Extension theorem for a pseudo-arc

by

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Abstract. It is proved that every mapping f from a subcontinuum F of a hereditarily indecomposable metric continuum X into a pseudo-arc P can be extended to a mapping f^* from X into P.

In this paper we will prove some similar extension theorem to that of [6]. These results were earlier announced by D. P. Bellamy. The author has obtained the generalization of Corollary 18 from that paper to the nonmetric case, but the proof is much more complicated. We extensively use the methods of [1].

1. Notation. If X is a topological space and $A \subset F \subset X$, then $\mathrm{cl}A$, $\mathrm{Int}A$, $\mathrm{Int}_F A$ denote respectively the closure of A in X, the interior of A in X and the interior of A with respect to F. If $x \in A$, then $K(x, A)$ means a component of x in A. For a metric space X and a real number $\varepsilon > 0$ we put $B(A, \varepsilon) = \{x \in X: \varrho(x, A) < \varepsilon\}$ for every $A \subset X$ where ϱ is a metric in X and $\varrho(x, A) = \inf\{\varrho(x, a): a \in A\}$. The closed unit interval $[0, 1]$ we denote by I.

If $f: X \to I$ and $t \in (0, 1)$, then $f^-(t) = \mathrm{cl} f^{-1}([0, t))$ and $f^+(t) = \mathrm{cl} f^{-1}((t, 1])$. By Q we always denote the set of all rational numbers in I which is arranged into an infinite sequence $0, 1, r_1, r_2, \ldots$ All mappings considered in this paper are continuous.

2. Separating functions. We say that a function f from X onto I is *separating* if $f^-(t) \cap f^+(t) = \emptyset$ for $t \in Q \setminus \{0, 1\}$.

PROPOSITION 1. *If $f: X \to Y$ is onto and $g: Y \to I$ is separating, then $g \circ f$ is separating.*

PROPOSITION 2. *There exists a separating function $g: I \to I$ which is onto and monotone and $g(Q) = Q$.*

In fact, let $f_1: I \to I$ be a homeomorphism which carries the set of rationals onto the set of triadic rationals (see [5], pp. 51–54), let $f_2: I \to I$ be the Cantor ternary function, and let $f_3: I \to I$ be a homeomorphism which carries the set of dyadic rationals onto the set of rationals. Then $g = f_3 \circ f_2 \circ f_1$ has the required properties.

THEOREM 3. *Let F be a closed subset of a normal space X. If $f: F \to I$ is separating, then there exists a separating extension $f^*: X \to I$ of f.*

Proof. We proceed as in the proof of Urysohn's Lemma. For every number $r_i \in Q$ we shall define open sets $V_i, U_i \subset X$ subject to the conditions

(1) $\operatorname{cl} V_i \subset U_i \subset \operatorname{cl} U_i \subset V_j$, whenever $r_i < r_j$,

(2) $f^-(r_i) \subset V_i$ and $f^+(r_i) \subset X \setminus \operatorname{cl} U_i$.

The sets V_i and U_i will be defined inductively. Since $f^-(r_1) \cap f^+(r_1) = \emptyset$, we can find open sets V_1 and U_1 (by the normality of X) satisfying (2) for $i = 1$ and $\operatorname{cl} V_1 \subset U_1$. Assume that the sets V_i and U_i are already defined for $i \leqslant n$ and satisfy (1) and (2) for $i, j \leqslant n$. Let us denote by r_l and r_m respectively those of the numbers $r_1, r_2, \ldots, r_n$ that are closest to r_{n+1} from the left and from the right. We have $\operatorname{cl} U_l \subset V_m$. From the normality of X we infer that there exist open sets G, H such that $\operatorname{cl} U_l \cup f^-(r_{n+1}) \subset G$, $(X \setminus V_m) \cup f^+(r_{n+1}) \subset H$ and $\operatorname{cl} G \cap \operatorname{cl} H = \emptyset$ because $f^-(r_{n+1}) \subset f^-(r_m) \subset V_m$ and $f^+(r_{n+1}) \subset f^+(r_l) \subset X \setminus \operatorname{cl} U_l$. This implies that $\operatorname{cl} U_l \subset G \subset \operatorname{cl} G \subset X \setminus \operatorname{cl} H \subset \operatorname{cl}(X \setminus \operatorname{cl} H) \subset X \setminus H \subset V_m$. Assuming $V_{n+1} = G$ and $U_{n+1} = X \setminus \operatorname{cl} H$, we obtain sets that satisfy the required conditions.

Put $V = V_1 \cup V_2 \cup \ldots$ The function f^* from X to I is defined by the formula

$$(3) \qquad f^*(x) = \begin{cases} \inf\{r_i \colon x \in V_i\} & \text{for } x \in V, \\ 1 & \text{for } x \in X \setminus V. \end{cases}$$

Now we have to prove that f^* is continuous. It suffices to show that the inverse images of intervals $[0, a)$ and $(b, 1]$ where $a \leqslant 1$ and $b \geqslant 0$, are open. The inequality $f^*(x) < a$ holds if and only if there exists an $r_i < a$ such that $x \in V_i$; hence the set $(f^*)^{-1}([0, a)) = \bigcup \{V_i \colon r_i < a\}$ is open. And the inequality $f^*(x) > b$ holds if and only if there exists an $r_i > b$ such that $x \notin V_i$, which — by virtue of (1) — means that there exists an $r_j > b$ such that $x \notin \operatorname{cl} V_j$. Hence the set

$$(f^*)^{-1}((b, 1]) = \bigcup \{X \setminus \operatorname{cl} V_j \colon r_j > b\} = X \setminus \bigcap \{\operatorname{cl} V_j \colon r_j > b\}$$

is open, too.

Moreover, by (1), we have

$$(f^*)^-(r_j) = \operatorname{cl}\big(\bigcup \{V_i \colon r_i < r_j\}\big) \subset \operatorname{cl} V_j$$

and

$$(f^*)^+(r_j) = \operatorname{cl}\big(\bigcup \{X \setminus \operatorname{cl} V_k \colon r_j < r_k\}\big) \subset \operatorname{cl}(X \setminus U_j) = X \setminus U_j .$$

But $\operatorname{cl} V_j \cap (X \setminus U_j) = \emptyset$, thus the function f^* is separating.

It remains to prove that $f^*|F = f$. Let $x \in F$ and suppose that $f(x) < f^*(x)$. Take r_i such that $f(x) < r_i < f^*(x)$. Then $f^-(r_i) \subset V_i$ by (2). The definition of f^* implies that $f^*(x) \leqslant r_i$, a contradiction. Suppose now that $f^*(x) < f(x)$ and take r_i such that $f^*(x) < r_i < f(x)$. Then $x \in f^+(r_i) \subset X \setminus \operatorname{cl} U_i$ by (2). Therefore $x \notin V_j$ for $r_j < r_i$ by (1). Hence $f^*(x) \geqslant r_i$, by the definition of f^*, a contradiction. The proof of Theorem 3 is complete.

3. Nice extensions. It is known (see [4], Lemma 1.2.8, p. 13) that:

Proposition 4. *If sets A and B are separated in a metric space X, then there is an open set U in X such that $A \subset U \subset \operatorname{cl} U \subset X \setminus B$.*

Now, we have

Proposition 5. *Let F be a closed subset of a metric space X and let $f \colon F \to I$ be separating. For each $\varepsilon > 0$ and each $t \in Q \setminus \{0, 1\}$ there is an open set V in X such that $\operatorname{Int}_F f^{-1}(t) \subset V \subset B(f^{-1}(t), \varepsilon)$ and $\operatorname{cl} V \cap F \subset f^{-1}(t)$.*

Indeed, the sets $f^{-1}([0, t)) \cup f^{-1}((t, 1])$ and $\operatorname{Int}_F f^{-1}(t)$ are separated; thus, by Proposition 4, there is an open set U such that

$$\operatorname{Int}_F f^{-1}(t) \subset U \subset \operatorname{cl} U \subset X \setminus \big(f^{-1}([0, t)) \cup f^{-1}((t, 1])\big).$$

If we take $V = U \cap B(f^{-1}(t), \varepsilon)$, then V has the required properties.

Let F be a closed subset of a space X. We say that $f^* \colon X \to I$ is a *nice extension of* $f \colon F \to I$ if $f^*|F = f$ and $\operatorname{Int}_F f^{-1}(t) \subset \operatorname{Int}(f^*)^{-1}(t)$ for $t \in Q \setminus \{0, 1\}$.

Theorem 6. *Let F be a closed subset of a metric space X. If $f \colon F \to I$ is separating, then there exists a separating nice extension $f^* \colon X \to I$ of f.*

Proof. For every number $r_i \in Q$ we can define, by Proposition 5, an open set V_i subject to the conditions: $\operatorname{Int}_F f^{-1}(r_i) \subset V_i \subset B(f^{-1}(r_i), 1/i)$, $\operatorname{cl} V_i \cap F \subset f^{-1}(r_i)$ and $\operatorname{cl} V_i \cap \operatorname{cl} V_j = \emptyset$ for $i \neq j$. Put $E = F \cup \operatorname{cl} V_1 \cup \operatorname{cl} V_2 \cup \ldots$ and define $f_0 \colon E \to I$ as follows:

$$f_0(x) = \begin{cases} f(x) & \text{if } x \in F, \\ r_i & \text{if } x \in \operatorname{cl} V_i . \end{cases}$$

The mapping f_0 is separating; thus, by Theorem 3, there is a separating extension $f^* \colon X \to I$ of f_0. It is easy to see that f^* is a separating nice extension of f.

Theorem 7. *Let F be a closed subset of a metric space X. If $f \colon X \to I$, $h \colon F \to I$ and $g \colon I \to I$ are separating functions such that g is monotone and $g(Q) = Q$ and f is a nice extension of $g \circ h$, then there is a separating nice extension $h^* \colon X \to I$ of h such that $f = g \circ h^*$.*

Proof. We can assume that $g(0) = 0$ and $g(1) = 1$. For every number $r_i \in Q$ we can define, by Proposition 5, an open set H_i subject to the conditions:

(4) $\operatorname{Int}_F h^{-1}(r_i) \subset H_i \subset B(h^{-1}(r_i), 1/i) \cap \{x \in X \colon \varrho(x, h^{-1}(r_i)) < \varrho(x, X \setminus \operatorname{Int} f^{-1}(g(r_i)))\}$,

(5) $\operatorname{cl} H_i \cap F \subset h^{-1}(r_i)$,

(6) $\operatorname{cl} H_i \cap \operatorname{cl} H_j = \emptyset$ for $i \neq j$.

Put $E = F \cup \operatorname{cl} H_1 \cup \operatorname{cl} H_2 \cup \ldots$ and define $h_0 \colon E \to I$ by

$$h_0(x) = \begin{cases} h(x) & \text{for } x \in F, \\ r_i & \text{for } x \in \operatorname{cl} H_i . \end{cases}$$

Then h_0 is separating, f is a nice extension of $g \circ h_0$ (this easily follows from (4), (5) and (6)), and

(7) $h_0^-(r_i) \cap f^+(g(r_i)) = \emptyset$ and $h_0^+(r_i) \cap f^-(g(r_i)) = \emptyset$.

Suppose that $x \in h_0^-(r_i) \cap f^+(g(r_i))$. Let W be an open neighborhood of x in X such that $W \cap (h_0^+(r_i) \cup f^-(g(r_i))) = \emptyset$. Then $W \cap E = W \cap h_0^{-1}([0, r_i])$; thus $f(W \cap E) \subset [0, g(r_i)]$ by the monotoneity of g. But $f(W) \subset [g(r_i), 1]$. Hence $f(W \cap E) = g(r_i)$. Since f is a nice extension of $g \circ h_0$, there is an open set W' in X containing x such that $f(W') = g(r_i)$. But then $x \notin f^+(g(r_i))$, a contradiction. Similarly one can prove the second equality.

Now, for every number $r_i \in Q$ we shall define open sets V_i, $U_i \subset X$ satisfying the conditions

(8) $\mathrm{cl}\, V_i \subset U_i \subset \mathrm{cl}\, U_i \subset V_j$, whenever $r_i < r_j$,

(9) $h_0^-(r_i) \subset V_i$ and $h_0^+(r_i) \subset X \setminus \mathrm{cl}\, U_i$,

(10) $f^-(g(r_i)) \subset V_i$ and $f^+(g(r_i)) \subset X \setminus \mathrm{cl}\, U_i$.

The sets V_i and U_i will be defined inductively. Since $(h_0^-(r_1) \cup f^-(g(r_1))) \cap (h_0^+(r_1) \cup f^+(g(r_1))) = \emptyset$ (compare (7)), we can find sets V_1 and U_1 (by the normality of X) satisfying (9) and (10) for $i = 1$ and $\mathrm{cl}\, V_1 \subset U_1$. Assume that the sets V_i and U_i are already defined for $i \leqslant n$ and satisfy (8), (9) and (10) for $i, j \leqslant n$. Let us denote by r_l and r_m respectively those of the numbers $r_1, r_2, \ldots, r_n$ that are closest to r_{n+1} from the left and from the right. We have $\mathrm{cl}\, U_l \subset V_m$. From the normality of X we infer that there exist open sets G, H such that

$$\mathrm{cl}\, U_l \cup f^-(g(r_{n+1})) \cup h_0^-(r_{n+1}) \subset G, \quad (X \setminus V_m) \cup f^+(g(r_{n+1})) \cup h_0^+(r_{n+1}) \subset H$$

and $\mathrm{cl}\, G \cap \mathrm{cl}\, H = \emptyset$ (compare (7)). Assuming $V_{n+1} = G$ and $U_{n+1} = X \setminus \mathrm{cl}\, H$, we obtain sets that satisfy the required conditions.

Put $V = V_1 \cup V_2 \cup \ldots$ The function h^* from X to I is defined by the formula

$$h^*(x) = \begin{cases} \inf\{r_i \colon x \in V_i\} & \text{for } x \in V, \\ 1 & \text{for } x \in X \setminus V. \end{cases}$$

As in the proof of Theorem 3, conditions (8) and (9) imply that h^* is a separating extension of h_0. Condition (10) guarantees the equality $f = g \circ h$. Moreover, the choice of h_0 shows that h^* is a nice extension of h. The proof of Theorem 7 is complete.

4. N-mappings. We say that (see [1]) a mapping $h\colon I \to I$ is an *N-mapping* if h satisfies the following conditions:

(i) $h(q)$ is rational if and only if q is rational.

(ii) There exist four rationals a, b, c, d with $0 < a < c < 1$ and $0 < d < b < 1$ and $h(a) = b$ and $h(c) = d$.

(iii) $h(0) = 0$ and $h(1) = 1$.

(iv) Each of the mappings $h|[0, a]$, $h|[a, c]$ and $h|[c, 1]$ is a homeomorphism.

We will prove

THEOREM 8. *Let F be a subcontinuum of a hereditarily indecomposable continuum X and let $f\colon X \to I$ and $g\colon F \to I$ be separating surjections. If $h\colon I \to I$ is an N-mapping such that f is a nice extension of $h \circ g$, then there is a separating nice extension $g^*\colon X \to I$ of g such that $f = h \circ g^*$.*

Proof. Let a, b, c, d be the rationals which describe h. Define $P \subset X$ and $R \subset X$ by:

$$P = \{x \in f^-(b)\colon K(x, f^-(b)) \cap f^-(d) \neq \emptyset\},$$
$$R = \{x \in f^+(d)\colon K(x, f^+(d)) \cap f^+(b) \neq \emptyset\}.$$

As in the proof of Lemma 3 in [1], p. 8, one can check that

(11) P and R are closed and disjoint.

Moreover

(12) $g^+(a) \cap P = \emptyset$ and $g^-(c) \cap R = \emptyset$.

We will show only the first equality (a parallel argument will show the second). Suppose that $x \in g^+(a) \cap P$ and $z \in K(x, f^-(b)) \cap f^-(d)$. Since X is hereditarily indecomposable, we infer that $K(x, f^-(b)) \subset F$. Since f is a nice separation extension of $h \circ g$ and $z \in f^-(d)$, we obtain $z \in (h \circ g)^-(d)$. Therefore $g(z) \in [0, a)$ because $f(z) \in [0, d]$. But $x \in g^+(a)$ implies that $K(x, f^-(b)) \cap g^{-1}(a) \neq \emptyset$. Since $\mathrm{Int}_F g^{-1}(a) \subset \mathrm{Int} f^{-1}(b)$ and $K(x, f^-(b)) \subset f^-(b)$, we conclude that $\mathrm{Int}_F g^{-1}(a) \cap K(x, f^-(b)) = \emptyset$. Then $(g^-(a) \cap K(x, f^-(b))) \cup (g^+(a) \cap K(x, f^-(b)))$ is a separation of $K(x, f^-(b))$, a contradiction.

Now we will prove that

(13) every component of $g^-(a)$ is a component of $f^-(b)$ and every component of $g^+(c)$ is a component of $f^-(d)$.

Indeed, let K be a component of $g^-(a)$ and let C be a component of $f^-(b)$ containing K. Then $C \subset F$, because X is hereditarily indecomposable. Since $C \cap \mathrm{Int}_F g^{-1}(a) = \emptyset$, we obtain $C = (C \cap g^-(a)) \cup (C \cap g^+(a))$. But $g^-(a) \cap g^+(a) = \emptyset$; thus $C \cap g^+(a) = \emptyset$ by the connectedness of C, i.e., $C = K$.

It follows from (11), (12), (13) and Lemma 2 in [1], p. 7 that there is a separation $A \cup M$ of $f^-(b)$ such that $P \cup g^-(a) \subset A$ and $(R \cup g^+(a)) \cap f^-(b) \subset M$. Similarly there is also a separation $B \cup N$ of $f^+(d)$ such that $R \cup g^+(c) \subset B$ and $(A \cup g^-(c)) \cap f^+(d) \subset N$. Then

(14) A and B are disjoint.

Moreover, as in the proof of Lemma 3 in [1], p. 9, we have

(15) $X = A \cup B \cup (M \cap N) \cup (f^{-1}(d) \cap M) \cup (f^{-1}(b) \cap N)$.

Put $J = A \cup (f^{-1}(b) \cap N)$, $K = (f^{-1}(b) \cap N) \cup (M \cap N) \cup (f^{-1}(d) \cap M)$ and $L = (f^{-1}(d) \cap M) \cup B$ and define $g^*\colon X \to I$ by

$$g^*(x) = \begin{cases} (h|[0, a])^{-1}(f(x)) & \text{for } x \in J, \\ (h|[a, c])^{-1}(f(x)) & \text{for } x \in K, \\ (h|[c, 1])^{-1}(f(x)) & \text{for } x \in L. \end{cases}$$

As in the proof of Lemma 3 in [1], using (14) and (15) one can easily check that g^* is continuous, $h \circ g^* = f$, g^* is separating and $g^*|F = g$. The fact that g^* is a nice extension of g also follows easily. The proof of Theorem 8 is complete.

5. Extension theorem. Let $g_n: I \to I$ be an arbitrary separating and monotone function such that $g_n(Q) = Q$ and $g_n(0) = 0$ (compare Proposition 2). We consider a continuum I_∞ defined as the limit of the inverse sequence $\{I_n, \alpha_n\}$ where for each $n \geq 1$ we have $I_n = I$, $\alpha_n = g_n \circ h_n$ and h_n are N-mappings from I onto I. Then we say that I_∞ is of type N^*. We denote the projection from the inverse limit I_∞ onto I_n by π_n. If $n < m$, then we put $\alpha_n^m = \alpha_{n+1} \circ \ldots \circ \alpha_{m-2} \circ \alpha_{m-1}$. It is known that

(16) $\pi_n = \alpha_n^m \circ \pi_m$.

In particular, $\pi_n = g_n \circ h_n \circ \pi_{n+1}$. Since g is separating, it follows by Proposition 1 that

(17) π_n and $h_n \circ \pi_{n+1}$ are separating.

We will prove

THEOREM 9. *If Y is a subcontinuum of a hereditarily indecomposable metric continuum X, and if f is a continuous mapping from Y onto I_∞, then there is a continuous mapping f^* from X onto I_∞ such that $f^*|Y = f$.*

Proof. It suffices to construct a sequence of continuous mappings $f_n: X \to I_n$ which satisfies the following conditions:

(18) f_n are separating,

(19) f_n are nice extensions of $\pi_n \circ f$,

(20) $f_{n-1} = \alpha_{n-1} \circ f_n$ for $n > 1$.

The mapping $\pi_1 \circ f: Y \to I_1$ is separating. It follows from Theorem 6 that there exists a separating nice extension $f_1: X \to I$ of $\pi_1 \circ f$. Then f_1 satisfies (18) and (19). Assume that the functions $f_1, f_2, \ldots, f_n$ have been constructed in such way that (18), (19) and (20) hold for $k \leq n$. Now, we will construct f_{n+1} by induction. Using Theorem 7 and (17) we can find a separating nice extension $f'_{n+1}: X \to I$ of $h_n \circ \pi_{n+1} \circ f$ such that $f_n = g_n \circ f'_{n+1}$. Therefore, by Theorem 8 and (17), there exists a separating nice extension $f_{n+1}: X \to I$ of $\pi_{n+1} \circ f$ such that $f'_{n+1} = h_n \circ f_{n+1}$. But then also $f_n = \alpha_n \circ f_{n+1}$, which completes the proof of Theorem 9.

6. Factorizations of separating functions. Now we will prove the following

THEOREM 10. *If $f: X \to I$ is a separating function from a normal space X and $g: I \to I$ is separating and monotone and $g(Q) = Q$, then there is a separating function $h: X \to I$ such that $f = g \circ h$. If $X = I$, f is monotone and $f(Q) = Q$, then h can be chosen to be monotone and $h(Q) = Q$.*

Proof. We can assume that $g(0) = 0$ and $g(1) = 1$. For every number $r_i \in Q$ we shall define open sets $V_i, U_i \subset X$ subject to the following conditions:

(21) $\mathrm{cl}\, V_i \subset U_i \subset \mathrm{cl}\, U_i \subset V_j$, whenever $r_i < r_j$,

(22) $f^-(g(r_i)) \subset V_i$ and $f^+(g(r_i)) \subset X \setminus \mathrm{cl}\, U_i$.

The sets V_i and U_i will be defined inductively in the same way as in the proof of Theorem 3.

Put $V = V_1 \cup V_2 \cup \ldots$ The function h from X to I is defined by the formula

$$h(x) = \begin{cases} \inf\{r_i: x \in V_i\} & \text{for } x \in V, \\ 1 & \text{for } x \in X \setminus V. \end{cases}$$

Using (21) and (22) as in the proof of Theorem 3, one can easily check that h is a separating function such that $f = g \circ h$.

If $X = I$ and f is monotone and such that $f(0) = 0$, then V_i and U_i can be chosen in the form of sets $[0, s)$, which implies the monotoneity of h and completes the proof of Theorem 10.

7. Mappings onto N-continua. It follows from Lemma 1 and Lemma 3 in [1], p. 7 that

PROPOSITION 11. *If X is a connected normal space, then there exists a separating function f from X onto I.*

PROPOSITION 12. *If X is a hereditarily indecomposable continuum, $f: X \to I$ is separating and $h: I \to I$ is an N-function, then there exists a separating function $g: X \to I$ such that $f = h \circ g$.*

From Theorem 10 and Propositions 11 and 12 we infer

THEOREM 13. *Any hereditarily indecomposable continuum can be mapped onto any continuum of type N^*.*

Proof. Let X be a hereditarily indecomposable continuum and let I_∞ be an arbitrary continuum of type N^* represented as an inverse limit of a sequence of intervals I_n and compositions $\alpha_n = g_n \circ h_n$ as bonding mappings where h_n are N-mappings from I onto I and $g_n: I \to I$ are separating monotone functions such that $g(Q) = Q$ (compare Section 5 here). From Proposition 11 we find $f_1: X \to I$, which is separating. Now, suppose that $f_1, \ldots, f_n$ have been selected such that they are separating and $f_{j-1} = \alpha_{j-1} \circ f_j$ for $1 < j \leq n$. Theorem 10 assures us that there is a separating function $f'_{n+1}: X \to I$ such that $f_n = g_n \circ f'_{n+1}$. It follows from Proposition 12 that there exists a separating function $f_{n+1}: X \to I$ such that $f'_{n+1} = h_n \circ f_{n+1}$. Then the sequence $f_1, f_2, \ldots$ induces a surjection $f: X \to I_\infty$, which completes the proof of Theorem 13.

8. N-continua. Recall that (see [1], p. 6) a compact metric continuum is of type N if it can be represented as an inverse limit of a sequence $\{I_n, h_n\}$ where $I_n = I$ and h_n are of type N (compare section 4 here). Firstly, we have

THEOREM 14. *If $h: I \to I$ is an N-mapping and $f: I \to I$ is a separating monotone function such that $f(Q) = Q$, then there are separating monotone functions $f': I \to I$ and $g: I \to I$ such that $f'(Q) = Q$ and $g(Q) = Q$ and there is an N-mapping $h': I \to I$ such that $f \circ h = h' \circ g \circ f'$.*

Proof. It follows from Whyburn's factorization theorem (see [5], Theorem 3–40, p. 137) that there are a monotone mapping $\alpha: I \to X$ and a light mapping $\beta: X \to I$

such that $\beta \circ \alpha = f \circ h$. Since α is monotone, it follows that X is homeomorphic to I. Assume $X = I$. It is easy to see that, for each $s \in \alpha(Q) \setminus \{0, 1\}$, we have $\alpha^-(s) \cap \cap \alpha^+(s) = \emptyset$. There is a homeomorphism δ of I onto I such that $\delta(\alpha(Q)) = Q$, because the set $\alpha(Q)$ is countable and dense in I. Therefore we can assume that $\alpha(Q) = Q$ and $\alpha(0) = 0$. Then α is a separating monotone function such that $\alpha(Q) = Q$. It follows from Proposition 2 and Theorem 10 (the additional assertion) that there are separating monotone functions $f'\colon I \to I$ and $g\colon I \to I$ such that $f'(Q) = Q$, $g(Q) = Q$ and $\alpha = g \circ f'$. Put $h' = \beta$. It is easy to check that the mappings f', g and h' satisfy the required conditions (it is possible for h' to be a homeomorphism preserving rationals — we treat such an h' also as a mapping of type N).

It is known that (see [7], Theorem 10, p. 69)

PROPOSITION 15. *Let $\{X_n, h_n\}$ and $\{Y_n, g_n\}$ be two inverse sequence of metric spaces X_n, and let $f_n\colon X_n \to Y_n$ be monotone surjections such that $f_n \circ h_n = g_n \circ f_{n+1}$. Then f_n induce a monotone surjection of an inverse limit of $\{X_n, h_n\}$ onto an inverse limit of $\{Y_n, g_n\}$.*

(This theorem is formulated in [7] in a more general form.)

Using theorem 14 and Proposition 15, we obtain the following

THEOREM 16. *Every metric continuum of type N can be mapped by a monotone mapping onto a continuum of type N^*.*

Remark that the converse theorem is also true.

9. Applications. The main results. Since hereditary indecomposability is preserved by monotone mappings, by using Theorem 2 in [1], p. 11, Theorem 1 in [2] and the theorems proved in this paper we obtain the following corollaries.

COROLLARY 17. *A pseudo-arc is of type N^*.*

COROLLARY 18. *Every mapping f from a subcontinuum F of a hereditarily indecomposable metric continuum X into a pseudo-arc P can be extended to a mapping f^* from X into P.*

COROLLARY 19. *If X is a hereditarily indecomposable continuum, then there is a mapping from X onto a pseudo-arc.*

COROLLARY 20. *Every subcontinuum of a pseudo-arc P is a retract of P.*

Corollary 19 has been proved in [1], and Corollary 20 was obtained in [3].

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