A generalization of abstract model theory

by

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Abstract. We generalize the traditional axioms of abstract model theory so as to include those logics which deal with enriched structures, e.g., topological, monotone, weak, uniform structures.

To these structures we then generalize a number of results in abstract model theory, among which the identity "Robinson Consistency = Compactness + Craig Interpolation" and the Duality Theorem between logics and equivalence relations.

0. Introduction. Together with a great number of logics for ordinary structures, i.e., structures with relations and functions only, in recent years several logics have been developed for structures enriched by some additional machinery, such as a topology (see, e.g., [FZ], [Ga], [MK], [MZ] and [Sg1]), a monotone system (see, e.g., [EZ] and [MT]), a measure (see [Mo]), a uniformity (see [FZ]), and so on. The main motivation for introducing such logics is clearly expressed, for the case of topological logics, in Abraham Robinson's pioneering paper [Ro, p. 504]:

> "What I have in mind is a theory which is related to algebraic-topological structures, such as topological groups and fields, as ordinary model theory is related to algebraic structures (e.g., groups and fields)."

In a few cases (see [Zi], [Sg2] and [FZ]) also Lindström-like characterization theorems for such logics have been proved, which makes it desirable to correspondingly generalize abstract model theory.

Before investigating the nature of a generalized logic L, one must however say which structures L is to speak about: for short, one must find a reasonable notion of "universe of discourse" $\mathscr{C}$ for logic L. As [Bro] and [Bru] show, if we let $\mathscr{C}$ be a mere *class* of structures, without giving in advance also a notion of embedding or of substructure, then we may encounter additional problems at the very beginning of our generalization. Therefore we prefer to start with a *category* $\mathscr{C}$ whose objects are (enriched) structures, and whose arrows are suitable isomorphic embeddings: for instance, topological logic shall have to deal not only with topological structures, but also with homeomorphic embeddings, i.e., homeomorphisms from one structure onto the topological substructure of another structure. We shall be careful to maintain the categorial notions used here at the most rudimentary level.

As $\mathscr{C}$ is devised to be a "universe of discourse", one naturally expects $\mathscr{C}$ to be sufficiently large, and closed under the basic model-theoretical operations, such as expansion, renaming, formation of pairs and disjoint unions. In our first Definition 1.1 we make precise the above requirements, and call a *semantic domain* any category satisfying them. Semantic domains encompass the most general categories of structures which are suitable as universes of discourse for logics. In many interesting cases, the isomorphic embeddings of $\mathscr{C}$ also enable one to unambiguously speak of the substructure of a structure in $\mathscr{C}$. We then simply say that *$\mathscr{C}$ has substructures* (Def. 1.2). Many classes of structures, upon being equipped with a suitable notion of isomorphic embedding, become semantic domains with substructures: for example, topological structures with the above mentioned homeomorphic embeddings, monotone structures with monotone embeddings 1.6.3 — as well as ordinary structures with ordinary isomorphic embeddings.

Having made precise what a universe of discourse is, in Section 2 we define logics. Our notion of a logic L on the semantic domain $\mathscr{C}$ is a faithful adaptation of the familiar definition in ordinary abstract model theory (see [Ba], [Fe], [Fl]), thus allowing for the greatest generality.

As a matter of fact, our definition encompasses all logics existing in the literature extending first-order logic $L_{\omega\omega}$, and whose universe of discourse is a semantic domain, e.g., topological, monotone, weak, uniform logics — as well as every ordinary extension of $L_{\omega\omega}$.

All the familiar properties of logics are now defined precisely as in the ordinary case: this holds in particular for *Robinson's consistency* (also called, *Robinson's property*) to the effect that whenever $\mathfrak{A}$ and $\mathfrak{B}$ are L-equivalent in type $\tau_{\mathfrak{A}} \cap \tau_{\mathfrak{B}}$, then they are jointly L-equivalent to some structure $\mathfrak{M}$ of type $\tau_{\mathfrak{A}} \cup \tau_{\mathfrak{B}}$. Naturally, we regard Robinson's consistency as a general property of equivalence relations, following also [Mu1-Mu6].

As for the crucial notion of *relativization*, to the effect that all that can be said in L about $\mathfrak{A}$, also can be said about a definable portion of $\mathfrak{A}$, we give a general Definition 2.3 entirely in terms of isomorphic embeddings, after assuming that the semantic domain of L has substructures.

The main results of this paper apply to logics with relativization: in Section 3, for any such logic we prove the identity:

"Robinson consistency = compactness + Craig interpolation",

thus generalizing the result proved for the ordinary case by the author in the Spring of 1979 (see [Mu1, Mu2]) and, independently, by Makowsky and Shelah (see [MS2]). The proof given in Section 3 is entirely self-contained, and has been written down for high-school students. As in the ordinary case, we require that $|\mathrm{Stc}(\tau)|$ always exists: the example of $L_{\infty\infty}$ shows that this requirement is indispensable. The discussion in [MS2, Section 5] also shows that some large-cardinal hypothesis is necessary for the above identity to hold: indeed, our proof depends on the following set-theoretical axiom (denoted by $\natural$ and provably weaker than $V = L$, or $\neg O^{\#}$, or $\neg L^{\mu}$):

every uniform ultrafilter on any regular cardinal $\varkappa$ is λ-descendingly incomplete for every infinite $\lambda \leqslant \varkappa$.

In Section 4 we generalize the Duality Theorem between logics and equivalence relations, proved by the author in [Mu3] for the ordinary case. For X a class of (generalized) structures of finite type τ, we let $\mathit{span}(X)$ be the smallest collection of classes of structures obtainable from X by repeated applications of the first-order operations (including relativization). For $\sim$ an equivalence relation on structures, $\mathit{hull}(\sim)$ is the collection of those classes X with finite type such that every $Y \in \mathrm{span}(X)$ is a union of equivalence classes of $\sim$. We say that $\sim$ is *separable* iff every two $\sim$-inequivalent structures can be separated by some $X \in \mathrm{hull}(\sim)$, and we say that $\sim$ is *bounded* iff the collection of $\sim$-equivalence classes of type τ has a cardinality, for each τ. The hard direction of our duality theorem yields in a rather concrete way, for any bounded separable equivalence relation $\sim$ with Robinson's property, finer than $\equiv$, coarser than $\cong$, and preserved under reduct, a unique logic L with relativization and whose sentences are all of finite type, such that $\sim\ = \ \equiv_L$. In addition, L is compact and satisfies interpolation, and $|\mathrm{Stc}(\tau)|$ always exists. The easy direction of the Duality Theorem is the converse of this fact. For the proof we use Theorem 3.1. The above results together with the remarkable fact that there are several logics with the Robinson property (see [FZ]) add new life to some old problems of abstract model theory, and might motivate further study of its generalizations. For example, consider the problem of which universes of discourse are suitable for which first-order logics. Here "first-order" might be tentatively approximated by "Robinson" (in the light of [Fe], [MS1, MS2], [Mu1-Mu6]), or by "Robinson + axiomatizable", and we might also try to replace "universe of discourse" by "semantic domain". Then the results of this paper show that the problem has an algebraic equivalent formulation in terms of equivalence relations on certain categories of structures.

Eventually, semantic domains might be replaced by purely categorial notions, perhaps calling for a global categorial formulation of Robinson's property and, why not, of the large-cardinal hypotheses used in this paper.

While looking forward to the time one will be able to study abstract model theory without even mentioning ordinals, cardinals and sets, throughout this paper we traditionally let α, β, γ and denote ordinals, and $\varkappa$, λ, μ, ν, and θ denote cardinals.

1. Axioms for universes of discourse. In this paper we use the name of *ordinary structures* for those structures having only relations, functions and constants. For $A \neq \emptyset$ a set, the *superstructure* V^A_ω of A is given by

$$V^A_0 = A, \quad V^A_{n+1} = V^A_n \cup PV^A_n, \quad V^A_\omega = \bigcup_{n<\omega} V^A_n,$$

where P is the power-set operation.

There is no change in the definition of (many-sorted, similarity) types, i.e., possibly infinite *sets* of sort, relation, function and constant symbols, as in [Fe].

A *(generalized) structure of type* τ is a function $\mathfrak{A}$ which sends each sort symbol σ of τ into a pair (A_σ, p_σ) with A_σ a nonempty *set* and $p_\sigma \in V_\omega^{A_\sigma}$, and sends each relation, function, or constant symbol of τ into a relation, function or constant with the same arity (i.e., number of places) and acting on the same sorts. Thus a structure looks like an ordinary structure enriched by some additional machinery taken from the superstructure of each universe. $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{D}$, $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{S}$ are structures with universe A, B, D, M, N, S respectively.

Most mathematical objects are structures in this sense, e.g., topological, uniform, monotone structures — as well as ordinary structures.

One can easily extend the above definition so as to include also structures (like measure spaces) where the additional machinery is a function from an element of the superstructure of some sort into an element of the superstructure of some other sort: indeed all the results of this paper apply to these structures as well; however, we shall not consider this further generalization for the sake of simplicity.

The *forgetful* function $||\cdot||$ sends each structure $\mathfrak{A}$ into the unique ordinary structure $||\mathfrak{A}||$ having the same type $\tau_{\mathfrak{A}}$ of $\mathfrak{A}$ and where the additional machinery p_σ $(\sigma \in \tau_{\mathfrak{A}})$ of $\mathfrak{A}$ is absent.

If $\tau \subseteq \tau_{\mathfrak{A}}$ then $\mathfrak{A} \upharpoonright \tau$, the *reduct of* $\mathfrak{A}$ *to* τ, is naturally obtained by adding to $||\mathfrak{A}|| \upharpoonright \tau$ the p_σ of $\mathfrak{A}$ only for $\sigma \in \tau$.

If $\varrho\colon \tau \to \tau'$ is a renaming, then $\mathfrak{A}^\varrho$ is naturally obtained by adding to $||\mathfrak{A}||^\varrho$ the extra machinery of $\mathfrak{A}$, which is now understood as pertaining to the superstructures of the renamed universes. For f a function, f^ϱ denotes the function acting on the sorts of τ' exactly as f acts on the sorts of τ.

$\mathfrak{A}'$ is called a *strict expansion of* $\mathfrak{A}$ iff $\mathfrak{A}'$ and $\mathfrak{A}$ have the same sorts and $p'_\sigma = p_\sigma$ for each sort $\sigma \in \tau_{\mathfrak{A}}$, and $||\mathfrak{A}'||$ is an expansion of $||\mathfrak{A}||$. In other words, a strict expansion of $\mathfrak{A}$ leaves unchanged all the universes and superstructures, and only adds ordinary machinery to $\mathfrak{A}$. An important case of a strict expansion of $\mathfrak{A}$ is the *diagram expansion* $\mathfrak{A}_A$, whose definition is the same as for ordinary structures; similarly, if $f\colon A \to B$, then $\mathfrak{B}_{fA}$ is defined as in the ordinary case (see [Ke2]). The second kind of expansion is obtained as follows: if $\mathfrak{A}$ and $\mathfrak{B}$ have $\tau_{\mathfrak{A}} \cap \tau_{\mathfrak{B}} = \emptyset$, then the *disjoint pair* $[\mathfrak{A}, \mathfrak{B}]$ is just $\mathfrak{A} \cup \mathfrak{B}$ (recall that structures are functions, hence, in particular, they are sets, since so are types and universes). Thus $[\mathfrak{A}, \mathfrak{B}]$ has type $\tau_{\mathfrak{A}} \cup \tau_{\mathfrak{B}}$ and its reduct to $\tau_{\mathfrak{A}}$ (resp., to $\tau_{\mathfrak{B}}$) is $\mathfrak{A}$ (resp., $\mathfrak{B}$). We say that $[\mathfrak{A}, \mathfrak{B}]$ is a *pair expansion* of $\mathfrak{A}$ (as well as of $\mathfrak{B}$).

As already remarked in the introduction, if $\mathscr{C}$ is the "universe of discourse" of a logic L, then $\mathscr{C}$ must be (i) sufficiently large, (ii) closed under the familiar model-theoretical operations, and (iii) equipped from the very start with some natural notion of isomorphic embedding. We let $\mathcal{O}$ denote the standard universe of discourse, i.e., the function which assigns to every type τ the category $\mathcal{O}(\tau) = (\mathrm{Str}(\tau), \mathrm{Emb}(\tau))$ whose objects are the ordinary structures of type τ, and whose arrows are the (ordinary) isomorphic embeddings from one structure into another

structure. For definiteness, let us call $\mathcal{O}$ the *ordinary semantic domain*. Generalizing this, we have the following definition (see [Go] or [ML] for the rudiments of category theory used in this paper):

1.1. Definition. A *semantic domain* is a function $\mathscr{C}$ which assigns to every type τ a category $\mathscr{C}(\tau) = (\mathrm{Ob}(\tau), \mathrm{Arr}(\tau))$ whose objects are (generalized) structures of type τ, and whose arrows, called the *isomorphic embeddings of* $\mathscr{C}$, are functions with ordinary composition, satisfying the following conditions:

functor: $||\cdot||$ preserves identities and compositions, i.e., if f is identity on $\mathfrak{A}$ in $\mathscr{C}$, then f is identity on $||\mathfrak{A}||$ in $\mathcal{O}$; if

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{f} & \mathfrak{B} \\ & {\scriptstyle h}\searrow \;\; \swarrow{\scriptstyle g} & \\ & \mathfrak{D} & \end{array} \quad \text{commutes in } \mathscr{C},$$

then

$$\begin{array}{ccc} ||\mathfrak{A}|| & \xrightarrow{f} & ||\mathfrak{B}|| \\ & {\scriptstyle h}\searrow \;\; \swarrow{\scriptstyle g} & \\ & ||\mathfrak{D}|| & \end{array} \quad \text{commutes in } \mathcal{O}.$$

closure: $\bigcup_\tau \mathrm{Ob}(\tau)$ is closed under reduct, renaming, strict and pair expansion, and under the formation of *disjoint union*, the latter meaning that for any set of structures $\{\mathfrak{A}_\alpha\}_{\alpha<\varkappa} \subseteq \mathrm{Ob}(\tau)$, τ without constants, there exist $\mathfrak{A} \in \mathrm{Ob}(\tau)$ and arrows $f_\alpha\colon \mathfrak{A}_\alpha \to \mathfrak{A}$ $(\alpha<\varkappa)$ having pairwise disjoint range whose union is A.

richness: each ordinary structure can be obtained from $\mathscr{C}$ via $||\cdot||$, i.e., $\forall \mathfrak{A} \in \mathrm{Str}(\tau) \exists \mathfrak{A}' \in \mathrm{Ob}(\tau)$ such that $||\mathfrak{A}'|| = \mathfrak{A}$;

finite reduct: $f\colon \mathfrak{A} \to \mathfrak{B}$ iff $[f\colon \mathfrak{A} \upharpoonright \tau_0 \to \mathfrak{B} \upharpoonright \tau_0, \forall \tau_0$ finite $\subseteq \tau_{\mathfrak{A}}$ $(=\tau_{\mathfrak{B}})]$;

renaming: $f\colon \mathfrak{A} \to \mathfrak{B} \Rightarrow f^\varrho\colon \mathfrak{A}^\varrho \to \mathfrak{B}^\varrho$;

pair: $f\colon \mathfrak{A} \to \mathfrak{M}$ and $g\colon \mathfrak{B} \to \mathfrak{N}$ (with $\tau_{\mathfrak{A}} \cap \tau_{\mathfrak{B}} = \emptyset$) $\Rightarrow f \cup g\colon [\mathfrak{A}, \mathfrak{B}] \to [\mathfrak{M}, \mathfrak{N}]$;

diagram: $f\colon \mathfrak{A} \to \mathfrak{B} \Rightarrow f\colon \mathfrak{A}_A \to \mathfrak{B}_{fA}$.

Intuitively, the above axioms say that in $\bigcup_\tau \mathrm{Ob}(\tau)$ there are many structures, that one can perform here the usual model-theoretical operations, and that the embeddings are well-behaved with respect to $||\cdot||$, $\upharpoonright$, ϱ, pair and diagram expansion. Notice the "concrete" character of $\mathscr{C}$ (see [ML, p. 26]).

As it will be evident below, many important semantic domains come equipped in advance with a natural notion of "substructure", in the sense of the following definition:

1.2. Definition. We say that a semantic domain $\mathscr{C}$ satisfies the *substructure axiom*, or, for short, $\mathscr{C}$ *has substructures*, iff $\mathscr{C}$ satisfies the following conditions:

existence: if A' $(\subseteq A)$ is the range of an isomorphic embedding into $||\mathfrak{A}||$ in $\mathcal{O}$, then A' is also the range of an isomorphic embedding into $\mathfrak{A}$ in $\mathscr{C}$.

factorization: if $\mathfrak{A} \underset{f}{\to} \mathfrak{M} \underset{g}{\leftarrow} \mathfrak{B}$ and $\mathrm{range}(f) \subseteq \mathrm{range}(g)$, then $\exists h\colon \mathfrak{A} \to \mathfrak{B}$

such that $f = gh$; in categorial terms: if $\mathrm{range}(f) \subseteq \mathrm{range}(g)$ then $f \subseteq g$ as subobjects (see [Go, p. 77]).

Here are some simple facts about semantic domains with substructures:

1.3. Proposition. *If $m\colon \mathfrak{M} \to \mathfrak{N}$ and m is onto N, then $m\colon \mathfrak{M} \cong \mathfrak{N}$ (where, of course, $\cong$ is isomorphism with respect to the category under consideration).*

Proof. We have the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{M} & \xrightarrow{m} & \mathfrak{N} \\ {\scriptstyle i_{\mathfrak{M}}}\uparrow & \nwarrow^{p} & \uparrow{\scriptstyle i_{\mathfrak{N}}} \\ \mathfrak{M} & \xleftarrow[p]{} & \mathfrak{N} \end{array}$$

where p is as given by the factorization axiom, since $\mathrm{range}(i_{\mathfrak{N}}) = N$ by the functor axiom. Therefore, $mp = i_{\mathfrak{N}}$ and $pm = i_{\mathfrak{M}}$ ($i_{\mathfrak{B}}$ denotes identity on $\mathfrak{B}$). ■

1.4. Proposition. *If $\mathfrak{A} \xrightarrow[f]{} \mathfrak{D} \xleftarrow[g]{} \mathfrak{B}$ and f, g have equal range, then $\mathfrak{A} \cong \mathfrak{B}$.*

Proof. By the factorization axiom we have the following commutative diagram:

$$\begin{array}{ccc} & \mathfrak{D} & \\ {\scriptstyle f}\nearrow & & \nwarrow{\scriptstyle g} \\ \mathfrak{A} & \xrightarrow[h]{} & \mathfrak{B} \end{array}$$

By the functor axiom, the following also commutes:

$$\begin{array}{ccc} & \|\mathfrak{D}\| & \\ {\scriptstyle f}\nearrow & & \nwarrow{\scriptstyle g} \\ \|\mathfrak{A}\| & \xrightarrow[h]{} & \|\mathfrak{B}\| \end{array}$$

The last diagram shows that f, g, h are 1-1 and h is onto B (otherwise $\mathrm{range}(f) \neq \mathrm{range}(gh)$, which is impossible). By the above Proposition 1.3 we have that $h\colon \mathfrak{A} \cong \mathfrak{B}$. ■

1.5. Remarks. (i) If $\mathscr{C}$ is a semantic domain, then all the implications ($\Rightarrow$) of the renaming, diagram, existence and factorization axiom can be reversed: to see this, it is sufficient to apply the renaming, reduct and functor axiom in a very direct way.

(ii) The existence axiom in Definition 1.2 has not yet been used, but will have an important role below: its effect is that there are in $\mathscr{C}$ sufficiently many substructures, and that the notion of a subset A' of A being the universe of some substructure $\mathfrak{A}'$ of $\mathfrak{A}$ is essentially the same as the familiar τ-closure notion (see [Fl], [Mu3], or the definition of relativization is Section 4 below): notice that, by Proposition 1.4, if A' is the range of some isomorphic embedding into $\mathfrak{A}$, say $f\colon \mathfrak{B} \to \mathfrak{A}$, then $\mathfrak{B}$ is unique up to isomorphism; in this sense we shall often speak of the *substructure* $\mathfrak{A}|A'$ with universe A', even if $\mathfrak{A}|A'$, strictly speaking, is only the isomorphism class to which $\mathfrak{B}$ belongs.

1.6. Examples of semantic domains.

1.6.1. $\mathscr{O}$, *the ordinary structures with isomorphic embeddings*. One immediately sees that $\mathscr{O}$ satisfies all the axioms for semantic domains with substructures. A disjoint union $\mathfrak{A}$ of structures $\{\mathfrak{A}_\alpha\}_{\alpha<\varkappa}$ of type τ can be obtained by first taking the disjoint union A of the universes A_α, then expanding A to some structure $\mathfrak{A} \in \mathrm{Str}(\tau)$ via the inclusion functions f_α from A_α into A. Notice that if constants were allowed in τ, then in $\mathfrak{A}$ each constant would have $\varkappa$-many interpretations, which is impossible. Also notice that if τ has relations or functions, then there are many degrees of freedom when defining them in $\mathfrak{A}$ (for n-tuples whose components are not all in the same A_α): however, our closure axiom does not require the uniqueness of the disjoint union.

1.6.2. $\mathscr{T}$, *the (many-sorted) topological structures*. Here the proper notion of isomorphic embedding is that of a *homeomorphic embedding* $f\colon \mathfrak{A} \to \mathfrak{B}$, i.e., a homeomorphism f from $\mathfrak{A}$ onto the topological substructure $\mathfrak{B}|f(A)$ of $\mathfrak{B}$ whose universe is $f(A)$ (perhaps composed with the natural inclusion function). $\mathscr{T}$ is closed under formation of disjoint union: given topological structures $\{\mathfrak{A}_\alpha\}_{\alpha<\varkappa}$ one equips their disjoint union $\mathfrak{A}$ with the direct sum topology (see [ML]). As in 1.6.1, there are many possibilities for $\mathfrak{A}$, if the $\mathfrak{A}_\alpha$ have relations or functions. It is easy to see that $\mathscr{T}$ satisfies the substructure axiom.

1.6.3. $\mathscr{M}$, *the monotone structures*. Here the proper notion of $f\colon \mathfrak{A} \to \mathfrak{B}$ being an isomorphic embedding is that f is a monotone isomorphism from $\mathfrak{A}$ onto the monotone substructure $\mathfrak{B}|f(A)$ of $\mathfrak{B}$ with universe $f(A)$. $\mathfrak{B}|f(A)$ is obtained by intersecting the monotone sets in $\mathfrak{B}$ with $f(A)$. A monotone isomorphism is an isomorphism which maps each monotone set of the input onto a monotone set of the output, and vice versa. $\mathscr{M}$ is closed under direct sum: given $\{\mathfrak{A}_\alpha\}_{\alpha<\varkappa}$, construct $\mathfrak{A}$, by letting the monotone sets in $\mathfrak{A}$ be all possible disjoint unions of monotone sets in the $\mathfrak{A}_\alpha$. See also [MT]. $\mathscr{M}$ has substructures, as is not hard to see.

1.6.4. $\mathscr{W}$, *the weak structures* (see [Ke1]). This is a semantic domain with substructures, as can be easily seen by arguing as in 1.6.2 and 1.6.3; see also [Bro] and [Bru].

Further examples of semantic domains with substructures can be found in the literature (see, e.g., [FZ]): roughly, every collection of (generalized) structures closed under direct sum (or coproducts) and having a natural notion of isomorphism and of substructure, will give rise to a semantic domain.

2. Axioms for logics. Our definition of a logic is an obvious generalization of the familiar abstract model theoretical notion (see [Ba], [Fe], [Fl]). A *logic (on $\mathscr{C}$)* is a triple $L = (\mathscr{C}, \models, \mathrm{Stc})$ where $\mathscr{C}$ is a semantic domain, Stc is a function giving, for each type τ, a class $\mathrm{Stc}(\tau)$, called the class of *sentences of type* τ, (in L), and $\models$ is a binary relation, called the *satisfaction* relation (in L) from objects of $\mathscr{C}$ into sentences, satisfying the following axioms:

type: $\mathfrak{A} \models \varphi$ only if $\varphi \in \mathrm{Stc}(\tau_{\mathfrak{A}})$;

occurrence: $\forall\varphi\,\exists\tau_\varphi\,\forall\tau[\varphi\in \mathrm{Stc}(\tau)$ iff $\tau\supseteq\tau_\varphi]$;

isomorphism: $\mathfrak{A}\cong\mathfrak{B} \Rightarrow [\mathfrak{A}\models\varphi$ iff $\mathfrak{B}\models\varphi]$;

reduct: $\forall\varphi,\mathfrak{A},\tau[\varphi\in\mathrm{Stc}(\tau)$ and $\tau\subseteq\tau_{\mathfrak{A}} \Rightarrow (\mathfrak{A}\models\varphi$ iff $\mathfrak{A}\restriction\tau\models\varphi)]$;

renaming: $\forall\varphi\,\forall\varrho\colon \tau_\varphi\to\tau'\,\exists\varphi'\in\mathrm{Stc}(\tau')\,\forall\mathfrak{A}\in\mathrm{Ob}(\tau_\varphi)\,[\mathfrak{A}\models\varphi$ iff $\mathfrak{A}^\varrho\models\varphi']$;

negation: $\forall\varphi\,\exists\psi\in\mathrm{Stc}(\tau_\varphi)\,\forall\mathfrak{A}\in\mathrm{Ob}(\tau_\varphi)\,[\mathfrak{A}\models\psi$ iff not $\mathfrak{A}\models\varphi]$;

conjunction: $\forall\varphi,\psi\,\exists\chi\in\mathrm{Stc}(\tau_\varphi\cup\tau_\psi)\,\forall\mathfrak{A}\in\mathrm{Ob}(\tau_\varphi\cup\tau_\psi)\,[\mathfrak{A}\models\chi$ iff $\mathfrak{A}\models\varphi$ and $\mathfrak{A}\models\psi]$;

existential: $\forall\varphi\,\forall c$ constant $\exists\psi\in\mathrm{Stc}(\tau_\varphi\setminus\{c\})\,\forall\mathfrak{A}\in\mathrm{Ob}(\tau_\varphi\setminus\{c\})\,[\mathfrak{A}\models\psi$ iff $\mathfrak{A}$ has an expansion $\mathfrak{B}\models\varphi]$;

atomic: for every atomic sentence α in $L_{\omega\omega}$, there is $\varphi\in\mathrm{Stc}(\tau_\alpha)$ such that $\forall\mathfrak{A}\in\mathrm{Ob}(\tau_\alpha)\,[\mathfrak{A}\models\varphi$ iff $||\mathfrak{A}||\models_{L_{\omega\omega}}\alpha]$.

2.1. Remarks. If there is danger of confusion, we write $L = (\mathscr{C}_L, \models_L, \mathrm{Stc}_L)$. All the logics existing in the literature and dealing with such semantic domains as topological structures, monotone, weak, or ordinary structures, evidently satisfy the above list of axioms. On our general logics we can already define all the familiar abstract model theoretical notions: we limit ourselves to a few examples:

$$\mathrm{mod}^\tau\varphi \text{ (for } \tau\supseteq\tau_\varphi) = \{\mathfrak{A}\in\mathrm{Ob}(\tau)|\mathfrak{A}\models\varphi\}.$$

(Here we may drop superscript τ if there is no risk of confusion.)

$$\mathrm{th}(\mathfrak{A}) = \{\varphi\in\mathrm{Stc}(\tau_{\mathfrak{A}})|\mathfrak{A}\models\varphi\}.$$

$$\mathfrak{A}\equiv_L\mathfrak{B} \quad \text{iff} \quad \mathrm{th}(\mathfrak{A}) = \mathrm{th}(\mathfrak{B}) \quad (\text{in } L).$$

In particular, $\equiv$ is an equivalence relation on $\bigcup_\tau \mathrm{Ob}(\tau)$ given by $\mathfrak{A}\equiv\mathfrak{B}$ iff $||\mathfrak{A}||\equiv_{L_{\omega\omega}}||\mathfrak{B}||$.

For β a first-order sentence of type $\tau_\beta\supseteq\{x\}$, with x a constant symbol, we let

$$\{x|\beta(x)\}^{\mathfrak{A}} = \{x|\beta(x)\}^{||\mathfrak{A}||} = \{a\in A|\,||\mathfrak{A}||\models_{L_{\omega\omega}}\beta[a]\}.$$

$L\geqslant L'$ means that for every sentence in L' there is in L a sentence having the same type and the same models. If, in addition, $L'\geqslant L$, then (by abuse of notation) we write $L' = L$: as a matter of fact, in this case L' and L have exactly the same expressive power and their only possible difference is in the way sentences are written down in L and L'. As usual (see [Mu3] or [MS2]) we say that L has the *Robinson property*, or L *satisfies Robinson's consistency* theorem iff so does $\equiv_L$ as an equivalence relation on $\bigcup_\tau\mathrm{Ob}(\tau)$, i.e., $\forall\mathfrak{M}\in\mathrm{Ob}(\tau')$, $\forall\mathfrak{N}\in\mathrm{Ob}(\tau'')$, if $\tau = \tau'\cap\tau''$ and $\mathfrak{M}\restriction\tau\equiv_L\mathfrak{N}\restriction\tau$, then there exists $\mathfrak{D}\in\mathrm{Ob}(\tau'\cup\tau'')$, with $\mathfrak{D}\restriction\tau'\equiv_L\mathfrak{M}$ and $\mathfrak{D}\restriction\tau''\equiv_L\mathfrak{N}$. Notice that τ, τ' and τ'' need not all have the same sorts. As the reader can see, the above definitions are the same as for the ordinary case: so we will not insist on repeating the definition of, say, L satisfying Craig's interpolation, Δ-closure, compactness, and refer the reader to, e.g., [MS1].

With respect to abstract logics $L'\geqslant L_{\omega\omega}$ in ordinary abstract model theory, one sees that in the above list of axioms there is nothing ensuring that $L = (\mathscr{C}, \models, \mathrm{Stc})$ can express the simplest algebraic properties of $\mathscr{C}$; in particular, there is no axiom to the effect that in L one can say that f is an isomorphic embedding of $\mathscr{C}$. By contrast, already in $L_{\omega\omega}$, hence in every extension of it on $\mathscr{O}$, one can express this fact. The following definition makes more precise our discussion:

2.2. Definition. We say that *isomorphic embedding is projective in* $L = (\mathscr{C}, \models, \mathrm{Stc})$ iff $\mathscr{C}$ has substructures and $\forall\tau$ s.t. $\exists\varphi$ with $\tau_\varphi = \tau$, $\forall\varrho\colon\tau\to\tau'$ with $\tau\cap\tau' = \varnothing$, $\forall$ function $f\notin\tau\cup\tau'$, there exists a sentence $\psi_{\tau,\varrho,f}\in\mathrm{Stc}(\tau\cup\tau'\cup\{f\})$ such that $\forall\mathfrak{S}\in\mathrm{Ob}(\tau\cup\tau'\cup\{f\})\,[\mathfrak{S}\models\psi_{\tau,\varrho,f}$ iff $\mathfrak{S} = [\mathfrak{A},\mathfrak{B},f]$ is a strict expansion of $[\mathfrak{A},\mathfrak{B}]$, for some $\mathfrak{A}\in\mathrm{Ob}(\tau)$, $\mathfrak{B}\in\mathrm{Ob}(\tau')$, and $f\colon\mathfrak{A}^\varrho\to\mathfrak{B}]$.

Roughly, the above sentence $\psi_{\tau,\varrho,f}$ expresses in L the fact that f is an isomorphic embedding (up to a renaming); compare with [Fe]. All $L\geqslant L_{\omega\omega}$ on $\mathscr{O}$ satisfy the above condition, if each sentence has finite type; under this clause, in topological model theory a logic L satisfies 2.2 whenever one can express in L the fact that f is continuous: examples of such logics are, among others, in [FZ] and [Sg1].

Despite its naturality, the above requirement is superseded in applications by the following one, dealing with the crucial notion of relativization, which naturally generalizes the usual notion:

2.3. Definition. We say that $L = (\mathscr{C}, \models, \mathrm{Stc})$ satisfies the *relativization axiom*, or, for short, L *has relativization*, iff $\mathscr{C}$ has substructures and:

for any boolean combination of atomic sentences α with $\tau_\alpha\supseteq\{x\}$, for any φ there is $\psi\in\mathrm{Stc}(\tau_\varphi\cup(\tau_\alpha\setminus\{x\}))$ such that $\forall\mathfrak{A}\in\mathrm{Ob}(\tau_\varphi\cup(\tau_\alpha\setminus\{x\}))$ we have that $[\mathfrak{A}\models\psi$ iff $\{x|\alpha(x)\}^{\mathfrak{A}}$ is the range of some isomorphic embedding $f\colon\mathfrak{B}\to\mathfrak{A}\restriction\tau_\varphi$, for some $\mathfrak{B}\models\varphi]$.

Roughly, ψ says that φ holds upon restriction to $\{x|\alpha(x)\}$ in $\mathfrak{A}$. Notice that we incorporate the so-called τ_φ-closure in the definition of relativization (see [Fl], [Mu3], and Section 4 below). By Proposition 1.4, the structure $\mathfrak{B}$ in the above definition is unique up to isomorphism. Instead of ψ we write $\varphi^{\{x|\alpha\}}$, and call the latter the *relativization of* φ *to* $\{x|\alpha\}$. For notational simplicity we limit ourselves to single-sorted relativization: actually, repeated relativization may result in many-sorted exponents, as in [Mu3]: the generalization is however obvious, so we omit it. (Incidentally, the same applies to Definition 2.2). Requirements 2.2 and 2.3 are related, as the following proposition shows:

2.4. Proposition. *If* $L = (\mathscr{C}, \models, \mathrm{Stc})$ *is* Δ*-closed and in* L *isomorphic embedding is projective, then* L *has relativization.*

Proof. Let α be a boolean combination of atomic sentences, of type $\tau_\alpha\supseteq\{x\}$. Let φ be a sentence of type τ_φ and $\mathfrak{A}\in\mathrm{Ob}(\tau_\varphi\cup(\tau_\alpha\setminus\{x\}))$. Now we have

(1) $\{x|\alpha(x)\}^{\mathfrak{A}}$ is the range of $f\colon\mathfrak{N}\to\mathfrak{A}\restriction\tau_\varphi$, for some $\mathfrak{N}\models\varphi$ for some isomorphic embedding f iff $\exists f\,\exists\tau\,(\tau\cap\tau_\varphi = \varnothing)\,\exists\varrho\colon\tau\to\tau_\varphi\,\exists\mathfrak{N}$ of type τ such that $[\mathfrak{N},\mathfrak{A},f]\models\psi_{\tau,\varrho,f}$ and $\mathfrak{N}\models\varphi^{\varrho^{-1}}$ and $\{x|\alpha(x)\}^{\mathfrak{A}} = \mathrm{range}(f)$.

On the other hand, by Proposition 1.4 and the isomorphism axiom we have

(2) $\{x|\alpha(x)\}^{\mathfrak{A}}$ is *not* as in (1) iff *either* $\{x|\alpha(x)\}^{\mathfrak{A}}$ is not the range of any isomorphic embedding altogether, into $\mathfrak{A}\restriction\tau_\varphi$, *or* $\{x|\alpha(x)\}^{\mathfrak{A}}$ is the range of some isomorphic embedding $g\colon \mathfrak{D}\to\mathfrak{A}\restriction\tau_\varphi$ with $\mathfrak{D}\models\neg\varphi$.

By the existence clause in the substructure axiom (and by 1.5 (i)), the first alternative in (2) is equivalent to $\{x|\alpha(x)\}^{\|\mathfrak{A}\|}$ being not the range of any isomorphic embedding in $\mathcal{O}$. This can be expressed by a first order sentence, hence by a sentence of L, since the latter is closed under negation, conjunction, existential quantifier and contains (equivalents of) the atomic sentences. The second alternative is, too, expressible by a sentence of L, by arguing as for (1), since isomorphic embedding is projective in L. Therefore the property in (1) is both PC and complement of PC (see [Fe]), hence, by Δ-closure, it is expressible by some sentence θ in L. Now θ is equivalent to $\varphi^{\{x|\alpha\}}$. ■

2.5. Remarks. Thus, topological logic L_t (see [FZ]) has relativization, as it can express continuity and is well known to be Δ-closed.

We now deal with the familiar fact that in a compact logic sentences depend on finitely many symbols only (see [Fl] or [MS2]).

2.6. Proposition. *Assume that $L=(\mathscr{C},\models,\mathrm{Stc})$ is a compact logic where isomorphic embedding is projective. Then we have*

$$\forall\varphi\,\exists\tau_0 \textit{ finite}\subseteq\tau_\varphi\,\forall\mathfrak{A},\mathfrak{B}\in\mathrm{Ob}(\tau_\varphi)\,[\mathfrak{A}\restriction\tau_0\cong\mathfrak{B}\restriction\tau_0\Rightarrow(\mathfrak{A}\models\varphi \textit{ iff } \mathfrak{B}\models\varphi)].$$

Proof (by analogy with [Fl]). Let $\mathfrak{M},\mathfrak{N}$ be arbitrary structures of type τ_φ and $f\colon\mathfrak{M}\cong\mathfrak{N}$. Let $\varrho\colon\tau\to\tau_\varphi$ with $\tau\cap\tau_\varphi=\emptyset$. Let $\mathfrak{M}'=\mathfrak{M}^{\varrho^{-1}}$. Notice that $f\colon(\mathfrak{M}')^\varrho\cong\mathfrak{N}$, hence

$$\mathfrak{S}=[\mathfrak{M}',\mathfrak{N},f]\models\psi_{\tau,\varrho,f}\quad\text{and}\quad f\text{ is onto}.$$

By the finite-reduct axiom, together with the functor axiom, we have

$$\mathfrak{S}\models f\text{ is onto}\quad\text{and}\quad\bigwedge_{\tau^0\text{ finite}\subseteq\tau}\psi_{\tau^0,\varrho,f}.$$

Let $\hat{\mathfrak{S}}$ be an arbitrary model of the above (consistent) theory. Then, by the finite-reduct axiom, $\hat{\mathfrak{S}}=[\hat{\mathfrak{M}}',\hat{\mathfrak{N}},f]$ is a strict expansion of the pair $[\hat{\mathfrak{M}}',\mathfrak{N}]$ with $f\colon(\hat{\mathfrak{M}}')^\varrho\cong\hat{\mathfrak{N}}$; here one also has to recall Proposition 1.3 and the functor axiom. Therefore, by the isomorphism axiom we have

$$f\text{ is onto and}\quad\bigwedge_{\tau^0\text{ finite}\subseteq\tau}\psi_{\tau^0,\varrho,f}\models\varphi^{\varrho^{-1}}\leftrightarrow\varphi.$$

Since τ is a set and L is compact, the above theory has a finite subtheory, e.g., "f is onto and $\psi_{\tau^0_1,\varrho,f}\wedge\ldots\wedge\psi_{\tau^0_n,\varrho,f}$", which still implies that $\varphi^{\varrho^{-1}}\leftrightarrow\varphi$. For a suitably large finite type τ_0 we have that

(1) $$f\text{ is onto}\quad\text{and}\quad\psi_{\tau_0,\varrho,f}\models\varphi^{\varrho^{-1}}\leftrightarrow\varphi.$$

Now let $\mathfrak{A},\mathfrak{B}\in\mathrm{Ob}(\tau_\varphi)$ be arbitrary, with $f\colon\mathfrak{A}\restriction\tau_0\cong\mathfrak{B}\restriction\tau_0$. Then we have

$$[(\mathfrak{A}\restriction\tau_0)^{\varrho^{-1}},\mathfrak{B}\restriction\tau_0,f]\models f\text{ is onto}\quad\text{and}\quad\psi_{\tau_0,\varrho,f}.$$

By reduct we can write

$$[\mathfrak{A}^{\varrho^{-1}},\mathfrak{B},f]\models\psi_{\tau_0,\varrho,f}\quad\text{and}\quad f\text{ is onto}.$$

Assume now $\mathfrak{A}\models\varphi$; then by (1) and renaming we have

$$[\mathfrak{A}^{\varrho^{-1}},\mathfrak{B},f]\models\varphi,\quad\text{since}\quad\mathfrak{A}^{\varrho^{-1}}\models\varphi^{\varrho^{-1}}$$

whence, by reduct, $\mathfrak{B}\models\varphi$. ■

Remark. The list of results from abstract model theory one can immediately generalize to the present context, is by no means limited to the above propositions: for instance, the familiar implications between different interpolation and definability properties of logics (see [MS1]) are easily proved also for generalized logics. In the following sections we shall devote our attention to two abstract model theoretical results whose generalization requires a more detailed analysis of the effect of our axioms for semantic domains, with particular reference to the substructure axiom and to closure under disjoint unions.

3. An abstract model theoretical identity. Following [Mu2], we denote by ♮ (read: *natural*) the set-theoretical axiom given by:

$\forall\mu$ regular $\geqslant\omega$, $\forall\lambda\geqslant\omega$, every uniform ultrafilter D on μ is λ-descendingly incomplete if $\lambda\leqslant\varkappa$.

See, e.g., [CK] for this terminology. In [DJK] it is proved that $\neg L^\mu$ (viz., there is no inner model with an uncountable measurable cardinal) implies ♮. A fortiori, $V=L$, or even $\neg O^\#$, are both stronger than ♮.

3.1. Theorem (♮). *Let $L=(\mathscr{C},\models,\mathrm{Stc})$ be a logic with relativization, where $|\mathrm{Stc}(\tau)|$ exists for every τ, and all sentences are of finite type. Then L satisfies Robinson's consistency iff L is compact and satisfies Craig's interpolation.*

Proof. ($\Leftarrow$): There is nothing new with respect to the argument for the ordinary case; (see [MS1]).

($\Rightarrow$): Since for compact L, Robinson's consistency is equivalent to interpolation, then it suffices to show that Robinson's consistency implies compactness. Therefore assume L satisfies Robinson's consistency and L is not compact (absurdum hypothesis). Then L is not (λ,ω)-compact for some infinite cardinal λ. Assume λ is the least such cardinal. Then there is a type τ' and an inconsistent theory $T'=\{\psi_\alpha\}_{\alpha<\lambda}$ of type τ', with each $T_\alpha=\{\psi_\beta\}_{\beta<\alpha}$ being a consistent subtheory, $(\alpha<\lambda)$. Let $\mathfrak{A}_\alpha\models T_\alpha$; we can safely assume that τ' has no constant symbols: one can always replace any such symbol c by a relation R_c, with the additional stipulation that

$$\exists!\,xR_cx\wedge\forall x(R_cx\to\psi_\alpha(x/c)),$$

for all $\alpha<\lambda$, using closure of L under $\exists$, $\neg$, $\wedge$, and renaming, and the fact that atomic sentences are in L. Notice that τ' is a set. For simplicity we also assume τ' is single-sorted: the proof is the same even if τ' has many sorts, but the notation becomes heavier.

Let $\mathfrak{A}$ be a disjoint union of the $\mathfrak{A}_\alpha$, with isomorphic embeddings $f_\alpha\colon \mathfrak{A}_\alpha \to \mathfrak{A}$ (as given by closure under disjoint union). Let $\mathfrak{N}$ of type $\{<', b_\alpha\}_{\alpha<\lambda}$ be such that $\|\mathfrak{N}\| = \langle\lambda, <, \alpha\rangle_{\alpha<\lambda}$ (as given by the richness axiom). Structure $[\mathfrak{A}, \mathfrak{N}]$ is still in $\mathscr{C}$, by the latter being closed under disjoint pair. Form the strict expansion $[\mathfrak{A}, \mathfrak{N}, f]$ where $f\colon A \to N$ is such that $f(a) =$ the only $\alpha<\lambda$ with $a \in \mathrm{range}(f_\alpha)$: here we use closure under strict expansion. Let $T_0 = \mathrm{th}[\mathfrak{A}, \mathfrak{N}, f]$ in type $\tau' \cup \cup \{<', b_\alpha\}_{\alpha<\lambda} \cup \{f\}$. The following sentences are in T_0 (for $\alpha<\beta<\lambda$):

(I) $<'$ is a linear ordering on N and $b_\alpha <' b_\beta$ and $\forall\gamma(b_\alpha <' \gamma \to \psi_\alpha^{\{x|f(x)=\gamma\}})$,

i.e., the γth component of A is the range of an isomorphic embedding from a model of T_γ. Here we use the fact that each sentence of L can be relativized to boolean combinations of atomic sentences, and closure of L under the other first-order operations.

CLAIM 1. *The b_α $(\alpha<\lambda)$ are unbounded in every model of T_0.*

Proof. Otherwise (absurdum hypothesis), let $\langle\mathfrak{M}, m\rangle \models T_0 \wedge b_\alpha <' m$ for each $\alpha<\lambda$. Then by (I) we have (for all $\alpha<\lambda$):

$$\langle\mathfrak{M}, m\rangle \models \psi_\alpha^{\{x|f(x)=m\}}.$$

This means that there are embeddings and structures $g_\alpha\colon \mathfrak{B}_\alpha \to \mathfrak{M}\restriction\tau'$ with $\mathfrak{B}_\alpha \models T_\alpha$, for each $\alpha<\lambda$. By Proposition 1.4 any two such $\mathfrak{B}_\alpha$'s are isomorphic, since the g_α's have their range equal to $\{x|\ f(x) = m\}$ in $\langle\mathfrak{M}, m\rangle$. By the isomorphism axiom of L every $\mathfrak{B}_\alpha$, hence, in particular, $\mathfrak{B}_0$ is a model of the whole theory $\{\psi_\alpha\}_{\alpha<\lambda}$, which is impossible.

Having proved our claim, let $\mu\geqslant\lambda$ be regular and otherwise arbitrary. Expand the ordinary structure $\langle\mu, <, c_\alpha\rangle_{\alpha<\mu}$ to some generalized structure $\mathfrak{M}'_\mu$ in $\mathscr{C}$ of type $\{<, c_\alpha\}_{\alpha<\mu}$ such that $\|\mathfrak{M}'_\mu\| = \langle\mu, <, c_\alpha\rangle_{\alpha<\mu}$: use the richness axiom. Form the strict expansion $\mathfrak{M}_\mu$ of $\mathfrak{M}'_\mu$ obtained by adding a unary relation symbol P_s for each $s\subseteq\mu$, and a unary function symbol f_r for each $r\colon \mu\to\mu$, and by giving them the natural interpretation. Form the disjoint pair $[[\mathfrak{A}, \mathfrak{N}, f], \mathfrak{M}_\mu]$, for short $[\mathfrak{A}, \mathfrak{N}, f, \mathfrak{M}_\mu]$. Form the strict expansion $\mathfrak{S}_0 = [\mathfrak{A}, \mathfrak{N}, f, \mathfrak{M}_\mu, j]$ where j is the natural embedding from $\|\mathfrak{N}\|$ into $\|\mathfrak{M}_\mu\|$. Let $T^\mu = \mathrm{th}[\mathfrak{A}, \mathfrak{N}, f, \mathfrak{M}_\mu, j]$.

CLAIM 2. *The c_α $(\alpha<\mu)$ are unbounded in each model of T^μ.*

Proof. Otherwise (absurdum hypothesis), let $\langle\mathfrak{S}, m\rangle \models T^\mu \wedge c_\alpha < m$, for every $\alpha<\mu$. Define a collection D of subsets of μ by $d\in D$ iff $d\subseteq\mu$, $|d| = \mu$, $\langle\mathfrak{S}, m\rangle \models P_d m$.

It can be shown without difficulty that D is a uniform ultrafilter on μ. To prove our claim we shall show that D is λ-descendingly complete, thus contradicting ♮. As a matter of fact, assume (absurdum hypothesis) that D is λ-descendingly incomplete, so that we have, for a suitable descending chain $d_0 \supseteq d_1 \supseteq \ldots \supseteq d_\alpha \supseteq \ldots$ $(\alpha<\lambda,\ d_\alpha\in D)$, that $\bigcap_{\alpha<\lambda} d_\alpha = \emptyset$. We may safely assume that for any limit ordinal $\gamma<\lambda$, $\bigcap_{\alpha<\gamma} d_\alpha = d_\gamma$. We are thus enabled to define function $t\colon \mu\to\lambda$ by

$$t(\beta) = \alpha \quad \text{iff} \quad \beta\in d_\alpha\setminus d_{\alpha+1}.$$

Function t says how long β stays in the descending chain. Now we have

(i) $$(\alpha<\lambda) \qquad \mathfrak{S}_0, \mathfrak{S} \models \forall x\big(t(x)\leqslant c_\alpha \to \neg P_{d_{\alpha+1}}(x)\big)$$

since $\mathfrak{S}_0 \equiv_L \mathfrak{S}$; we also can write (for each $\alpha<\lambda$)

(ii) $$\langle\mathfrak{S}, m\rangle \models t(m)\leqslant c_\alpha \to \neg P_{d_{\alpha+1}}(m)$$

and

(iii) $$\langle\mathfrak{S}, m\rangle \models t(m) > c_\alpha.$$

From (iii) we can thus write

(iv) $$\langle\mathfrak{S}, m\rangle \models \forall x(x<c_\lambda \to t(m)>x);$$

as a matter of fact, if (iv) were false, then we would have, for some m' in the universe of $\mathfrak{S}$:

(v) $$\langle\mathfrak{S}, m, m'\rangle \models m'<c_\lambda \wedge t(m)\leqslant m'.$$

By Claim 1, m' can be replaced by some $c_\alpha<c_\lambda$ (recalling the definition of j), so that we have

(vi) $$\langle\mathfrak{S}, m, c_\alpha\rangle \models c_\alpha<c_\lambda \wedge t(m)\leqslant c_\alpha$$

which contradicts (iii). Thus (iv) is true. Therefore we have

(vii) $$\mathfrak{S}_0, \mathfrak{S} \models \exists y\,\forall x\big(x<c_\lambda \to t(y)>x\big),$$

which shows that $\bigcap_{\alpha<\lambda} d_\alpha \neq \emptyset$, a contradiction.

Having proved our second claim, we conclude the proof of the theorem as follows. Consider the family of the T^μ as μ ranges over the regular cardinals $\geqslant\lambda$; it is no loss of generality (in view of the renaming axioms) to assume that for any μ', μ'', the only common symbol in $T^{\mu'}$ and $T^{\mu''}$ is $<$.

CLAIM 3. *$T^{\mu'} \cup T^{\mu''}$ is inconsistent whenever $\mu' \neq \mu''$.*

Proof. Otherwise (absurdum hypothesis) let

$$\mathfrak{B} \models T^{\mu'} \cup T^{\mu''} \quad \text{with} \quad \mathfrak{B} = \langle B, <, c'_\alpha, c''_\beta, \ldots\rangle_{\alpha<\mu', \beta<\mu''}.$$

Assume $\mu'<\mu''$. By Claim 2 (and by reduct axiom) the c'_α are unbounded in B; thus, in the ordinal $(\cong\mu'')$ given by the interpretations in $\mathfrak{B}$ of the c''_β, we can find an unbounded chain of length μ', which contradicts the assumed regularity of μ''. This proves our third claim.

Let now $T_0^\mu = T^\mu \cap \mathrm{Stc}\{<\}$. By Robinson's consistency, and by Claim 3, $T_0^{\mu'} \cup T_0^{\mu''}$ is an inconsistent theory of type $\{<\}$, while $T_0^{\mu'}$ is complete and consistent. By the assumption about $|\mathrm{Stc}(\tau)|$, the collection of complete consistent theories of type $\{<\}$ has a cardinality. On the other hand, the above construction yields as many different T_0^μ as there are regular cardinals $\mu\geqslant\lambda$. This is a contradiction, and shows that L must be compact. ■

3.2. Remarks. In [FZ] one can find examples of logics (other than first-order logic) satisfying the assumptions of Theorem 3.1, in the light of Remark 2.5. Theorem 3.1 was originally proved for the ordinary case in [Mu2] and, independently, in [MS2]. Notice that for the theorem to hold, L need only be closed under relativization to boolean combinations of atomic sentences. The examples of $L_{\infty\infty}$ and $L^{\mathrm{II}}_{\varkappa\varkappa}$ (see [MS2]) show that one cannot dispense with imposing restrictions on the size of $\mathrm{Stc}(\tau)$, or of the type of sentences in L.

The above proof reveals a rather unexpected interplay between the categorial notion of coproduct and the model-theoretical notions of compactness and interpolation: the question naturally arises whether Theorem 3.1 still holds for logics on categories of structures which are not closed under coproducts.

In the following section Theorem 3.1 will be used to generalize the Duality Theorem between logics and equivalence relations, originally proved for ordinary logics in [Mu3].

4. A duality theorem. If $L = (\mathscr{C}, \models, \mathrm{Stc})$ is a compact logic with relativization and interpolation, and where $|\mathrm{Stc}(\tau)|$ always exists, and τ_φ is finite for every φ, then by the easy direction of Theorem 3.1, $\equiv_L$ is an equivalence relation on $\bigcup_\tau \mathrm{Ob}(\tau)$ satisfying Robinson's consistency, finer than $\equiv$ (by L being closed under the first-order operations), coarser than $\cong$ (by the isomorphism axiom satisfied by L); in addition, $\equiv_L$ is *bounded*, i.e., $\forall\tau$ the collection of equivalence classes of $\equiv_L$ on $\mathrm{Ob}(\tau)$ has a cardinality; also, $\equiv_L$ is *preserved under reduct*, i.e., $\mathfrak{A} \equiv_L \mathfrak{B} \Rightarrow \mathfrak{A}\restriction\tau \equiv_L \mathfrak{B}\restriction\tau$, by the reduct axiom. Considering now the collection $\mathfrak{h}$ given by the classes $\mathrm{mod}^{\tau_\varphi}\varphi$ (as φ ranges over the sentences of L), we see that $\mathfrak{h}$ has the property that whenever $\mathfrak{A} \not\equiv_L \mathfrak{B}$, there is $X \in \mathfrak{h}$ of some finite type τ_0 such that $\mathfrak{A}\restriction\tau_0 \in X$, but $\mathfrak{B}\restriction\tau_0 \notin X$. We describe this by saying that X *separates* $\mathfrak{A}$ and $\mathfrak{B}$.

The duality theorem proved in this section yields a converse of the above fact: namely, any arbitrary equivalence relation $\sim$ with the above properties can be written as $\equiv_L$ for precisely one logic L with relativization and sentences of finite type. And L is Robinson's!

The preliminary span-hull machinery developed below is just the same as in [Mu3]:

Let $\mathscr{C}$ be a semantic domain with substructures; a *finite-type class* X is a pair $X = (S, \tau)$ with τ a finite type and $S \subseteq \mathrm{Ob}(\tau)$. X is *elementary* iff it is the class of models (in $\mathscr{C}$) of some first-order sentence (in the sense of Section 2).

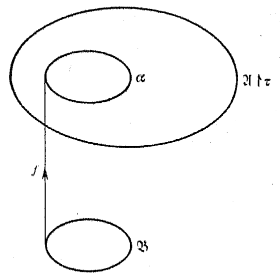
Given finite-type classes X, Y respectively of type τ_X, τ_Y, and given renaming ϱ and constant symbol c, one naturally defines ϱX, $X \wedge Y$, $\neg X$, $\exists cX$, $\forall cX$ as for the ordinary case (see [Mu3] for details). A *boolean function* $B(X_1, \ldots, X_r)$ is a composition of $\neg$, $\wedge$, and $\vee$ acting on the $X_1, \ldots, X_r$. A *prenex function* $\vec{Q} \circ \vec{c}$ is a composition of the form $Q_1c_1 \ldots Q_sc_s$ where each c_j is a constant and each Q_j is either $\exists$ or $\forall$ (depending on j). To complete our list of first-order operations on finite-type classes, we now deal with relativization; following [Mu3], we shall limit ourselves to relativization to boolean combinations of atomic sentences: see [Mu3] for a discussion on relativization. Notice that in our definition below we shall incorporate τ-closure; also, for notational simplicity, we shall limit ourselves to single-sorted relativization. Again let $\mathscr{C}$ be a semantic domain with substructures:

4.1. Definition. For α a boolean combination of atomic sentences with type $\tau_\alpha \supseteq \{x\}$, and for an arbitrary type τ we define $C_\tau^{\{x|\alpha\}}$ by

$\mathfrak{A} \in C_\tau^{\{x|\alpha\}}$ iff $\{x|\alpha\}^{\mathfrak{A}}$ is the range of some isomorphic embedding $f\colon \mathfrak{B} \to \mathfrak{A}\restriction\tau$ for some $\mathfrak{B}$ of type τ,

(for $\mathfrak{A} \in \mathrm{Ob}(\tau \cup (\tau_\alpha \setminus \{x\}))$), and we say that $C_\tau^{\{x|\alpha\}}$ is the class of *τ-closed structures upon restriction to* $\{x|\alpha\}$.

We picture the above situation as follows:



For X an arbitrary finite-type class of type τ_X, we further define $X^{\{x|\alpha\}}$ (called the *relativization of X to* $\{x|\alpha\}$) by

$\mathfrak{A} \in X^{\{x|\alpha\}}$ iff $\{x|\alpha\}^{\mathfrak{A}}$ is the range of some isomorphic embedding $f\colon \mathfrak{B} \to \mathfrak{A}\restriction\tau_X$, for some $\mathfrak{B} \in X$

(for $\mathfrak{A} \in \mathrm{Ob}(\tau_X \cup (\tau_\alpha \setminus \{x\}))$).

Notice that by Proposition 1.4, if $g\colon \mathfrak{D} \to \mathfrak{A}\restriction\tau_X$, too, and g has the same range as f, then automatically $\mathfrak{D} \in X$, provided X does not separate isomorphic structures.

Having completed our list of first-order operations, we now define the *span* of the finite-type class X (for short, *span*(X)) by

$Y \in \mathrm{span}(X)$ iff Y has the form

$$Y = \vec{Q} \circ \vec{c}\, B\big((\varrho_1 X)^{\{x_1|\alpha_1\}}, \ldots, (\varrho_r X)^{\{x_r|\alpha_r\}}, E_1, \ldots, E_p\big)$$

where $\vec{Q} \circ \vec{c} = Q_1c_1 \ldots Q_nc_n$ is a prenex function, B is a boolean function, $E_1, \ldots, E_p$ are elementary classes, each ϱ_i is a renaming with domain τ_X, each α_i is a boolean combination of atomic sentences, and the x_i are constants. Thus, the span of X is the smallest collection of finite-type classes one can get from X and the elementary classes by repeated applications of the first-order operations, including relativization.

Let now $\sim$ be an arbitrary equivalence relation on $\bigcup \mathrm{Ob}(\tau)$; we define *hull*$(\sim)$ as in [Mu3] by

$X \in \mathrm{hull}(\sim)$ iff X is a finite-type class and each $Y \in \mathrm{span}(X)$ is a union of equivalence classes of $\sim$.

Thus $X \in \mathrm{hull}(\sim)$ iff no finite-type class Y which is obtainable from X by the first-order operations can separate two equivalent structures.

We finally say that $\sim$ is *separable* iff for every type τ and structures $\mathfrak{A}, \mathfrak{B} \in \mathrm{Ob}(\tau)$ with $\mathfrak{A} \nsim \mathfrak{B}$, $\exists \tau_0$ finite $\subseteq \tau$ and $\exists X \in \mathrm{hull}(\sim)$ of type τ_0 such that $\mathfrak{A} \upharpoonright \tau_0 \in X$ and $\mathfrak{B} \upharpoonright \tau_0 \notin X$. Thus $\sim$ is separable iff every two inequivalent structures can be separated by some class in hull$(\sim)$. Now we can state our generalized duality theorem:

4.2. THEOREM (♮). *Let $\mathscr{C}$ be a semantic domain with substructures. Let $\sim$ be an arbitrary equivalence relation on $\bigcup_\tau \mathrm{Ob}(\tau)$. Then the following are equivalent:*

(i) *$\sim$ is bounded, separable, has the Robinson property, is preserved under reduct, finer than $\equiv$ and coarser than $\cong$;*

(ii) *$\sim\, = \,\equiv_L$ for precisely one logic L with relativization and having all its sentences of finite type; in addition, L satisfies compactness and Craig's interpolation, and $|\mathrm{Stc}(\tau)|$ exists for every τ.*

For the proof we prepare some lemmas, among which one might find one or two results of independent interest. Our first lemma is the generalization to $\mathscr{C}$ of Proposition 5.1 in [Mu3]:

4.3. LEMMA. *Let X, Y be finite-type classes which do not separate isomorphic structures; let α, β be boolean combinations of atomic sentences, ϱ a renaming, x, c, y constant symbols. Then we have:*

(1) $$(\neg X)^{\{x|\alpha\}} = C_{\tau_X}^{\{x|\alpha\}} \wedge \neg(X^{\{x|\alpha\}});$$

(2) $$\varrho(X^{\{x|\alpha\}}) = (\varrho X)^{\{x|\alpha^\varrho\}};$$

(3) $$(\exists c X)^{\{x|\alpha\}} = \exists c(X^{\{x|\alpha\}} \wedge \alpha(c));$$

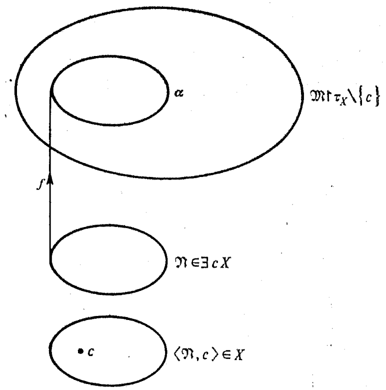
(4) $$(X \wedge Y)^{\{x|\alpha\}} = X^{\{x|\alpha\}} \wedge Y^{\{x|\alpha\}} \wedge C_{\tau_X \cup \tau_Y}^{\{x|\alpha\}};$$

(5) $$(X^{\{x|\alpha\}})^{\{y|\beta\}} = X^{\{x|\alpha \wedge \beta\}} \wedge C_{\tau_X \cup (\tau_\alpha \setminus \{x\})}^{\{y|\beta\}}.$$

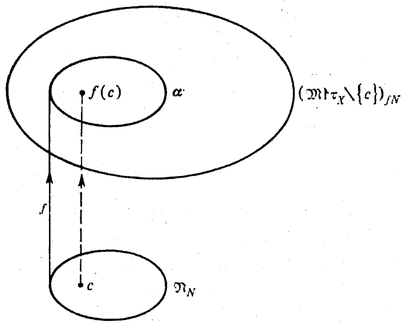
Proof. (1) If $\mathfrak{A}$ is τ_X-closed upon restriction to $\{x|\alpha\}$, then $\mathfrak{A}|\{x|\alpha\}^{\mathfrak{A}}$ is uniquely defined up to isomorphism, by Proposition 1.4. Notice now that X does not separate isomorphic structures, and that $(\neg X)^{\{x|\alpha\}} \subseteq C_{\tau_X}^{\{x|\alpha\}}$ by definition.

(2) By a simple application of the renaming axiom for $\mathscr{C}$. As usual, α^ϱ denotes the sentence obtained from α by renaming ϱ.

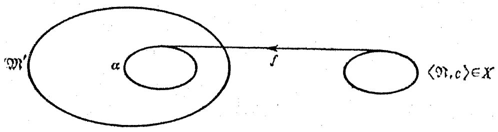
(3)′ We prove that $(\exists c X)^{\{x|\alpha\}} \subseteq \exists c(X^{\{x|\alpha\}} \wedge \alpha(c))$. We can draw the following picture:



By the diagram axiom we have

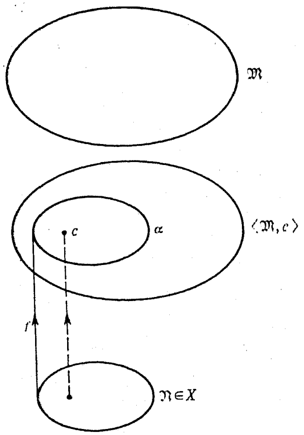


Now, $f(c)$ satisfies α in $\mathfrak{M}_{fN}$; hence $\mathfrak{M}$ has an expansion $\mathfrak{M}' = \langle \mathfrak{M}, f(c) \rangle$ such that the following holds:

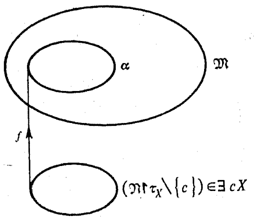


Therefore $\mathfrak{M} \in \exists d(X^{\{x|\alpha\}} \wedge \alpha(d))$, by just letting $d = f(c)$, and the proof becomes complete by renaming bound variables.

(3)″ We prove the converse of (3)′: by hypothesis, we have that $\mathfrak{M} \in \exists c(X^{\{x|\alpha\}} \wedge \alpha(c))$, i.e.,

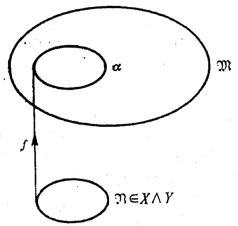


By the reduct axiom satisfied by $\mathscr{C}$ we have

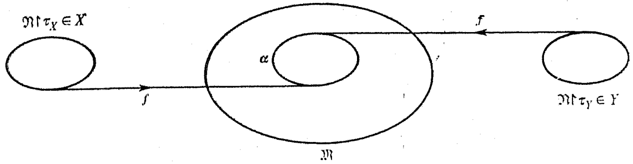


Therefore $\mathfrak{M} \in (\exists c X)^{\{x|\alpha\}}$ and the proof of (3) is complete.

(4)′ Assume $\mathfrak{M} \in (X \wedge Y)^{\{x|\alpha\}}$, so that we have the following:

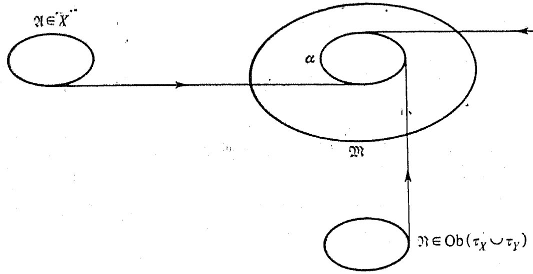


By reduct we also have the following (together with $\mathfrak{M} \in C^{\{x|\alpha\}}_{\tau_X \cup \tau_Y}$):

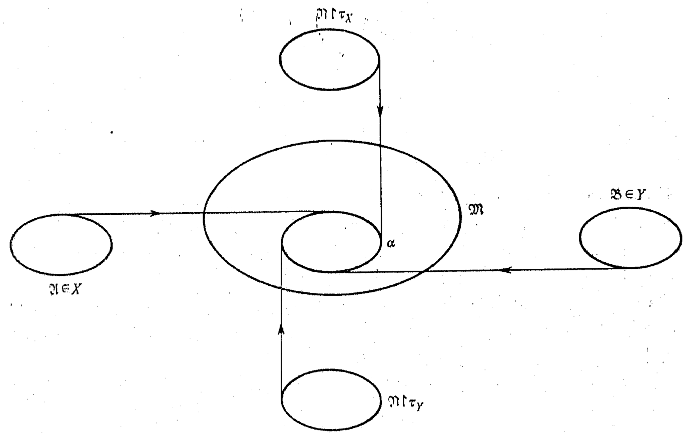


so that $\mathfrak{M} \in X^{\{x|\alpha\}} \wedge Y^{\{x|\alpha\}}$.

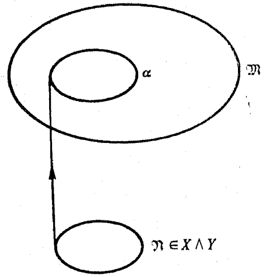
(4)″ Assume now $\mathfrak{M} \in X^{\{x|\alpha\}} \wedge Y^{\{x|\alpha\}} \wedge C^{\{x|\alpha\}}_{\tau_X \cup \tau_Y}$; then we can write:



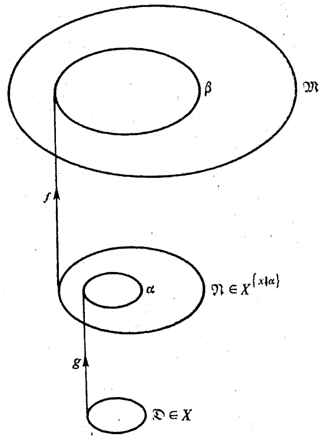
By making use of the reduct axiom satisfied by $\mathscr{C}$ we have:



Now, by Proposition 1.4, $\mathfrak{A} \cong \mathfrak{N} \upharpoonright \tau_X$ and $\mathfrak{B} \cong \mathfrak{N} \upharpoonright \tau_Y$, whence $\mathfrak{N} \upharpoonright \tau_X \in X$ and $\mathfrak{N} \upharpoonright \tau_Y \in Y$, by X and Y not separating isomorphic structures, so that $\mathfrak{N} \in X \wedge Y$ and we finally can write the following picture



(5)′ Assume $\mathfrak{M} \in (X^{\{x|\alpha\}})^{\{y|\beta\}}$, so that we have the following picture:



CLAIM. $\mathrm{range}(fg) = \{y|\beta\}^{\mathfrak{M}} \cap \{x|\alpha\}^{\mathfrak{M}}$.

Proof. As a matter of fact, if $x \in \mathrm{range}(fg)$ then x satisfies β in $\mathfrak{M}$, $f^{-1}(x)$ satisfies α in $\mathfrak{N}$, so that x also satisfies α in $\mathfrak{M}$, since, by the functor axiom, f preserves validity of atomic sentences: therefore $\langle \mathfrak{M}, x\rangle \models \alpha \wedge \beta$.

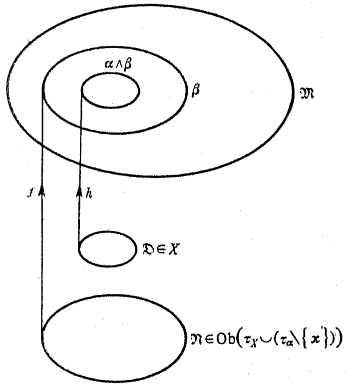
Conversely, if x satisfies both α and β in $\mathfrak{M}$, then x satisfies α in $\mathfrak{N}$ because f preserves validity of atomic sentences, whence $g^{-1}(x)$ is in $\mathfrak{D}$ because $\mathrm{range}(g) = \{x|\alpha\}^{\mathfrak{N}}$. Therefore $x \in \mathrm{range}(fg)$, which proves our claim.

Now, $\{x|\alpha \wedge \beta\}^{\mathfrak{M}}$ is the range of fg, the latter being an isomorphic embedding from some model $\mathfrak{D}$ of X, and $\{y|\beta\}^{\mathfrak{M}}$ is the range of some isomorphic embedding f from a structure $\mathfrak{N}$ of type $\tau_X \cup (\tau_\alpha \setminus \{x\})$. In symbols,

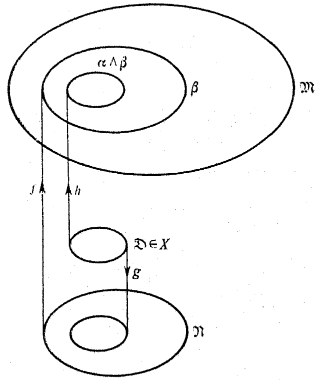
$$\mathfrak{M} \in X^{\{x|\alpha\wedge\beta\}} \wedge C^{\{y|\beta\}}_{\tau_X \cup (\tau_\alpha \setminus \{x\})},$$

which yields the desired conclusion.

(5)″ Assume now $\mathfrak{M} \in X^{\{x|\alpha\wedge\beta\}} \wedge C^{\{y|\beta\}}_{\tau_X \cup (\tau_\alpha \setminus \{x\})}$, so that we have the following picture:



By the factorization axiom, there exists g such that $h = fg$, so that we have the following picture:



By arguing as in (5)′, we see that $f^{-1}\{x|\alpha\wedge\beta\}^{\mathfrak{M}} = \{x|\alpha\}^{\mathfrak{N}}$; therefore, $\mathrm{range}(g) = f^{-1}\{x|\alpha\wedge\beta\}^{\mathfrak{M}}$ by the functor axiom, and $\{x|\alpha\}^{\mathfrak{N}}$ is the range of an isomorphic embedding g from some model $\mathfrak{D}$ of X, whence $\mathfrak{N} \in X^{\{x|\alpha\}}$ and $\mathfrak{M} \in (X^{\{x|\alpha\}})^{\{y|\beta\}}$, as required. ■

4.4. LEMMA. *Let L be a logic on the semantic domain $\mathscr{C}$, having relativization. Assume that each sentence of L is of finite type. Let φ and ψ be sentences of L, α be a boolean combination of atomic sentences of type $\tau_\alpha \supseteq \{x\}$, x, c constants. Then we have:*

(1) $(\exists c\varphi)^{\{x|\alpha\}}$ *is equivalent to* $\exists c(\varphi^{\{x|\alpha\}} \wedge \alpha(c))$;

(2) $(\forall c\varphi)^{\{x|\alpha\}}$ *is equivalent to* $\forall c(\alpha(c) \to \varphi^{\{x|\alpha\}}) \wedge (\theta \vee \neg\theta)^{\{x|\alpha\}}$, *where* θ *is an arbitrary sentence of type* $\tau_\varphi \setminus \{c\}$;

(3) $(\varphi \wedge \psi)^{\{x|\alpha\}}$ *is equivalent to* $\varphi^{\{x|\alpha\}} \wedge \psi^{\{x|\alpha\}} \wedge (\theta \vee \neg\theta)^{\{x|\alpha\}}$, *with* θ *arbitrary and of type* $\tau_\varphi \cup \tau_\psi$;

(4) $(\varphi \vee \psi)^{\{x|\alpha\}}$ *is equivalent to* $(\varphi^{\{x|\alpha\}} \vee \psi^{\{x|\alpha\}}) \wedge (\theta \vee \neg\theta)^{\{x|\alpha\}}$, *with* θ *arbitrary and of type* $\tau_\varphi \cup \tau_\psi$.

Proof. Apply 4.3 to $\mathrm{mod}^{\tau_\varphi}\varphi$ as φ ranges over the sentences of L. Notice that there is always in L a sentence expressing τ-closure (for τ finite), in view of the existence part of the substructure axiom for $\mathscr{C}$: one can take, for instance, the relativization to $\{x|\alpha\}$ of any valid sentence of type τ. Also notice that $(\varphi \vee \psi)^{\{x|\alpha\}}$ is not merely $\varphi^{\{x|\alpha\}} \vee \psi^{\{x|\alpha\}}$, as our axioms for $\mathscr{C}$ do not ensure that τ'-closure together with τ''-closure imply $(\tau' \cup \tau'')$-closure: a similar remark applies to (2) and (3). ■

4.5. Lemma. *Let L' and L'' be two logics on the same semantic domain $\mathscr{C}$, both with relativization and with every sentence being of finite type. Let $L = L' \cup L''$ be their union, i.e., the weakest logic stronger than both L' and L''. Then the sentences of L are (up to equivalence) precisely those having the following form*:

$$(\circ)\qquad \psi = \vec{Q} \circ \vec{k}\, B(\varphi_1', \ldots, \varphi_n';\ \varphi_1'', \ldots, \varphi_m'')$$

with $\vec{Q} \circ \vec{k}$ a prenex function, B a boolean function, every $\varphi_i' \in L'$ and every $\varphi_j'' \in L''$.

Proof. Clearly every ψ as given by $(\circ)$ must be equivalent to some sentence of L, as the latter is closed under prenex and boolean functions. Conversely, the class C of sentences given by $(\circ)$ is closed under the prenex and boolean functions, and encompasses both L' and L''. Closure of C under relativization to boolean combinations of atomic sentences can be proved, by induction on the complexity of ψ, using Lemma 4.4. ■

Remark. If the sentences in L' and L'' above were no longer assumed to have a finite type, then, when applying, say, 4.4(3) there might not exist any sentence θ of type $\tau_{\varphi'} \cup \tau_{\varphi''}$ (in either of L' and L'') for some $\varphi' \in L'$ and $\varphi'' \in L''$, and L would not be closed under relativization.

4.6. Lemma. *Let L' and L'' be as in Lemma 4.5. Assume further that both logics have the Robinson property, and that $\equiv_{L'} = \equiv_{L''}$. Let ψ, a sentence of $L' \cup L''$, be given by $(\circ)$ in 4.5. Then ψ does not separate L'-equivalent structures, i.e., for all $\mathfrak{A} \equiv_{L'} \mathfrak{B}$ we have that $\mathfrak{A} \models_{L'} \psi$ iff $\mathfrak{B} \models_{L'} \psi$.*

Proof. Word by word as in Proposition 2.2 of [Mu3]. ■

4.7. Lemma. *Under the hypotheses of 4.6, let $L = L' \cup L''$; then $\equiv_{L'} = \equiv_L$ and, in particular, L has relativization, Robinson's property, and each sentence of L is of finite type.*

Proof. First notice that $\equiv_L$ is finer than $\equiv_{L'}$; by 4.6, it is also coarser, whence $\equiv_{L'} = \equiv_L$. So L has Robinson's property, as the latter only depends on $\equiv_L$. L has relativization and its sentences are all of finite type by Lemma 4.5. ■

4.8. Lemma (♮). *Let $\sim$ be a bounded equivalence relation on semantic domain $\mathscr{C}$ with substructures: assume $\sim$ has Robinson's property. Let $\equiv_{L'} = \sim = \equiv_{L''}$ for L', L'' two logics with relativization, and whose sentences have finite type. Then $L' = L''$, and both logics satisfy compactness and interpolation, and $|\mathrm{Stc}(\tau)|$ exists for every τ, in both logics.*

Proof. The collection of sentences of L' with type τ has (mod. logical equivalence) a cardinality, for every τ, by $\sim$ being bounded. L' satisfies Robinson's consistency, as so does $\equiv_{L'}$. By our identity in Section 3, L' satisfies compactness and interpolation (here we use ⊐ and the assumption that L' has relativization). Notice that in the proof of 3.1 we use only relativization to boolean combinations of atomic sentences. Let $L = L' \cup L''$. By 4.7, L has relativization, Robinson's property and $|\mathrm{Stc}_L(\tau)|$ always exists (by 4.5). Again by applying Theorem 3.1 we see that L is compact. By a familiar finite-cover argument (see [Fl]), since $L \geqslant L'$, it must be that $L = L'$. Similarly, $L = L''$ and the lemma is proved. ■

4.9. Lemma. *Let $\sim$ be an equivalence relation on semantic domain $\mathscr{C}$ with substructures; assume $\sim$ satisfies* (i) *in 4.2: then $\mathrm{hull}(\sim)$ is closed under renaming, boolean and prenex operations, as well as under relativization to boolean combinations of atomic sentences.*

Proof. Exactly as the proof of Lemmas 6.1–6.7 in [Mu3]. However, instead of Proposition 5.1 therein, one must now use Lemma 4.3 above.

4.10. Proof of Theorem 4.2. The easier direction (ii) ⇒ (i) has already been proved in the discussion at the beginning of this section, except for separability. To see that $\sim\, = \,\equiv_L$ is indeed separable, let $\mathfrak{A} \not\sim \mathfrak{B}$; then some sentence φ of L separates $\mathfrak{A}$ and $\mathfrak{B}$, i.e., $\mathfrak{A} \in \mathrm{mod}\,\varphi$ and $\mathfrak{B} \notin \mathrm{mod}\,\varphi$. Now observe that $X = (\mathrm{mod}\,\varphi, \tau_\varphi)$ is in $\mathrm{hull}(\sim)$, since for every $Y \in \mathrm{span}(X)$, Y is $\mathrm{mod}\,\psi$ for some sentence ψ in L, by definition of span, and by L being closed under the first-order operations (including relativization to boolean combinations of atomic sentences). Therefore, Y is a union of equivalence classes of $\sim$, and $X \in \mathrm{hull}(\sim)$ separates $\mathfrak{A}$ and $\mathfrak{B}$.

Conversely, let us now prove (i) ⇒ (ii). As in [Mu3], define $L = (\mathscr{C}, \models, \mathrm{Stc})$ by

$$(+)\qquad \varphi \in \mathrm{Stc}(\tau) \quad \text{iff} \quad \varphi \in \mathrm{hull}(\sim) \text{ and } \tau_\varphi \subseteq \tau,$$

and

$$(++)\qquad \mathfrak{A} \models \varphi \quad \text{iff} \quad \tau_\varphi \subseteq \tau_{\mathfrak{A}} \text{ and } \mathfrak{A} \restriction \tau_\varphi \in \varphi,$$

for any structure $\mathfrak{A}$ in $\mathscr{C}$, for any type τ.

L satisfies the isomorphism axiom, since $\sim$ is coarser than $\cong$; the occurrence axiom follows from direct inspection of $(+)$ and $(++)$, the same applies to the type axiom; L satisfies the reduct axiom, as $\sim$ is preserved under reduct. L is closed under the first-order operations, including relativization to boolean combinations of atomic sentences, by Lemma 4.9; each sentence of L is of finite type, by direct inspection of (+); $|\mathrm{Stc}(\tau)|$ always exists by the assumed boundedness of $\sim$. One proves that $\sim\ =\ \equiv_L$ by first noting that $\equiv_L$ is coarser than $\sim$ (here one again uses the assumption that $\sim$ is preserved under reduct); then one shows that $\equiv_L$ is finer than $\sim$, by using separability. If necessary, see [Mu3, Claim 2 in 6.8] for details. The fact that L obeys compactness and interpolation now follows from $\equiv_L$ having Robinson's property, in the light of Theorem 3.1 (here one uses $\natural$). Uniqueness of L among logics with relativization, and with sentences having finite type, is given by Lemma 4.8 (here, again, one uses $\natural$). ■

4.10. COROLLARY ($\natural$). *Let* $\equiv_t$ *be the equivalence relation on the topological semantic domain* $\mathscr{T}$ *described in* 1.6.2, *given by topological logic* L_t (see [FZ]). *Then* L_t *is the only logic* L *with relativization and with sentences of finite type such that* $\equiv_t\ =\ \equiv_L$.

Proof. Use Theorem 4.2 and Remark 2.5, and recall that L_t, hence $\equiv_t$, satisfy Robinson's consistency. ■

4.11. Remarks. The assumption about every sentence of L having a finite type in the easy direction of Theorem 4.2, might be rephrased in (essentially) weaker terms, in the light of Proposition 2.6, once we assume that in L isomorphic embedding is projective. As for the harder direction, notice that with respect to [Mu3], here L is asserted to be unique among logics (with relativization and) whose sentences have a finite type: this latter requirement was not imposed in [Mu3]. As a matter of fact, in the ordinary case τ'-closure together with τ''-closure imply $(\tau' \cup \tau'')$-closure, and the remark following Lemma 4.5 above is no more necessary. By adding some sort of regularity requirement for $\cong$ as a basic axioms for $\mathscr{C}$ (for instance, the requirement that $\cong$ has the Robinson property, as in the ordinary case) one should be able to drop the assumption about the sentences of L being of finite type, and prove the exact correspondent of the duality theorem in [Mu3]. However, we have preferred to give the present list of axioms for $\mathscr{C}$, due to its simplicity.

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