A generalization of the Shoenfield theorem on Σ_2^1 sets

by

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Abstract. We generalize the Shoenfield theorem about the absoluteness of Σ_2^1 predicates in the Baire space with respect to inner models to the case of certain Σ_2^1 predicates in certain unseparable spaces. We show that there is a similarity between the Shoenfield proof and some forcing constructions.

The theorem of Shoenfield is the following: If A is Π_1^1 or Σ_2^1, $A \subseteq \omega^\omega$ then there is a tree $T \subseteq \omega^{<\omega} \times \omega_1^{<\omega}$ such that A is a projection of T, i.e.

$$x \in A \equiv (\mathrm{E}f)(\langle x, f\rangle \text{ is a branch of } T) \quad \text{see [3]}.$$

All known generalizations of this theorem concern higher projective classes and higher ordinals than ω_1 for example under the assumption of Δ_2^1 determinancy we have: If A is Π_3^1 or Σ_4^1, $A \subseteq \omega^\omega$ then there is a $\varkappa$ and a tree $T \subseteq \omega^{<\omega} \times \varkappa^{<\omega}$ such that A is a projection of T see [1].

Another example is a theorem of Mansfield [2]: If $\varkappa$ is measurable and A is $\Pi_2^1(\Sigma_3^1)$, then A is a projection of a tree on $\omega \times \varkappa$. Our generalization is in another direction. It deals with a class of predicates A being of a Π_2^1 form but in other topological spaces (not necessarily the Baire space). The class of possible spaces will be defined. For a special class of predicates A (containing the class of Π_1^1 predicates) we shall prove that they are projections of a tree in another topological space (see § 1), not necessarily the space $\omega^\omega \times \varkappa^\omega$ for a $\varkappa$.

We shall not use any set-theoretical assumptions except of ZFC.

We discuss the topological notions that we need in the paper, especially the notion of the continuity of a relation in § 1. In § 2 we prove our theorem. In § 3 we give applications of our theorem. The main application is to show a connection between the Shoenfield theorem and forcing constructions. We fix ourselves on the exposition of this connection and we are not interested here in new independence proofs.

§ 1. In this section we discuss the topological notions used in the paper. Let $\varkappa \in On$. Let $\mathscr{X} \subseteq \varkappa^\omega$, $\mathscr{Y} \subseteq \varkappa^\omega$, $\mathcal{O}$ be a basis of a topology in $\mathscr{X}$, $\mathcal{O}'$ in $\mathscr{Y}$. Assume that $\mathcal{O}$, $\mathcal{O}'$ contain usual Baire basis restricted to $\mathscr{X}$, $\mathscr{Y}$.

DEFINITION 1.1. Let $R \subseteq \mathscr{X} \times \mathscr{Y}$. Let $\mathrm{Dom} R = \{x \in \mathscr{X}\colon (\mathrm{E}y)(\langle x, y\rangle \in R)\}$. We shall say that R is *continuous in* $\langle \mathscr{X}, \mathcal{O}\rangle$ *w.r.t.* $\langle \mathscr{Y}, \mathcal{O}'\rangle$ iff for every $q \in \mathcal{O}'$, $R^{-1}(q)$ is relatively open in $\mathrm{Dom} R$ in $\mathscr{X}$.

Remark 1.1. If R is a function, then this is the usual notion of continuity.

DEFINITION 1.2. Let $T \subseteq \mathcal{O} \times \mathcal{O}'$. Let $\langle p', q'\rangle, \langle p, q\rangle \in T$. Let $\langle p', q'\rangle \leqslant \langle p, q\rangle$ if $p' \subseteq p$, $q' \subseteq q$. Then T with the ordering $\leqslant$ is called a *tree* in the space $\langle \mathscr{X}, \mathcal{O}\rangle \times \times \langle \mathscr{Y}, \mathcal{O}'\rangle$.

DEFINITION 1.3. Let T be a tree. Let $\langle x, y\rangle \in \mathscr{X} \times \mathscr{Y}$ be called a *branch of* T iff $(p)_{\mathcal{O}}(q)_{\mathcal{O}'}$,

$$(x \in p \ \&\ y \in q) \to (\mathrm{E}p')_{\mathcal{O}}(\mathrm{E}q')_{\mathcal{O}'}\ (x \in p' \ \&\ y \in q' \ \&\ p' \subseteq p \ \&\ q' \subseteq q \ \&\ \langle p', q'\rangle \in T).$$

DEFINITION 1.4. Let T be a tree. Let T be called *continuous* if $\langle p, q\rangle \in T \,\&$ $\&\, p' \subseteq p \,\&\, (\mathrm{E}x, y)(\langle x, y\rangle$ is a branch of $T \,\&\, x \in p') \to \langle p', q\rangle \in T$.

DEFINITION 1.5. Let $R \subseteq \mathscr{X} \times \mathscr{Y}$. Let us say that R has a *continuous tree* if there is a continuous tree T such that $R(x, y) \equiv \langle x, y\rangle$ is a branch of T.

LEMMA 1.1. *If R is closed in $\mathscr{X} \times \mathscr{Y}$ in $\mathcal{O} \times \mathcal{O}'$ and continuous, then R has a continuous tree.*

EXAMPLE 1.1. Let $\mathscr{X} = \omega^{\omega}$. Let $p \in \mathcal{O}$ iff $(\mathrm{E}s)_{\omega^{<\omega}}\ (p = \{x \in \mathscr{X}\colon s \subseteq x\})$. In the sequel let this topology be denoted by $\omega^{<\omega}$. Let $\mathscr{Y} = \omega$ and let $\mathcal{O}'$ be discrete. Let $R_0 \subseteq \omega^{<\omega} \times \omega$ and let $R(x, n) \equiv R_0(x{\restriction}n, n)$. Then R is closed and continuous.

We need two topological notions.

DEFINITION 1.6. Let $\mathcal{O}$ be a basis of a topology in $\mathscr{X}$. We say that $\mathcal{O}$ has the *Moore property* if there is a sequence $(O_n)_{n\in\omega}$ such that $O_n \subseteq \mathcal{O}$ for every n and $O_{n+1} \subseteq O_n$ and

1) $\bigcup O_n = \mathscr{X}$ for every n,

2) for every $p \in \mathcal{O}$, $x \in p$ there is an n such that $(p')O_n(x \in p' \to p' \subseteq p)$. This is a slight modification of the Moore property defined in the literature. Notice that $\bigcup\limits_n O_n$ is a basis of the same topology, and so without loss of generality we can always identify $\mathcal{O}$ with $\bigcup\limits_n O_n$.

Remark 1.2. If $\mathcal{O}$ has the Moore property in M, then there is a function rank: $\mathcal{O} \to \omega \cup \{\omega\}$ in M such that

1) if $x \in \mathscr{X}$ then $(n)(\mathrm{E}p)(x \in p \,\&\, \mathrm{rank}\, p = n)$,

2) if $x \in \mathscr{X}$ and $(p_n)_{n\in\omega}$ is a descending sequence of neighbourhoods of x such that $\mathrm{rank}\, p_{n+1} > \mathrm{rank}\, p_n$ or $\mathrm{rank}\, p_n = \omega$, then $(p_n)_{n\in\omega}$ is a basis in x.

Proof. Define

$$\mathrm{rank} p = \begin{cases} \max\{n\colon p \in O_n\} & \text{if defined}, \\ \omega & \text{if, for every } n,\ p \in O_n. \end{cases}$$

We have: 1) follows from 1) of Def. 1.6. To see 2) let $(p_n)_{n\in\omega}$ be a sequence with the required properties, and let $x \in p_n$ for every n. Let p be any neighbourhood of.

Then there is an n_0 such that $(p')_{O_{n_0}}(x \in p' \to p' \subseteq p)$. Let n be such that $\mathrm{rank}\, p_n \geqslant n_0$. Then $p_n \in O_{n_0}$ and so $p_n \subseteq p$. This proves 2). ■

EXAMPLE 1.3. Let $\mathscr{X} = \omega^{\omega}$, $\mathcal{O} = \omega^{<\omega}$. Then $\mathcal{O}$ has the Moore property. Indeed, let $O_n = \{s\colon lhs \geqslant n\}$. Then ranks $= lhs$.

DEFINITION 1.7. Let $\langle \mathscr{X}, \mathcal{O}\rangle$ be a topological space. Then $\mathcal{O}$ has the centralization property iff every subfamily $\mathcal{O}'$ of $\mathcal{O}$ which is centralized (finite intersections of elements from $\mathcal{O}'$ are non-empty) has a non-empty intersection.

Let us consider a few facts about the notions that we have introduced.

LEMMA 1.2. Let $\langle \mathscr{X}, \mathcal{O}\rangle, \langle \mathscr{Y}, \mathcal{O}'\rangle$ be complete metric spaces. Let $R \subseteq \mathscr{X} \times \mathscr{Y}$ be G_δ and continuous in $\mathcal{O}$ w.r.t. $\mathcal{O}'$. Then $\mathrm{Dom} R$ is G_δ.

This lemma is due to A. Louveau.

Remark 1.3. Let $\langle \mathscr{X}, \mathcal{O}\rangle, \langle \mathscr{Y}, \mathcal{O}'\rangle$ have the Moore property. Let $R \subseteq \mathscr{X} \times \mathscr{Y}$ have a continuous tree T and let R be continuous in $\mathcal{O}$ w.r.t. $\mathcal{O}'$. Then R is G_δ.

Proof. Indeed define $A_n = \{\langle p, q\rangle \in T\colon \mathrm{rank} p, \mathrm{rank} q > n\}$. Then $\bigcup A_n$ is open in $\mathscr{X} \times \mathscr{Y}$. It is clear that $R = \bigcap\limits_n \bigcup A_n$. ■

In the sequel we will be interested in spaces $\langle \mathscr{X}_0, \mathcal{O}_0\rangle$, $\langle \mathscr{X}_1, \mathcal{O}_1\rangle$ $\langle \mathscr{Y}, \mathcal{O}_2\rangle$ with the Moore property and the centralization property, relations $R \subset \mathscr{X}_0 \times \mathscr{X}_1 \times \mathscr{X}_2$ with a tree and continuous in $\mathcal{O}_0 \times \mathcal{O}_1$ w.r.t. $\mathcal{O}_2$ and in predicates $A \subset \mathscr{X}_0$ defined as $A(x_0) \equiv (x_1)(\mathrm{E}y) R(X_0 X_1 \mathscr{Y})$.

We shall prove our generalization of the Shoenfield theorem for this class of predicates.

By Remark 1.3 and Lemma 1.2 if $\langle \mathscr{X}_i, \mathcal{O}_i\rangle$, $\langle \mathscr{Y}, \mathcal{O}_2\rangle$ are metric complete, then A is of the form

$$A(x_0) \equiv (x_1) R_1(x_0 x_1)$$

where R_1 is G_δ.

Thus if $\langle \mathscr{X}_i, \mathcal{O}_i\rangle$, $\langle \mathscr{Y}, \mathcal{O}_2\rangle$ are subspaces of the Baire space $\langle \omega^{\omega}, \omega^{<\omega}\rangle$ then we can use classical Shoenfield methods to show that A is a projection of a tree $\boldsymbol{P}$. In this case we have:

$A(x_0) \equiv (T_{x_0}$ is well founded) for an appropriate tree T_{x_0}.

This is essential to define $\boldsymbol{P}$ and then we have

(**) $\quad \mathrm{E}x_0 A(x_0) \equiv \boldsymbol{P}$ is not well founded.

However for arbitrary spaces satisfying our assumptions (**) is no longer true for any $\boldsymbol{P}$ as we shall see in § 3. Hence classical methods do not suffice for our theorem. As we shall see in § 3 predicates A from the class that we have defined occur in mathematical practice, hence it is justified to study them. They are interesting mainly in the case where the spaces are not separable.

Let us restrict ourselves to the case where $\mathscr{X}_0 = \mathscr{X}_1 = \mathscr{Y} = \varkappa^{\omega}$ and $\mathcal{O}_0 = \mathcal{O}_1 = \mathcal{O}_2 = \varkappa^{<\omega}$. As we have said if A is of our form then A is of the form

$$(x_1) R_1(x_0 x_1) \quad \text{where } R_1 \text{ is } G_\delta.$$

Also conversely a predicate $A(x_0) \equiv (x_1)R_1(x_0x_1)$ where R_1 is G_δ is of our form.

Indeed let $(s_\xi)_{\xi\in\varkappa}$ be an enumeration of $\varkappa^{<\omega}\times\varkappa^{<\omega}$. Let us identify s_ξ with the neighbourhood that it determines. Then we have

$$A(x_0) \equiv (x_1)(n)(E_\xi)R'(x_0x_1n, \xi)$$

where

$$R_1 = \bigcap_n A_n \quad \text{and} \quad R'(x_0x_1n, \xi) \equiv s_\xi \subseteq A_n \,\&\, \langle x_0x_1\rangle \in s_\xi .$$

Hence there is an $R''\subseteq\varkappa^{<\omega}\times\varkappa^{<\omega}\times\omega\times\varkappa$ such that

$$R'(x_0x_1n, \xi) \equiv R''(x_{0\restriction \mathrm{dom}\, s_\xi}, x_{1\restriction \mathrm{dom}\, s_\xi}, n, \xi) .$$

Then $A(x_0) \equiv (x_1)(\mathrm{E}y)(n)R''(x_{0\restriction \mathrm{dom}\, Sy(n)}, x_{1\restriction \mathrm{dom}\, Sy(n)}, n, y(n))$.

If we define $R(x_0x_1y)$ as $(n)R''(x_{0\restriction \mathrm{dom}\, Sy(n)}, x_{1\restriction \mathrm{dom}\, Sy(n)}, n, y(n))$, then R is closed and continuous in $\varkappa^{<\omega}\times\varkappa^{<\omega}$ w.r.t. $\varkappa^{<\omega}$ and hence it has a continuous tree $T\subseteq\varkappa^{<\omega}\times\varkappa^{<\omega}\times\varkappa^{<\omega}$.

Now we shall prove our theorem and in § 3 we shall see examples of predicates A satisfying our assumptions and we shall see that they go beyond the case where one can use classical methods. Our proof does not use the Shoenfield theorem and that theorem hence follows.

§ 2.

THEOREM 2.1. *Let $\langle\mathscr{X}_0, \mathscr{O}_0\rangle$, $\langle\mathscr{X}_1, \mathscr{O}_1\rangle$, $\langle\mathscr{Y}_1, \mathscr{O}_2\rangle$ be topological spaces. Let $\mathscr{O}_i$ have the centralization property and the Moore property. Let $R\subseteq\mathscr{X}_0\times\mathscr{X}_1\times\mathscr{Y}$ have a continuous tree $T\subseteq\mathscr{O}_0\times\mathscr{O}_1\times\mathscr{O}_2$ and let R be continuous in $\mathscr{O}_0\times\mathscr{O}_1$ w.r.t. $\mathscr{O}_2$. Define*

$$A(x_0) = (x_1)_{\mathscr{X}_1}(E_y)_{\mathscr{Y}} R(x_0x_1y) .$$

Let $\overline{\overline{\mathscr{O}_0\times\mathscr{O}_1\times\mathscr{O}_2}} = \varkappa$. Consider the space $(\varkappa^+)^\varkappa$ with the Tichonow topology $\mathscr{O}$, where in $\varkappa^+$ we consider the discrete topology. Then there is a tree $\boldsymbol{P}$ in $\langle\omega^\omega, \mathscr{O}_0\rangle\times\langle\varkappa^{+\varkappa}, \mathscr{O}\rangle$ such that

$$x_0\in A \equiv (\mathrm{E}f)(\langle x_0, f\rangle \textit{ is a branch of } \boldsymbol{P}) .$$

Remark 2.1. The above theorem generalizes the case where $\mathscr{O}_0 = \mathscr{O}_1 = \omega^{<\omega}$ and $\mathscr{X}_0 = \mathscr{X}_1 = \omega^\omega$, $\langle\mathscr{Y}, \mathscr{O}_2\rangle$ is ω with the discrete topology and A is Π^1_1. Then $\varkappa = \omega$ and $\boldsymbol{P}$ is a tree in $\langle\omega^\omega, \omega^{<\omega}\rangle\times\langle\omega_1^\omega, \omega_1^{<\omega}\rangle$ the Tichonow topology.

Remark 2.2. If $\langle\mathscr{X}_i, \mathscr{O}_i\rangle$ are Polish with the centralization property, then they are as required in the Theorem by § 1. In this case the theorem can be simplified to the following: if R is G_δ and continuous then ΣR is G_δ; hence A is Π^1_1.

Remark 2.3. All spaces $\varkappa^\omega$ for $\varkappa\in On$ with the Tichonow topology satisfy the assumptions.

Proof of Theorem 2.1. First notice that we can assume that the tree T for R has the following property $\langle p, q\rangle\in T \Rightarrow p\wedge\mathrm{Dom}R\subseteq R^{-1}(q)$ because if T is a tree for R and

$$T' = \{\langle p, q\rangle\in T\colon p\cap\mathrm{Dom}R\subseteq R^{-1}(q)\} ,$$

then T' is a tree for T. Indeed let $R(x, y)$ and $\langle x, y\rangle\in\langle p, q\rangle$. Let $\langle p', q'\rangle\in T$ be such that $\langle x, y\rangle\in\langle p', q'\rangle\subseteq\langle p, q\rangle$. Let $p''\subseteq p'$ be such that $x\in p''$ and $p''\cap\mathrm{Dom}R\subseteq R^{-1}(q')$. Then by the continuity of T, $\langle p'', q'\rangle\in T$ and hence $\langle p'', q'\rangle\in T'$. Thus $\langle x, y\rangle$ is a branch of T'. Conversely if $\langle x, y\rangle$ is a branch of T' then it is a branch of T and thus $R(x, y)$.

For the sequel assume that T has the property: $\langle p, q\rangle\in T\to p\cap\mathrm{Dom}R\subseteq R^{-1}(q)$.

We have to define several new objects.

DEFINITION 2.1.

Let $T = \{\langle p, q, m\rangle\colon \langle p, q\rangle\in T \,\&\, \mathrm{rank}\, p\geqslant m, \mathrm{rank}\, q\geqslant m\}$.

Let $\langle p', q', m'\rangle < \langle p, q, m\rangle \equiv p'\leqslant p, q'\leqslant q, m' > m$.

Let $T(x_0, x_1) = \{\langle q, m\rangle\colon (\mathrm{E}p)(\langle x_0, x_1\rangle\in p \,\&\, \langle p, q, m\rangle\in T)\}$.

Let $p^1\in O_1$.

Let $T(x_0, p^1) = \{\langle q, m\rangle\colon (\mathrm{E}p^0)(x_0\in p^0) \,\&\, \langle\langle p^0p^1\rangle, q, m\rangle\in T\}$.

Let $T(x_0) = \{\langle p^1, q, m\rangle\colon (\mathrm{E}p^0)(x_0\in p^0) \,\&\, \langle\langle p^0p^1\rangle, q, m\rangle\in T\}$.

Let $\langle p^{-1}, q', m'\rangle < \langle p^1, q, m\rangle \equiv p^{-1}\leqslant p^1, q'\leqslant q, m' > m$.

Let $p\in O_0\times O_1$.

Let $p_0\in O_0$.

Let $T(p) = \{\langle q, m\rangle\colon \langle p, q, m\rangle\in T\}$. $T(p_0) = \{\langle p^1qm\rangle\colon \langle p_0p^1qm\rangle\in T\}$.

Let $\langle q'm'\rangle < \langle q, m\rangle = q'\leqslant q \,\&\, m' > m$.

The next definition is less typical:

Let $T'(x_0, p^1, q, k)$

$= \{\langle\bar{p}^1, m\rangle\colon \bar{p}^1\leqslant p^1 \,\&\, \neg(E\langle q', k'\rangle)_{T(x_0,\bar{p}^1)}(\langle q', k'\rangle < \langle q, k\rangle) \,\&\, \mathrm{rank}\,\bar{p}^1\geqslant m\}$.

Let $\langle\bar{\bar{p}}^1, m'\rangle < \langle\bar{p}^1, m\rangle \equiv \bar{\bar{p}}^1\leqslant\bar{p}^1 \,\&\, m' > m$.

Let $T'(p^0p^1, q, k)$

$= \{\langle\bar{p}^1, m\rangle\colon \bar{p}^1\leqslant p^1 \,\&\, \neg(E\langle q', k'\rangle)_{T(p_0\,\bar{p}^1)}(\langle q', k'\rangle < \langle q, k\rangle) \,\&\, \mathrm{rank}\,\bar{p}^1\geqslant m\}$.

LEMMA 2.1. *We have $(x_1)(\mathrm{E}y)R(x_0, x_1, y) = (\langle p^1, q, k\rangle)_{T(x_0)}T'(x_0, p^1, q, k)$ is well-founded.*

This lemma is essential for our considerations.

Proof. Assume that $(x_1)(\mathrm{E}y)R(x_0, x_1, y)$. Suppose that $T'(x_0, p^1, q, k)$ is not well-founded and $\langle p^1, q, k\rangle\in T(x_0)$. Let $(\langle p^1_m, m_n\rangle)_{n\in\omega}$ be a descending sequence in $T'(x_0, p^1, q, k)$. Let $x_1\in\bigcap_n p^1_n$. We have $(\mathrm{E}y)R(x_0, x_1, y)$. By the fact that $\langle p^1, q, k\rangle\in T(x_0)$, that $x_1\in p^1$ and by our assumptions on T, we have $(\mathrm{E}y)_q R(x_0, x_1, y)$. Let $\langle q', k'\rangle$ be such that $k' > k$, $\mathrm{rank}\, q'\geqslant k'$, $q'\leqslant q$ and $y\in q'$. By the fact that $(p^1_n)_{n\leqslant\omega}$ is a basis in x_1 (we have $\mathrm{rank}\, p^1_n\geqslant m_n$), there is an n such that $\langle q', k'\rangle\in T(x_0, p^1_n)$.

But then $\langle p^1_n, m_n\rangle\notin T'(x_0, p^1, q, k)$. Contradiction.

Assume conversely that $(\mathrm{E}x_1)(y)\neg R(x_0, x_1, y)$. We shall show that $T'(x_0, p^1, q, k)$ is not well-founded for a $\langle p^1, q, k\rangle$ in $T(x_0)$.

Remark 2.4. There is no descending sequence $\langle p^1_n, q_n, k_n\rangle$ in $T(x_0)$ such that $x_1\in\bigcap_n p^1_n$. Indeed, suppose that $\langle p^1_n, q_n, k_n\rangle$ form such a sequence. Let p^0_n be such that $\langle\langle p_n^0 p_n^1\rangle, q_n, k_n\rangle \in T \,\&\, x_0 \in p_n^0$. Let $y \in \bigcap_n q_n$. We shall show that $\langle x_0, x_1, y\rangle$ is a branch of T. Indeed, let $\langle\langle x_0 x_1\rangle, y\rangle \in \langle p, q\rangle$. By the properties of ranks there is an n such that $\langle\langle p_n^0 p_n^1\rangle, q_n\rangle \leqslant \langle p, q\rangle$ (we have $\operatorname{rank} p_n^0 \geqslant k_n$, $\operatorname{rank} p_n^1 \geqslant k_n$, $\operatorname{rank} q_n \geqslant k_n$ for every n). Hence $\langle x_0 x_1, y\rangle$ is a branch of T and thus $R(x_0 x_1, y)$. Contradiction. By the remark there is a $\langle p^1, q, k\rangle$ which is minimal with the property: $\langle p^1, q, k\rangle \in T(x_0) \,\&\, x_1 \in p^1$. Let us show that $T'(x_0, p^1, q, k)$ is not well-founded. Let $\langle p_n^1, m_n\rangle$ be such that $\operatorname{rank} p_n^1 \geqslant m_n$, $x_1 \in p_n^1$, $p_{n+1}^1 \leqslant p_n^1$, $m_{n+1} > m_n$, $p_0^1 = p^1$. Suppose

$$(\mathrm{E}\langle q', k'\rangle)_{T(x_0, p_n^1)}(\langle q', k'\rangle < \langle q, k\rangle) \quad \text{for} \quad n \geqslant 1 .$$

Then $\langle p_n^1, q', k'\rangle \in T(x_0)$, $x_1 \in p_n^1$ and $\langle p_n^1, q', k'\rangle < \langle p^1, q, k\rangle$. Contradiction. Thus for $n \geqslant 1$, $\langle p_n^1, m_n\rangle \in T'(x_0, p^1, q, k)$ and thus $T'(x_0, p^1, q, k)$ is not well-founded.

Now we are ready to define the required tree $\boldsymbol{P}$.

Let for a $p_0 \in \mathcal{O}_0$, $K_{p_0} \subseteq (\mathcal{O}_1 \times \mathcal{O}_2 \times \omega) \times (\mathcal{O}_1 \times \omega)$ be defined as follows:

$$K_{p_0} = \{\langle\langle p^1, q, m\rangle, \langle \bar{p}^1, \overline{m}\rangle\rangle \colon \langle p^1, q, m\rangle \in T(\bar{p}_0) \text{ for a } \bar{p}_0 \geqslant p_0 \text{ and } \langle \bar{p}^1, \overline{m}\rangle \in T'(p^0, p^1, q, m)\}.$$

Let $(V_\eta)_{\eta \in \varkappa}$ be an enumeration of $\bigcup_{p_0 \in \mathcal{O}_0} K_{p_0}$.

Let $p \in \boldsymbol{P}$ iff $p = \langle p_0, t\rangle$ where $p_0 \in \boldsymbol{P}_0$, $rgt \subseteq \varkappa^+$ $\operatorname{dom} t \subseteq \varkappa$, $\operatorname{dom} t$ is finite and if $\xi, \eta \in \operatorname{dom} t$ and $V_\xi, V_\eta \in Kp_0$ and $(V_\xi)_0 = (V_\eta)_0$, $(V_\xi)_1 < (V_\eta)_1$ in $T'(p_0, (V_\xi)_0)$, then $t(\xi) < t(\eta)$.

LEMMA 2.2. *We have $T'(x_0 p^1, q, k)$ is well founded for every*

$$\langle p^1, q, k\rangle \in T(x_0) \equiv (\mathrm{E}g)_{\varkappa^{+\varkappa}}(\langle x_0, g\rangle \text{ is a branch of } \boldsymbol{P}) .$$

Proof. Assume first that $\langle x_0, g\rangle$ is a branch of $\boldsymbol{P}$. Let $\langle p^1, q, k\rangle \in T(x_0)$. Define

$$g\langle p^1, q, k\rangle \quad \text{as} \quad g_{\langle p^1, q, k\rangle}(\langle \bar{p}^1, \overline{m}\rangle) = g(\eta)$$

where $V_\eta = \langle\langle p^1 q k\rangle, \langle \bar{p}^1 \overline{m}\rangle\rangle$. Let us show that $T'(\langle x_0 p^1, q, k\rangle) \subseteq \operatorname{dom} g_{\langle p^1, q, k\rangle}$. Let $\langle \bar{p}^1, \overline{m}\rangle \in T'(x_0, p^1, q, k)$. Let p^0 be such that $x_0 \in p^0$ and $\langle p^1, q, k\rangle \in T(p^0)$. Let $\bar{p}^0$ be such that $\langle \bar{p}^1, \overline{m}\rangle \in T'(p^0, p^1, q, k)$ and $x_0 \in p^0$ and $\bar{p}^0 \subseteq p^0$. Then $\langle\langle p^1, q, k\rangle, \langle \bar{p}^1, \overline{m}\rangle\rangle \in K_{\bar{p}^0}$. Hence there is an η such that

$$\langle\langle p^1, q, k\rangle, \langle \bar{p}^1, \overline{m}\rangle\rangle = V_\eta .$$

Thus $g_{\langle p^1, q, k\rangle}$ is defined at $\langle \bar{p}^1, \overline{m}\rangle$ as $g(\eta)$.

We shall show now that $T'(x_0, p^1, q, k)$ is well founded. We shall show that $g_{\langle p^1, q, k\rangle} \colon T'(x_0 p^1 q, k) \to \varkappa^+$ order preserving. Let $\langle \bar{p}^1, \overline{m}\rangle, \langle \bar{p}^1, \overline{m}\rangle \in T'(x_0 p^1, q, k)$ & $\langle \bar{p}^1, \overline{m}\rangle < \langle \bar{p}^1, \overline{m}\rangle$. Let p^0 be such that

$$\langle\langle p^1, q, k\rangle, \langle \bar{p}^1, \overline{m}\rangle\rangle,\ \langle\langle p^1, q, k\rangle, \langle \bar{p}^1, \overline{m}\rangle\rangle \in K_{p^0}$$

and $x_0 \in p^0$. Using the fact that $\langle x_0 g\rangle$ is a branch of $\boldsymbol{P}$, there is a $\bar{p}^0 \subseteq p^0$ and a $t \subseteq g$ such that $\langle \bar{p}^0, t\rangle \in \boldsymbol{P}$ and $\xi, \eta \in \operatorname{dom} t$ where $\langle\langle p^1, q, k\rangle, \langle \bar{p}^1, \overline{m}\rangle\rangle = V_\xi$, $\langle\langle p^1, q, k\rangle, \langle \bar{p}^1, \overline{m}\rangle\rangle = V_\eta$. But then $V_\xi, V_\eta \in K_{\bar{p}^0}$. Hence $t(\xi) < t(\eta)$ and thus $g_{\langle p^1, q, k\rangle}(\langle \bar{p}^1, \overline{m}\rangle) < g_{\langle p^1, q, k\rangle}(\langle \bar{p}^1, \overline{m}\rangle)$. That is what we wanted to show.

Assume conversely that $(\langle p^1, q, k\rangle)_{T(x_0)} T'(x_0, p^1, q, k)$ is well-founded. Let $g_{\langle p^1, q, k\rangle} \colon T'(x_0, p^1, q, k) \to \varkappa^+ - \{0\}$ order preserving. Define $g(\eta) = g_{(V_\eta)_0}((V_\eta)_1)$ if $V_\eta \in K_{p_0}$ for a p_0 such that $x_0 \in p_0$ and $(V_\eta)_1 \in T'(x_0, (V_\eta)_0)$. Otherwise define $g(\eta) = 0$. Let us show that $\langle x_0, g\rangle$ is a branch of $\boldsymbol{P}$. Indeed let $x_0 \in p^0$, $t \subseteq g$. Let $\operatorname{dom} t = \{\eta_1 \ldots \eta_k\}$. Assume that $\eta_1 \ldots \eta_j$ are all such that $t(\eta_1) = \ldots t(\eta_j) = 0$. Then either $V_{\eta_i} \notin K_{\bar{p}^0}$ for any neighbourhood $\bar{p}^0$ of x_0 or $(V_{\eta_i})_1 \notin T'(x_0, (V_{\eta_i})_0)$. In any case there is a $\bar{p}^0 \subseteq p^0$ such that $x_0 \in \bar{p}^0$ and $V_{\eta_i} \notin K_{\bar{p}^0}$ for $1 \leqslant i \leqslant j$. For other η_i for $j < i \leqslant k$ there is a p_0^i such that $V_\eta^i \in K_{p_0^i}$.

Let $\tilde{p}^0 \subseteq \bar{p}^0 \cap p_{j+k}^0 \cap \ldots \cap p_k^0$, and $x_0 \in \tilde{p}^0$. Then $V_{\eta_i} \in K_{\tilde{p}^0}$ for $j+1 \leqslant i \leqslant k$, $V_{\eta_i} \in K_{\tilde{p}^0}$ for $1 \leqslant i \leqslant j$. Let us show that $\langle \tilde{p}^0, t\rangle \in \boldsymbol{P}$.

Indeed let $\xi, \eta \in \operatorname{dom} t$ be such that $V_\xi, V_\eta \in K_{\tilde{p}^0}$, $(V_\xi)_0 = (V_\eta)_0$, $(V_\xi)_1 < (V_\eta)_1$ in $T'(\tilde{p}_0, (V_\xi)_0)$. Then $\xi, \eta \in \{\eta_{j+1} \ldots \eta_k\}$. Then by the definition of j, $(V_\xi)_1 \in T'(x_0, (V_\xi)_0)$ and $(V_\eta)_1 \in T'(x_0, (V_\eta)_0)$. Hence $(V_\xi)_1, (V_\eta)_1 \in T'(x_0, (V_\xi)_0)$ and $(V_\xi)_1 < (V_\eta)_1$ in $T'(x_0, (V_\xi)_0)$. By the fact that $g_{(V_\xi)_0}$ is order preserving at $T'(x_0, (V_\xi)_0)$ we have $t(\xi) < t(\eta)$. ∎

By combining Lemma 2.1 and Lemma 2.2 we obtain the theorem. ∎

§ 3. In this section we give two applications of our theorem — one direct and the other concerning forcing.

EXAMPLE 3.1. Let $\langle \mathscr{X}_0, \mathcal{O}_0\rangle = \langle \omega_1^\omega$ with the Tichonow topology$\rangle = \langle \mathscr{Y}, \mathcal{O}_2\rangle$.

Let $\mathscr{X}_1$ be a subset of ω^ω consisting of well-orderings, $\mathcal{O}_1$ be discrete. $R(n, x_0, x_1, y) \equiv y$ is an isomorphism of $x_0(n)$ and x_1.

Let $A(x_0) \equiv (x_1)(\mathrm{E}y)(\mathrm{E}n) R(n, x_0, x_1, y)$.

It is easy to see that A can be reduced to a predicate satisfying the assumptions of the theorem. Thus the set of sequences A majorizing $\mathscr{X}_1$ is a projection of an appropriate tree.

Let us consider the second application. Assume we have an inner model M, e.g. $M = L$. Then the classical Shoenfield theorem can be strengthened to the following theorem:

If A is $\Pi_1^1(M)$, then there is a tree T in M $T \subseteq \omega \times \omega_1$ such that $x \in A \equiv (\mathrm{E}f)$ $(\langle x, f\rangle$ is a branch of $T)$.

Our theorem has also an appropriate strengthened form under certain assumptions about the topologies. Let us make a few definitions and observations. We have to ensure the absoluteness of A and in a certain sense of the topologies. One can see that the topologies given by finite sets as in Example 3.1 are absolute enough.

Next we shall define another class of topologies and we formulate Theorem 3.1 for that class, although one can remember that it is true for a larger class of topologies.

Assume $\mathscr{X}_0, \mathscr{X}_1, \mathscr{Y} \subseteq \omega^\omega$.

DEFINITION 3.1. Let $\mathcal{O}$ be a basis of a topology in $\mathscr{X}$, and let M be an inner model. Then we say that $\mathcal{O}$ is *M-codable* if there is a partially ordered set $\langle O, \leqslant\rangle$ in M such that $O \subseteq \omega^\omega \cap M$ and there is an isomorphism φ of $\langle O, \leqslant\rangle$ and $\langle \mathcal{O}, \subseteq\rangle$ such that for a $p \in O$ the relation $x \in \varphi(p)$ is $\Sigma_1^1(M)$, i.e., Σ_1^1 with a parameter from M, uniformly in p.

In the sequel let us identify p with $\varphi(p)$, $\mathcal{O}$ with O.

DEFINITION 3.2. Let $\langle \mathscr{X}, \mathcal{O}\rangle$, $\langle \mathscr{Y}, \mathcal{O}'\rangle$ be given, $R \subseteq \mathscr{X} \times \mathscr{Y}$. Let us say that R is $\tilde{\Sigma}_1^1(M)$ if for a given $p \in O$ the relation

$$R_p(x, y) \equiv x \in p \,\&\, R(x, y)$$

is $\Sigma_1^1(M)$ uniformly in p.

Remark 3.1. If R is $\tilde{\Sigma}_1^1(M)$ then R is absolute in w.r.t. M.

Indeed, assume that $\langle x, y\rangle \in M$. Let $R(x, y)$. By the assumption $R \subseteq \mathscr{X} \times \mathscr{Y}$ there is a $p \in O$ such that $R_p(x, y)$. But then $R_p^M(x, y)$; thus $R^M(x, y)$. Conversely, if $R^M(x, y)$ then $R_p^M(x, y)$ for a p and hence $R(x, y)$.

LEMMA 3.1. *Let $\langle \mathscr{X}, \mathcal{O})$, $\langle \mathscr{Y}, \mathcal{O}'\rangle$ be given, let $T \subseteq \mathcal{O} \times \mathcal{O}'$, and let $\mathcal{O}, \mathcal{O}'$ be M codable. Let R be $\tilde{\Sigma}_1^1(M)$ continuous in $\mathcal{O}$ w.r.t $\mathcal{O}'$ and closed. Then R has a continuous tree $T \in M$.*

Proof. Consider the tree defined for R in Lemma 1.1. We have for $\langle p, q\rangle \in O \times O'$:

$$\langle p, q\rangle \in T \equiv (E\langle x, y\rangle)(x \in p \,\&\, y \in q \,\&\, R(x, y)) \,\&$$
$$\&(p \cap \operatorname{Dom} R \subseteq R^{-1}(q)) \equiv (E\langle x, y\rangle)(R_p(x, y) \,\&\, y \in q)(x) \,\&\, ((Ey)R_p(x, y)$$
$$\rightarrow (Ey)(y \in q \,\&\, R_p(x, y)))\,.$$

Thus if $\langle p, q\rangle \in O \times O'$ then the relation $\langle p, q\rangle \in T$ is $\Pi_2^1(M)$ uniformly in $\langle p, q\rangle$. Hence T is the same as the appropriate T defined in M. Hence $T \in M$. The equivalence "$R(x, y) \equiv \langle x, y\rangle$ is a branch of T" holds both in the world and in M by the absoluteness of R and Lemma 1.1. ■

LEMMA 3.2. *Let $\langle \mathscr{X}_0, \mathcal{O}_0\rangle$, $\langle \mathscr{X}_1, \mathcal{O}_1\rangle$, $\langle \mathscr{Y}, \mathcal{O}_2\rangle$ be given, let $\mathcal{O}_i$ be M-codable and have the Moore property in M and let them have the following property: if $p_n \in O_i$, $p_{n+1} \leqslant p_n$ is a descending chain, then $\bigcap_n p_n \neq \emptyset$ in $\mathscr{X}_i$ or $\mathscr{Y}$. Let $R \subseteq \mathscr{X}_0 \times \mathscr{X}_1 \times \mathscr{Y}$ have a tree in M. Then the predicate $A'(x_0) = (Ex_1)(y)\neg R(x_0 x_1 y)$ is absolute w.r.t. M and so is $A(x_0) = (x_1)(Ey)R(x_0, x_1, y)$.*

Proof.

Remark 3.2. We have $R(x_0, x_1, y) \equiv \langle x_0, x_1, y\rangle$ is a branch of

$$T \equiv (\langle p^0 p^1 q\rangle)_{\mathcal{O}_0 \times \mathcal{O}_1 \times \mathcal{O}_2}(\langle x_0 x_1 y\rangle \in \langle p^0 p^1 q\rangle$$
$$\rightarrow (E\langle \bar{p}^0 \bar{p}^1 q'\rangle)(\langle x_0 x_1 y\rangle \in \langle \bar{p}^0 \bar{p}^1 q'\rangle \leqslant \langle p^0 p^1 q\rangle))$$
$$\equiv (n)(E\langle p^0 p^1 q\rangle)_T(\operatorname{rank} p^0, \operatorname{rank} p^1, \operatorname{rank} q \geqslant n \,\&$$
$$\&\, \langle x_0 x_1 y\rangle \in \langle p^0 p^1 q\rangle).$$

Let us prove the last equivalence. Indeed, let $\langle x_0 x_1 y\rangle$ be a branch of T. Let n be given. Let $\langle p^0 p^1 q\rangle \in O_0 \times O_1 \times O_2$ be such that $\operatorname{rank} p^0 = \operatorname{rank} p^1 = \operatorname{rank} q = n \,\&\, \langle x_0 x_1 y\rangle \in \langle p^0 p^1 q\rangle$. Let $\langle \bar{p}^0 \bar{p}^1 q'\rangle$ be chosen as above. Then $\operatorname{rank} \bar{p}^0$, $\operatorname{rank} \bar{p}^1$, $\operatorname{rank} q' \geqslant n$ and $\langle \bar{p}^0 \bar{p}^1 q'\rangle$ has the required properties.

Assume conversely that

$$(n)(E\langle p^0 p^1 q\rangle)_T(\operatorname{rank} p^0, \operatorname{rank} p^1, \operatorname{rank} q \geqslant n \,\&\, \langle x_0 x_1 y\rangle \in \langle p^0 p^1 q\rangle)\,.$$

Let $\langle p^0 p^1 q\rangle$ be such that $\langle x_0 x_1 y\rangle \in \langle p^0 p^1 q\rangle$. Choose n such that

$$\{\langle \bar{p}^0 \bar{p}^1 q'\rangle \in (O_0)_n \times (O_1)_n \times (O_2)_n\colon \langle x_0 x_1 y\rangle \in \langle \bar{p}^0 \bar{p}^1 q'\rangle\} \subseteq P(\langle p^0 p^1 q\rangle).$$

Let $\langle \bar{p}^0 \bar{p}^1 q'\rangle \in ((O_0)_n \times (O_1)_n \times (O_2)_n) \cap T$ and $\langle x_0 x_1 y\rangle \in \langle \bar{p}^0 \bar{p}^1 q'\rangle$. Then $(\bar{p}^0 \bar{p}^1 q'\rangle \leqslant \langle \bar{p}^0 \bar{p}^1 q\rangle$ and $\langle \bar{p}^0 \bar{p}^1 q'\rangle$ is as required.

Remark 3.3. Let $T(x_0 x_1) = \{\langle q, n\rangle\colon (Ep^0 p^1)(\langle p^0 p^1 q\rangle \in T \,\&\, x_0 \in p^0 \,\&\, x_1 \in p^1) \,\&\, \&\, \operatorname{rank} p^0,\ \operatorname{rank} p^1,\ \operatorname{rank} q \geqslant n\}$. Let $\langle q' n\rangle \leqslant \langle q, n\rangle \equiv (q' \leqslant q \,\&\, n' > n)$. We have $((Ey)R(x_0, x_1, y) = T(x_0 x_1)$ is not well-founded).

Indeed, let y be such that $R(x_0, x_1, y)$. Let $p_n^0 p_n^1 q_n$ be such that $\langle x_0 x_1 y\rangle \in \langle p_n^0 p_n^1 q_n\rangle$, $\operatorname{rank} p_n^0, \operatorname{rank} p_n^1, \operatorname{rank} q_n \geqslant n$ and $\langle p_n^0 p_n^1 q_{nn}\rangle \in T$. Then $(\langle q_n, n\rangle)_{n\in\omega}$ is a descending chain in $T(x_0 x_1)$. Conversely, let $\langle q_n, n\rangle_{n\in\omega}$ be a descending sequence in $T(x_0 x_1)$. Let $\langle p_n^0 p_n^1\rangle$ be such that $\operatorname{rank} p_n^i \geqslant n$, $\langle p_n^0 p_n^1 q_n\rangle \in T$, $\langle x_0 x_1\rangle \in \langle p_n^0 p_n^1\rangle$. Let $y \in \bigcap_n q_n$. Then by Remark 3.2 $\langle x_0 x_1 y\rangle$ is a branch of T, thus $R(x_0, x_1, y)$.

Let us prove the lemma.

We have $A'(x_0) = (Ex_1)(y)\neg R(x_0, x_1, y) = (Ex_1)T(x_0 x_1)$ is well-founded. Then by standard methods we define a tree $\boldsymbol{P}$ in M such that $A'(x_0) = (Eg)(\langle x_0, g\rangle$ is a branch of $\boldsymbol{P}) \equiv \boldsymbol{P}$ is not well-founded. Hence follows the required absoluteness of A', A. ■

Then our theorem can be strengthened to the following:

THEOREM 3.1. *Let $\langle \mathscr{X}_0, \mathcal{O}_0\rangle, \langle \mathscr{X}_1, \mathcal{O}_1\rangle, \langle \mathscr{Y}_1, \mathcal{O}_2\rangle$ be topological spaces, $\mathcal{O}_i$ be M-codable. Let $\mathcal{O}_i$ have the centralization property and the Moore property in M. Let $\overline{\overline{O_0 \times O_1 \times O_2}}^M = \varkappa$. Let $R \subseteq \mathscr{X}_0 \times \mathscr{X}_1 \times \mathscr{Y}$ have a continuous tree T in M, let R be continuous in $\mathcal{O}_0 \times \mathcal{O}_1$ w.r.t. $\mathcal{O}_2$. Define*

$$A(x_0) = (x_1)_{\mathscr{X}_1}(Ey)_{\mathscr{Y}} R(x_0 x_1 y)\,.$$

Then there is a tree $\boldsymbol{P} \in M$, $\boldsymbol{P} \subseteq O_0 \times \bigcup_{\xi_1 \ldots \xi_n \in \varkappa} (\varkappa^+)^{\{\xi_1 \ldots \xi_n\}}$ such that

$$x_0 \in A \equiv (Ef)_{(\varkappa+)^\varkappa}(\langle x_0 f\rangle \text{ is a branch of } \boldsymbol{P}).$$

We can prove this theorem by analyzing the proof of Theorem 2.1.

Let us observe now that Theorem 3.1 can be applied to forcing constructions over M. Indeed, an M generic filter over $\boldsymbol{P}$ induces a branch of $\boldsymbol{P}$, and thus an element of A. It follows that if A is non-empty then it has M-generic members.

This extends the classical Shoenfield result on $\Pi_1^1(M)$ sets — if it is non-empty then it has a member in M.

Notice that there is no tree $\boldsymbol{P}$ in M such that A is a projection of $\boldsymbol{P}$ and $\boldsymbol{P}$ satisfies (*) from the remarks following Remark 1.3.

Indeed if there was such a P then the sentence $(Ex_0)A(x_0)$ would be absolute w.r.t. M and this is not the case if $\mathscr{X}_1 = \mathrm{Bord} M$. Thus the classical Shoenfield method can not be applied here.

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Continuous relations and generalized G_δ sets

by

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Abstract. In the paper some purely topological analogues of the main notions connected with forcing are considered. We compare the properties of the topological notions with the properties of their forcing counterparts.

Introduction. The paper was inspired by studies on forcing.

If we consider the first original notion of forcing, i.e., the set of Cohen forcing conditions $2^{<\omega}$, then $2^{<\omega}$ codes a base of the natural product topology in 2^ω. If $\underset{\sim}{x}$ is a fixed real in the Shoenfield universe $V^{2^{<\omega}}$, then the relation R, $s \Vdash (\check{t} \subseteq \underset{\sim}{x})$, is a relation between finite sequences s and t. If $\mathscr{D}$ is a family of dense subsets of $2^{<\omega}$, then the set X of reals $\mathscr{D}$-generic over $2^{<\omega}$ is a subset of 2^ω. If $\mathscr{D}$ is so large that for $\alpha \in X$, $i_\alpha(\underset{\sim}{x})$ can be defined then the function $f(\alpha) = i_\alpha(\underset{\sim}{x})$ defined for $\alpha \in X$ is a function from X into 2^ω. Since all R, X and f are objects connected with the topological space $\langle 2^\omega, 2^{<\omega}\rangle$, we can ask about their topological characterizations. Moreover, we can study their topological properties. This leads us to the notions of a regular relation, a g. G_δ set, a forcing function, and a continuous relation. These notions are not restricted to the case of the Cantor space $\langle 2^\omega, 2^{<\omega}\rangle$ but the reader should always have in mind this space or the Baire space $\langle \omega^\omega, \omega^{<\omega}\rangle$ as the main illustration. The mentioned notions are inspired respectively by

1) sets of the form $\{x\colon x \text{ is } \boldsymbol{P}\text{-generic over } M\}$ for given $\boldsymbol{P}$, M,

2) relations of the form $\{\langle p, q\rangle\colon p \in \boldsymbol{P}, p \Vdash (\underset{\sim}{x} \in \check{q}), q \in \boldsymbol{Q}\}$ for given $\boldsymbol{P}, \boldsymbol{Q}, M, \underset{\sim}{x} \in M^{\boldsymbol{P}}$,

3) functions of the form $f(\alpha) = i_\alpha(\underset{\sim}{x})$ for given $\boldsymbol{P}, M, \underset{\sim}{x} \in M^{\boldsymbol{P}}$,

4) relations of the form $\{\langle x, y\rangle\colon \langle x, y\rangle \text{ is generic over } \boldsymbol{P} \times \boldsymbol{Q}, M\}$ for given $\boldsymbol{P}$, $\boldsymbol{Q}$, M.

In certain cases our topological notions characterize the appropriate forcing notions, then we indicate it — Fact 2, Corollary 4 but in general the correspondence is not strict. However, it turns out that certain topological theorems about our notions have analogues in the forcing theory. We prove a few such theorems, mainly Fact 1 and Theorem 1. Indeed Fact 1, especially the fact that $\mathrm{Dom} f$ is g. G_δ for f satisfying certain assumptions corresponds to the fact that the function $f(\alpha) = i_\alpha(\underset{\sim}{x})$ is defined for all generic α. Theorem 1 corresponds to the fact that if $\langle x, y\rangle$ is generic