R. W. Hansell, *Borel measurable mappings for nonseparable metric spaces*, Trans. Amer. Math. Soc. 161 (1971), pp. 145–169.

[5] — *On characterizing non-separable analytic and extended Borel sets as types of continuous images*, Proc. London Math. Soc. 28 (1974), pp. 683–699.

[6] K. Kuratowski, *Topology*, Vol. I, New York–London–Warszawa 1966.

[7] S. E. Mosiman and R. F. Wheeler, *The strict topology in a completely regular setting: relations to topological measure theory*, Canad. J. Math. 24 (1972), pp. 873–890.

[8] J. C. Oxtoby, *Homeomorphic measures in metric spaces*, Proc. Amer. Math. Soc. 24 (1970), pp. 419–423.

[9] R. Pol, *Note on decompositions of metrizable spaces* II, Fund. Math. 100 (1978), pp. 129–143.

[10] D. Preiss, *Metri spaces in which Prohorov's theorem is not valid*, Z. Wahrscheinlichkeistheorie verw. Geb. 27 (1973), pp. 109–116.

[11] Ju. V. Prohorov, *Convergence of random processes and limit theorems in probability theory*, Theor. Prob. Appl. 1 (1956), pp. 157–214.

[12] C. A. Rogers and J. E. Jayne, *K-analytic sets*, in: *Analytic Sets*, New York 1980, pp. 1–181.

[13] A. H. Stone, *Kernel constructions and Borel sets*, Trans. Amer. Math. Soc. 107 (1963), pp. 58–70.

[14] — *On σ-discreteness and Borel isomorphism*, Amer. J. Math. 85 (1963), pp. 655–666.

[15] F. Topsøe, *Compactness and tightness in a space of measures with the topology of weak convergence*, Math. Scand. 34 (1974), pp. 187–210.

[16] R. F. Wheeler, *A survey of Baire measures and strict topologies*, Expositiones Math. 1 (1983), pp. 97–190.

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