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Remarks on characterization of dimension of separable metrizable spaces *

by

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Abstract. We establish some characterizations of dimension of separable metrizable spaces. For instance, it is shown that a separable metrizable space X is of dimension $\leqslant n$ if and only if X is homeomorphic to a subset S of the $(2n+1)$-dimensional cube I^{2n+1} such that

$$\lim_{\varepsilon\to 0} k(\varepsilon, \|\cdot\|)\varepsilon^p = 0 \quad \text{for} \quad p>n$$

where

$$k(\varepsilon, \|\cdot\|) = \inf\{n\colon \text{ there exists an } \varepsilon\text{-net}\{x_1, \ldots, x_n\} \text{ for } S\}.$$

Dimension is a topological concept, but in many cases it can be characterized by metrics or pseudometrics, [8], [10], [5]. In [12] Szpilrajn established some connections between the concept of dimension and the classical concept of Hausdorff measure. Borsuk [3] has constructed, for each $n \in N$, an n-dimensional pseudomeasure V_n^B of compacta lying in the Hilbert space l_2. This concept is a topological invariant, i.e. if $V_n^B(X)>0$ then $V_n^B(Y)>0$ for every compactum Y homeomorphic to X, [4]. Several connections between dimension and Borsuk pseudomeasure are given in [3], [4]. Since the Borsuk pseudomeasure is defined only on compacta isometrically embedable into l_2, we construct in § 1 of this note a pseudomeasure for the class of all compacta similar to the Borsuk pseudomeasure. This pseudomeasure is shown to have many of the properties possessed by the Borsuk pseudomeasure. In § 2 we establish certain characterizations of dimension of separable metrizable spaces which are related to old results of Szpilrajn [12] and Pontrjagin and Schinirelman [11].

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§ 1. Pseudomeasure and dimension of separable metric spaces. Given a separable metrizable space X. Let $M_{tb}(X)$ (resp. $P_{tb}(X)$) denote the set of all totally

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bounded compatible metrics (resp. totally bounded continuous pseudometrics) on X. For every $d, \varrho \in P_{tb}(X)$ and $p \geqslant 0$, put

$$\|d-\varrho\| = \sup\{|d(x,y)-\varrho(x,y)|:\ x, y \in X\},$$

$$m_p(X,d) = \liminf_{\varepsilon\to 0}\{\sum_{i=1}^{n} d(A_i)^p:\ X = \bigcup_{i=1}^{n} A_i:\ d(A_i) \leqslant \varepsilon \text{ for } i = 1, \dots, n\},$$

$$V_p(X,d) = \liminf_{\varepsilon\to 0}\{m_p(X,\varrho):\ \|d-\varrho\| \leqslant \varepsilon \text{ and } \varrho \text{ is uniformly continuous with respect to } d\}.$$

We call $V_p(X, d)$ the *p-dimensional pseudomeasure of* (X, d).

1-1. Remark. Obviously if (X, d) is a compact metric space then $m_p(X, d)$ is identical with the p-dimensional Hausdorff measure $m_p^H(X, d)$ of (X, d) as defined e.g. in [9]. In general we have

$$m_p(X,d) \geqslant m_p^H(X,d) \quad \text{for each separable metric space } (X,d).$$

Let l^∞ denote the Banach space of all bounded sequences of real numbers equipped with the supremum norm. Let $\{a_i\}_{i\in N}$ be a dense sequence in X. For every $d \in P_{tb}(X)$, put

(1) $$T_d(x) = \{d(x, a_i)\}_{i\in N} \quad \text{for every } x \in X.$$

Obviously T_d is an isometry of X into l^∞.

Now let X be a subset of l^∞ and $\varepsilon > 0$. A map $f\colon X \to l^\infty$ is called an *ε-push* iff f is a uniformly continuous map satisfying the condition $\|x-f(x)\| \leqslant \varepsilon$ for each $x \in X$.

Let us prove the following

1-2. Proposition. *For every $d \in P_{tb}(X)$ we have*

(2) $$V_p(X,d) = \liminf_{\varepsilon\to 0}\{m_p(fT(X), \|\cdot\|):\ f \text{ is an } \varepsilon\text{-push}\}$$

where T is an arbitrary isometric embedding of (X, d) into l^∞ and $\|\cdot\|$ is the norm of l^∞.

Proof. Given $d \in P_{tb}(X)$ and an isometry $T\colon (X, d) \to l^\infty$. Denote

$$V = \operatorname{Lim\,inf}_{\varepsilon\to 0}\{m_p(fT(X), \|\cdot\|):\ f \text{ is an } \varepsilon\text{-push}\}.$$

If $\alpha > V$ then for each $\varepsilon > 0$ there is an ε-push $f\colon T(X) \to l^\infty$ such that $m_p(fT(X), \|\cdot\|) \leqslant \alpha$. Define a pseudometric ϱ on X by the formula

$$\varrho(x,y) = \|fT(x)-fT(y)\| \quad \text{for} \quad x, y \in X.$$

It is easy to see that $\|d-\varrho\| \leqslant 2\varepsilon$ and $m_p(X,\varrho) = m_p(fT(X), \|\cdot\|)$. Thus $V_p(X, d) \leqslant \alpha$ and hence we get

(3) $$V_p(X,d) \leqslant V.$$

Conversely let $\alpha > V_p(X, d)$. Then for each $\varepsilon > 0$ there exists $\varrho \in P_{tb}(X)$ uniformly continuous with respect to d such that

$$\|d-\varrho\| \leqslant \varepsilon \quad \text{and} \quad m_p(X,\varrho) \leqslant \alpha.$$

Let $H\colon T(X) \to T_d(X)$ be an isometry defined by the formula

$$HT(x) = T_d(x) \quad \text{for each } x \in X.$$

By [1], [2] there are 1-Lipschitz maps $H', H''\colon l^\infty \to l^\infty$ such that $H'|T(X) = H$ and $H''|T_d(X) = H^{-1}$. We define $f\colon T(X) \to l^\infty$ by the formula

$$fT(x) = H''T_\varrho(x) \quad \text{for each } x \in X.$$

Since H'' is a 1-Lipschitz map we have

$$m_p(fT(X), \|\cdot\|) = m_p(H''T_\varrho(X), \|\cdot\|) \leqslant m_p(X,\varrho) \leqslant \alpha.$$

On the other hand for each $x \in X$ we have

$$d(fT(x), T(x)) = d(H''T_\varrho(x), H''H'T(x)) \leqslant d(T_\varrho(x), H'T(x)) = d(T_\varrho(x), HT(x)) = d(T_\varrho(x), T_d(x)) \leqslant 2\varepsilon.$$

Thus f is an 2ε-push. Thus we get

(4) $$V \leqslant V_p(X,d).$$

From (3) and (4) we get the assertion.

Let $V_n^B(X)$ denote the n-dimensional Borsuk pseudomeasure of a compactum X lying in the Hilbert space l_2 defined by the formula (see [3])

$$V_n^B(X) = \liminf_{\varepsilon\to 0}\{m_n(Q, \|\cdot\|):\ f\colon X \to Q \text{ is an } \varepsilon\text{-push of } X \text{ into a polyhedron } Q \subset l_2\}.$$

From Proposition 1-2 we get

1-3. Corollary. *For every compactum X lying in l_2 we have*

$$V_n(X, \|\cdot\|) \leqslant V_n^B(X) \quad \textit{for every } n = 1, 2, \dots$$

where $\|\cdot\|$ denotes the norm of l_2.

1-4. Remark. The author does not know whether V_n and V_n^B actually coincide on compacta in l_2. It can however be shown that they coincide on polyhedra in l_2 (where they coincide also with m_n (cf. [3])).

Let us note that $V_p(X, d)$ has many of the properties possessed by $V_n^B(X)$. For instance (compare [3]),

(1-5) If $\dim X < p$ then $V_p(X, d) = 0$ and if $\dim X > p$ then $V_p(X, d) = \infty$ for every $d \in M_{tb}(X)$.

(1-6) If X is a continuum and $d \in M_{tb}(X)$ then $V_1(X, d) \geqslant \operatorname{diam}(X, d)$.

(1-7) If X is an arc and $d \in M_{tb}(X)$ then

$$V_1(X,d) = \operatorname{length}(X,d) = \inf\{\sum_{i=1}^{k} d(s(t_i), s(t_{i+1})):\ 0 = t_0 < t_1 < \ldots < t_{k+1} = 1\}$$

for any homeomorphism $s\colon [0,1] \to X$.

From Proposition 1-2 we get

(1-8) If $X \subset l^\infty$ and $V_p(X, ||\cdot||) > 0$ then for every $\alpha < V_p(X, ||\cdot||)$ there is an $\varepsilon > 0$ such that $V_p(f(X), ||\cdot||) > \alpha$ for each ε-push $f\colon X \to l^\infty$.

(1-9) If $f\colon (X,d) \to (Y,\varrho)$ is a homeomorphism of X onto Y satisfying the condition $\varrho(f(x), f(y)) \leqslant Kd(x,y)$ for $x, y \in X$ then $V_p(Y,\varrho) \leqslant K^p V_p(X,d)$.

1-10. Remark. In (1-9) the assumption on f to be a homeomorphism rather than any surjective K-Lipschitz map, is essential (take $Y = [0,1]$ and $X =$ the graph of a surjection of a Cantor set on $[0,1]$ and $f =$ projection of X onto Y then $V_0(X) = 0$ and $V_0(Y) = \infty$).

1-11. Remark. In [4] it is shown that there exist compacta X, Y lying in the interval $[0,1]$ such that

$$V_1^B(X \cup Y) > V_1^B(X) + V_1^B(Y).$$

This example also yields that

$$V_1(X \cup Y, |\cdot|) > V_1(X, |\cdot|) + V_1(Y, |\cdot|).$$

Thus V is not a measure.

Let us prove the following theorem analogue of the basic result of [4].

1-12. Theorem. *Let (X,d) be a compact metric space. Then* $\dim X \leqslant n$ *if and only if $V_p(X,d) = 0$ for $p > n$.*

Proof. Identifying (X,d) with $T_d(X)$ we may assume that $X \subset L^\infty$. If $\dim X \leqslant n$ then for each $\varepsilon > 0$ there exists an ε-push $f\colon X \to l^\infty$ such that $f(X)$ is contained in a polyhedron of dimension $\leqslant n$. Since m_p vanishes on such a polyhedron for $p > n$ we infer that $V_p(X,d) = 0$ if $\dim X \leqslant n$ and $p > n$.

The proof of the converse involves the following fact proved in [4].

1-13. Lemma. *Given an m-dimensional Banach space E^m and $\varepsilon > 0$. Then there exists an $\delta = \delta(\varepsilon, m)$ such that for every compactum $Y \subset E^m$ with $m_{n+1}(X, ||\cdot||) < \delta$ there exists an ε-push $g\colon Y \to E^m$ such that $\dim g(Y) \leqslant n$.*

Proof. Since E^m is isomorphic to R^m, it suffices to consider the case $E^m = R^m$, and this is done in [4].

Using Lemma 1-13 we are able to complete the proof of Theorem 1-12.

Assume that $V_{n+1}(X,d) = 0$. We have to show that $\dim X \leqslant n$.

For each $m \in N$, put

$$l_m^\infty = \{x = (x_i) \in l^\infty\colon x_i = 0 \text{ for } i > m\}$$

and let $P_m\colon l^\infty \to l_m^\infty$ denote the natural projection. Since $T_d(X)$ is a compact in l^∞, for every $\varepsilon > 0$ there exists an $m \in N$ such that

$$||x - P_m(x)|| \leqslant \varepsilon \quad \text{for every } x \in T_d(X).$$

Take $\delta = \delta(\varepsilon, m)$ with $E^m = l_m^\infty$ from Lemma 1-13. Since $V_{n+1}(X,d) = 0$ there exists ε-push $f\colon T_d(X) \to l^\infty$ such that $m_{n+1}(fT_d(X), ||\cdot||) < \delta$. Whence $P_m f$ is an 2ε-push. Since

$$m_{n+1}(P_m f T_d(X), ||\cdot||) \leqslant m_{n+1}(fT_d(X), ||\cdot||) < \delta$$

by Lemma 1-13 there exists an ε-push $g\colon P_m f T_d(X) \to l_m^\infty$ such that $\dim g P_m f T_d(X) \leqslant n$. Since T_d is an isometry and $gP_m f$ is an 3ε-push it follows that $\dim X \leqslant n$.

This completes the proof of Theorem 1-12.

1-14. Corollary. *A separable metrizable space X is of dimension $\leqslant n$ if and only if for every $d \in M_{\rm tb}(X)$ and $p > n$ we have $V_p(X,d) = 0$.*

Proof. Let $\dim X \leqslant n$ and $d \in M_{\rm tb}(X)$. We may consider (X,d) as a totally bounded subset of l^∞. Then for each $\varepsilon > 0$ there exists an ε-push $f\colon X \to l^\infty$ such that $f(X)$ is contained in a polyhedron of dimension $\leqslant n$. Thus $V_p(X,d) = 0$ for $p > n$.

Conversely assume that $V_{n+1}(X,d) = 0$ for every $d \in M_{\rm tb}(X)$. By [7] there exists a $\tilde d \in M_{\rm tb}(X)$ such that $\dim \tilde X = \dim X$, where $\tilde X$ denotes the completion of X with respect to the metric $\tilde d$. Since $V_{n+1}(X, \tilde d) = 0$ we infer that $V_{n+1}(\tilde X, \tilde d) = 0$. Thus by Theorem 1-12 we have $\dim X = \dim \tilde X \leqslant n$.

1-15. Corollary (cf. Szpilrajn [12]). *If X is an n-dimensional separable metrizable space then for each $d \in M_{\rm tb}(X)$ we have $m_n(X,d) > 0$.*

Proof. Let $(\tilde X, d)$ denote the completion of X with respect to the metric d. Since

$$m_n(X,d) = m_n(\tilde X, d) \geqslant V_n(\tilde X, d) \quad \text{for every } d \in M_{\rm tb}(X)$$

the assertion follows from Theorem 1-12.

§ 2. A metric characterization of dimension of separable metrizable spaces. Let X be a separable metrizable space. For every $d \in M_{\rm tb}(X)$ and $\varepsilon > 0$, put

$$k(\varepsilon, d) = \inf\{n\colon \text{there exists an } \varepsilon\text{-net } \{x_1, \ldots, x_n\} \text{ for } (X,d)\}.$$

In this section we prove the following

2-1. Theorem. *Let X be a separable metrizable space. Then*

(i) *If $\dim X \geqslant n$ then for every $d \in M_{\rm tb}(X)$ we have*

$$\liminf_{\varepsilon \to 0} k(\varepsilon, d)\varepsilon^n > 0.$$

(ii) *If $\dim X \leqslant n$ then there exists a $d \in M_{\rm tb}(X)$ such that*

$$\lim_{\varepsilon \to 0} k(\varepsilon, d)\varepsilon^p = 0 \quad \text{for} \quad p > n.$$

Proof. (i) Assume that $\dim X \geqslant n$. By Corollary 1-15 for every $d \in M_{tb}(X)$ we have $\gamma(d) = m_n(X, d) > 0$. Thus for each $\alpha \in (0, \gamma(d))$ there is an $\delta > 0$ such that for every $\varepsilon \in (0, \delta)$ we have

$$\inf\{\sum_{i=1}^{m} d(A_i)^n\colon X = \bigcup_{i=1}^{m} A_i,\ d(A_i) \leqslant \varepsilon\} > \alpha .$$

Hence

$$k(\varepsilon, d)\varepsilon^n \geqslant \alpha \quad \text{for every } \varepsilon \in (0, \tfrac{1}{2}\delta).$$

This proves (i).

(ii) Let $M_n^{2n+1} \subset I^{2n+1}$, where $I = [0, 1]$, denote the n-dimensional Menger universal space constructed as follows (see Engelking [7], p. 121):

For every $i = 0, 1, \ldots$ divide the interval $[0, 1]$ into $3^{i(i+1)/2}$ equal intervals. One gets a subdivision of the cube I^{2n+1} into $3^{(2n+1)i(i+1)/2}$ small cubes with the length of the edges $3^{-i(i+1)/2}$.

Let $\mathcal{K}_i$ denote the family of all such cubes. For every family $\mathcal{K}$ of cubes, put

$$|\mathcal{K}| = \bigcup\{Q\colon Q \in \mathcal{K}\}, \qquad \mathcal{S}_n(\mathcal{K}) = \bigcup\{\mathcal{S}_n(Q)\colon Q \in \mathcal{K}\},$$

where $\mathcal{S}_n(Q)$ denotes the family of all faces of Q which have dimension $\leqslant n$. Moreover for each $\mathcal{K} \subset \mathcal{K}_i$, put

$$\mathcal{K}' = \{Q \in \mathcal{K}_{i+1}\colon Q \subset |\mathcal{K}|\}.$$

Let

$$\mathcal{F}_0 = \{I^{2n+1}\}, \qquad F_0 = |\mathcal{F}_0| = I^{2n+1}$$

and for every $i = 1, 2, \ldots$ define $\mathcal{F}_i$ and F_i by induction

$$\mathcal{F}_i = \{Q \in \mathcal{F}'_{i-1}\colon Q \cap \mathcal{S}_n(\mathcal{F}_{i-1}) \neq \emptyset\}; \qquad F_i = |\mathcal{F}_i| .$$

Then $\{\mathcal{F}_i,\ i = 0, 1, \ldots\}$ is a sequence of finite collections of cubes, $\mathcal{F}_i \subset \mathcal{K}^i$ for every $i = 0, 1, \ldots$ and $F_0 \supset F_1 \supset \ldots$ is a decreasing sequence of closed subsets of I^{2n+1}. We define M_n^{2n+1} by the formula

$$M_n^{2n+1} = \bigcap_{i=0}^{\infty} F_i \subset I^{2n+1}.$$

By the universal space theorem [7] there exists an embedding of X into M_n^{2n+1}. Thus it suffices to prove the theorem for $X = M_n^{2n+1}$ and d is the metric of M_n^{2n+1} induced by the norm of the $(2n+1)$-dimensional Euclidean space R^{2n+1}.

Let us note that each cube Q with the length of edges $3^{-i(i+1)/2}$ contains at most $A3^{n(i+1)}$ cubes with the length of edges $3^{-(i+1)(i+2)/2}$ which intersect the n-dimensional faces of Q, where A is the number of the n-dimensional faces of Q. There are at most

$$A^i 3^{n+2n+\ldots+(i+1)n} = A^i 3^{n(i+1)(i+2)/2}$$

cubes with the length of edges $3^{-(i+1)(i+2)/2}$ intersecting M_n^{2n+1}. Since these cubes

form a cover of M_n^{2n+1}, for every $\varepsilon > 0$, say

$$\varepsilon \in [(2n+1)^{1/2}3^{-(i+1)(i+2)/2}, (2n+1)^{1/2}3^{-i(i+1)/2}]$$

we have

$$k(\varepsilon, d)\varepsilon^p \leqslant A^i 3^{n(i+1)(i+2)/2}(2n+1)^{p/2}3^{-i(i+1)p/2} = A^i(2n+1)^{p/2}3^{(i+1)(ni+2n-pi)/2}.$$

Since $p > n$ we infer that

$$\lim_{\varepsilon \to 0} k(\varepsilon, d)\varepsilon^p \leqslant \lim_{i \to \infty} (2n+1)^{p/2}A^i 3^{(i+1)(ni+2n-pi)/2} = 0 .$$

This completes the proof of Theorem 2-1.

In [12] Szpilrajn has shown that a separable metrizable space X is of dimension $\leqslant n$ if and only if X is homeomorphic to a subset S of I^{2n+1} with $m_p(S) = 0$ for every $p > n$. Let us note that the proof of Theorem 2-1 gives the following

2-2. Corollary. *A separable metrizable space X is of dimension $\leqslant n$ if and only if X is homeomorphic to a subset S of the cube I^{2n+1} such that*

$$\lim_{\varepsilon \to 0} k(\varepsilon, \|\cdot\|)\varepsilon^p = 0 \quad \textit{for every } p > n$$

where $\|\cdot\|$ denotes the norm of the Euclidean space R^{2n+1}.

2-3. Remark. Obviously if $\lim_{\varepsilon \to 0} k(\varepsilon, d)\varepsilon^p = 0$ then $m_p(X, d) = 0$.

The following example shows that the converse does not hold true even for compact metric spaces.

2-4. Example. *For every integer $n \in N$ there exists a compact metric space (X, d) such that $m_p(X, d) = 0$ for every $p > 0$ and*

$$\lim_{\varepsilon \to 0} k(\varepsilon, d)\varepsilon^p = \infty \quad \textit{for} \quad p < n .$$

Proof. In the Euclidean space R^n consider the set $X = A^n$, where

$$A = \left\{0, \frac{1}{\ln 2}, \frac{1}{\ln 3}, \ldots\right\}.$$

Since X is a countable compact set we have $m_p(X, \|\cdot\|) = 0$ for every $p > 0$. Now let $p < n$. For every $\varepsilon > 0$ take $k \in N$ such that

$$\frac{1}{\ln(k+1)} - \frac{1}{\ln(k+2)} \leqslant \varepsilon \leqslant \frac{1}{\ln k} - \frac{1}{\ln(k+1)} .$$

Then we have

$$k(\varepsilon, \|\cdot\|)\varepsilon^p \geqslant k^n \left(\frac{\ln\left(1 + \frac{1}{k+1}\right)}{\ln(k+1)\ln(k+2)}\right)^p = \frac{k^n}{(k+1)^p}\left(\frac{\ln\left(1 + \frac{1}{k+1}\right)^{k+1}}{\ln(k+1)\ln(k+2)}\right)^p .$$

Since $p<n$ we infer that

$$\lim_{\varepsilon\to 0} k(\varepsilon, \|\cdot\|)\varepsilon^p = \infty$$

From Theorem 2-1 we get also

2-5. COROLLARY (Pontrjagin–Schinirelman [11], Bruijning [5]). *For every separable metrizable space X we have*

$$\dim X = \inf\{k(d)\colon d \in M_{tb}(X)\}$$

where

$$k(d) = \liminf_{\varepsilon\to 0}\{\log_2 k(\varepsilon, d)/\log_2(\varepsilon^{-1})\}.$$

Proof. Assume that $p>\dim X$. By Theorem 2-1 there exists a metric $d\in M_{tb}(X)$ such that $\lim_{\varepsilon\to 0} k(\varepsilon, d)\varepsilon^p = 0$. Thus there exists an $\delta>0$ such that

$$k(\varepsilon, d)\varepsilon^p<1 \quad \text{for every } \varepsilon\in(0,\delta).$$

Hence

$$\log_2 k(\varepsilon, d)<p\log_2(\varepsilon^{-1}) \quad \text{for every } \varepsilon\in(0,\delta).$$

Therefore $k(d)\leqslant p$. Thus $k(d)\leqslant \dim X$.

Conversely assume that $p>k(d)$ for some metric $d\in M_{tb}(X)$. Take an r such that $p>r>k(d)$. Thus there exists a decreasing sequence of positive numbers $\{\varepsilon_n\}$ tending to zero such that

$$\log_2 k(\varepsilon_n, d)/\log_2(\varepsilon_n^{-1})<r \quad \text{for every } n\in N.$$

This implies that

$$k(\varepsilon_n, d)<\varepsilon_n^{-r} \quad \text{for every } n\in N.$$

Since $p>r$ we have

$$\lim_{\varepsilon\to 0} k(\varepsilon_n, d)\varepsilon_n^p = 0.$$

Consequently by Theorem 2-1(i) we get $\dim X\leqslant p$. Thus

$$\dim X\leqslant k(d) \quad \text{for some metric } d\in M_{tb}(X).$$

This completes the proof of Corollary 2-5.

2-6. Remark. Corollary 2-5 has been established originally by Pontrjagin and Schinirelman [11] for compact metrizable spaces. Bruijning [5] extended this result for separable metrizable spaces. The proof of Bruijning [5] is based on the Pontrjagin–Schinirelman theorem.

2-7. Remark. Let us put

$$K(d) = \limsup_{\varepsilon\to 0}\{\log_2 k(\varepsilon, d)/\log_2(\varepsilon^{-1})\}.$$

Bruijning [6] has provided an example of a metric space (X, d) for which $k(d) = 0$ whereas $K(d) = \infty$. He asked whether Corollary 2-5 still holds if $k(d)$ is replaced by $K(d)$? Is there a metric $d\in M_{tb}(X)$ for which $K(d) = \dim X$? (see [6], p. 45). The following corollary answers affirmatively his questions

2-8. COROLLARY. *For any separable metrizable space X we have*

$$\dim X = \inf\{K(d)\colon d\in M_{tb}(X)\}.$$

Moreover this infimum is attained.

Proof. Let d denote the metric obtained in Theorem 2-1(ii). Then we have

$$k(d) = K(d) = \lim_{\varepsilon\to 0}\log_2 k(\varepsilon, d)/\log_2(\varepsilon^{-1})\leqslant \dim X.$$

Therefore the result follows from Corollary 2-5.

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