Remarks on characterization of dimension of separable metrizable spaces

by

Nguyen To Nhu (Warszawa)

Abstract. We establish some characterizations of dimension of separable metrizable spaces. For instance, it is shown that a separable metrizable space $X$ is of dimension $\leq n$ if and only if $X$ is homeomorphic to a subset of the $(2n+1)$-dimensional cube $I^{2n+1}$ such that

$$\lim_{\varepsilon \to 0} k(\varepsilon, || \cdot ||) \varepsilon^p = 0 \quad \text{for} \quad p > n$$

where

$$k(\varepsilon, || \cdot ||) = \inf \{ n : \text{there exists an $\varepsilon$-net} (x_1, \ldots, x_n) \text{ for}\ S \}$$

Dimension is a topological concept, but in many cases it can be characterized by metrics or pseudometrics, [8], [10], [5]. In [12] Szpilrajn established some connections between the concept of dimension and the classical concept of Hausdorff measure. Borsuk [3] has constructed, for each $n \in N$, an $n$-dimensional pseudo-measure $\nu^n$ of compacts lying in the Hilbert space $I_n$. This concept is a topological invariant, i.e. if $\nu^n(Y) > 0$ then $\nu^n(Y') > 0$ for every compact $Y'$ homeomorphic to $Y$, [4]. Several connections between dimension and Borsuk pseudomeasure are given in [3], [4]. Since the Borsuk pseudomeasure is defined only on compacts isometrically embeddable into $I_n$, we construct in § 1 of this note a pseudomeasure for the class of all compacts similar to the Borsuk pseudomeasure. This pseudomeasure is shown to have many of the properties possessed by the Borsuk pseudomeasure. In § 2 we establish certain characterizations of dimension of separable metrizable spaces which are related to old results of Szpilrajn [12] and Pontryagin and Schinzel [11].

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§ 1. Pseudomeasure and dimension of separable metric spaces. Given a separable metrizable space $X$, let $M_d(X)$ (resp. $P_d(X)$) denote the set of all totally

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bounded compatible metrics (resp. totally bounded continuous pseudometrics) on $X$. For every $d, q \in P_0(X)$ and $p \geq 0$, put
\[ |d-q| = \sup \{d(x, y) - q(x, y) : x, y \in X\}. \]

\[ m_p(X, d) = \lim_{n \to \infty} \inf \left\{ \frac{1}{n} \sum_{i=1}^{n} d(A_i) : X = \bigcup_{i=1}^{n} A_i \right\} \text{ for } i = 1, \ldots, n, \]

\[ V_{p}(X, d) = \inf \{m_p(X, q) : |d-q| \leq \varepsilon \text{ and } q \text{ is uniformly continuous with respect to } d \}. \]

We call $V_{p}(X, d)$ the $p$-dimensional pseudomcurve of $(X, d)$.

1-1. Remark. Obviously if $(X, d)$ is a compact metric space then $m_p(X, d)$ is identical with the $p$-dimensional Hausdorff measure $m_p^H(X, d)$ of $(X, d)$ as defined e.g. in [9]. In general we have
\[ m_p(X, d) \geq m_p^H(X, d) \quad \text{ for each separable metric space } (X, d). \]

Let $l^\infty$ denote the Banach space of all bounded sequences of real numbers equipped with the supremum norm. Let $(a_n)_{n=1}^\infty$ be a dense sequence in $X$. For every $d \in P_0(X)$, put
\[ (1) \quad T_{d}(x) = \{d(x, a_n)\}_{n=1}^\infty \quad \text{ for } x \in X. \]

Obviously $T_d$ is an isometry of $X$ into $l^\infty$. Now let $X$ be a subset of $l^\infty$ and $\varepsilon > 0$. A map $f : X \to l^\infty$ is called an $\varepsilon$-push iff $f$ is a uniformly continuous map satisfying the condition $|x-f(x)| \leq \varepsilon$ for each $x \in X$.

Let us prove the following

1-2. Proposition. For every $d \in P_0(X)$ we have
\[ (2) \quad V_{d}(X, d) = \liminf_{n \to \infty} m_p(T_{d}(X), ||\cdot||) : f \text{ is an } \varepsilon \text{-push} \]

where $T$ is an arbitrary isometric embedding of $(X, d)$ into $l^\infty$ and $||\cdot||$ is the norm of $l^\infty$.

Proof. Given $d \in P_0(X)$ and an isometry $T : (X, d) \to l^\infty$. Denote
\[ V = \liminf_{n \to \infty} m_p(T(X), ||\cdot||) : f \text{ is an } \varepsilon \text{-push}. \]

If $x > \varepsilon$ then for each $\varepsilon > 0$ there is an $\varepsilon$-push $f : X \to l^\infty$ such that
\[ m_p(T(X), ||\cdot||) \leq \varepsilon. \]

Define a pseudometric $\varrho$ on $X$ by the formula
\[ \varrho(x, y) = ||T(x) - T(y)|| \quad \text{ for } x, y \in X. \]

It is easy to see that $|d-q| \leq 2\varepsilon$ and $m_p(X, q) = m_p(T(X), ||\cdot||)$. Thus
\[ V_{\varrho}(X, d) \leq \varepsilon \text{ and hence we get } (3) \quad V_{\varrho}(X, d) \leq V. \]

Conversely let $\varepsilon > V_{\varrho}(X, d)$. Then for each $\varepsilon > 0$ there exists $p \in P_0(X)$ uniformly continuous with respect to $d$ such that
\[ |d-q| \leq \varepsilon \quad \text{ and } m_p(X, q) \leq \varepsilon. \]

Let $H : T(X) \to T(X)$ be an isometry defined by the formula
\[ H T(x) = T_{\varrho}(x) \quad \text{ for each } x \in X. \]

By [1], [2] there are 1-Lipschitz maps $H' : l^\infty \to l^\infty$ such that $H'(T(X)) = H$ and $H''(T(X)) = H^{-1}$. We define $f : T(X) \to l^\infty$ by the formula
\[ f(T(x)) = H'' T(x) \quad \text{ for each } x \in X. \]

Since $H'$ is a 1-Lipschitz map we have
\[ m_p(T(X), ||\cdot||) = m_p(H'' T(X), ||\cdot||) \leq m_p(X, q) \leq \varepsilon. \]

On the other hand for each $x \in X$ we have
\[ d(f(T(x)), T(x)) = d(H'' T(x), H' H'' T(x)) \leq d(T_{\varrho}(x), H'' T(x)) = d(T_{\varrho}(x), H T(x)) \leq d(T_{\varrho}(x), T(x)) \leq 2\varepsilon. \]

Thus $f$ is an $2\varepsilon$-push. Thus we get
\[ (4) \quad V \leq V_{\varrho}(X, d). \]

From (3) and (4) we get the assertion.

Let $V_{n}(X)$ denote the $n$-dimensional Borsuk pseudomeasure of a compactum $X$ lying in the Hilbert space $l_2$ defined by the formula (see [3])
\[ V_{n}(X) \leq \liminf_{n \to \infty} m_p(Q, ||\cdot||) : f \text{ is an } \varepsilon \text{-push of } X \text{ into a polyhedron } Q = l_2. \]

From Proposition 1-2 we get

1-3. Corollary. For every compactum $X$ lying in $l_2$ we have
\[ V_{n}(X, ||\cdot||) \leq V_{n}(X) \quad \text{for every } n = 1, 2, \ldots \]

where $||\cdot||$ denotes the norm of $l_2$.

1-4. Remark. The author does not know whether $V_{n}$ and $V_{n}$ actually coincide on compacta in $l_2$. It can however be shown that they coincide on polyhedra in $l_2$ (where they coincide also with $m_n$ (cf. [3])).

Let us note that $V_{n}(X, d)$ has many of the properties possessed by $V_{n}(X)$. For instance (compare [3]),

1-5. If $\dim X < p$ then $V_{n}(X, d) = 0$ and if $\dim X > p$ then $V_{n}(X, d) = \infty$ for every $d \in M_0(X)$.

1-6. If $X$ is a continuum and $d \in M_0(X)$ then $V_{n}(X, d) \geq \text{diam}(X, d)$.

1-7. If $X$ is an arc and $d \in M_0(X)$ then
for any homeomorphism $x:[0,1] \rightarrow X$.

From Proposition 1-2 we get

(1-8) If $x \in \mathcal{T}$ and $V_f(X, ||x||)>0$ then for every $x \in V_f(X, ||x||)$ there is an $\varepsilon>0$ such that $V_f(x, ||x||)+\varepsilon>|x|$ for each $e$-push $f:X \rightarrow I^n$.

(1-9) If $f:(X, d) \rightarrow (Y, d)$ is a homeomorphism of $X$ onto $Y$ satisfying the condition $g(x, y) < Kd(x, y)$ for $x, y \in X$ then $V_f(Y, d) < K^2 V_f(X, d)$.

1-10. Remark. In (1-9) the assumption on $f$ to be a homeomorphism rather than any surjective $K$-Lipschitz map, is essential (take $Y=[0,1]$ and $X$ the graph of a surjection of a Cantor set on $[0,1]$ and $f=projection$ of $X$ onto $Y$ then $P_f(X) = 0$ and $P_f(Y) = \infty$).

1-11. Remark. In [4] it is shown that there exist compacts $X, Y$ lying in the interval $[0,1]$ such that

$V_f(X \cup Y) > V_f(X) + V_f(Y)$.

This example also yields that

$V_f(X \cup Y, ||x||) > V_f(X, ||x||) + V_f(Y, ||x||)$.

Thus $V$ is not a measure.

Let us prove the following theorem analogue of the basic result of [4].

1-12. THEOREM. Let $(X, d)$ be a compact metric space. Then dim $X \leq n$ if and only if $V_f(X, d) = 0$ for $p > n$.

Proof. Identifying $(X, d)$ with $T_d(X)$ we may assume that $X \in \mathcal{T}_n$. If dim $X \leq n$ then for each $e > 0$ there exists an $e$-push $f:X \rightarrow I^n$ such that $f(X)$ is contained in a polyhedron of dimension $\leq n$. Since $m_p$ vanishes on such a polyhedron for $p > n$ we infer that $V_f(X, d) = 0$ if dim $X \leq n$ and $p > n$.

The proof of the converse involves the following fact proved in [4].

1-13. LEMMA. Given an $m$-dimensional Banach space $E^m$ and $\varepsilon > 0$. Then there exists an $\varepsilon$-net on $X$, such that for every compact $Y \subseteq E^m$ with $m_{\varepsilon+1}(Y, ||y||) < \delta$ there exists an $e$-push $g:Y \rightarrow E^m$ such that dim $g(Y) < \varepsilon$.

Proof. Since $E^m$ is isomorphic to $R^m$, it suffices to consider the case $E^m = R^m$, and this is done in [4].

Using Lemma 1-13 we are able to complete the proof of Theorem 1-12.

Assume that $V_{\varepsilon+1}(X, d) = 0$. We have to show that dim $X \leq n$.

For each $m \in N$, put

$I^m = \{x = (x_i) \in I^m: x_i = 0 \ for \ i > m\}

and let $P_m:I^m \rightarrow I^m$ denote the natural projection. Since $T_d(X)$ is a compact in $I^m$,

for every $\varepsilon > 0$ there exists an $m \in N$ such that

$\|x - P_m(x)\| < \varepsilon$ for every $x \in T_d(X)$.

Take $\delta = \delta(e, m)$ with $E^m = R^m$ from Lemma 1-13. Since $V_{\varepsilon+1}(X, d) = 0$ there exists $e$-push $f: T_d(X) \rightarrow I^m$ such that $m_{\varepsilon+1}(f(T_d(X), ||x||) < \delta$. Whence $P_m f$ is an $2e$-push. Since

$m_{\varepsilon+1}(P_m f(T_d(X), ||x||) < \delta m_{\varepsilon+1}(f(T_d(X), ||x||) < \delta

by Lemma 1-13 there exists an $e$-push $g: P_m f(T_d(X) \rightarrow I^m$ such that dim $g P_m f(T_d(X) \leq n$. Since $T_d$ is an isometry and $g P_m f$ is an $3e$-push it follows that dim $X \leq n$.

This completes the proof of Theorem 1-12.

1-14. COROLLARY. A separable metrizable space $X$ is of dimension $\leq n$ if and only if for every $d \in M_n(X)$ and $p > n$ we have $V_f(X, d) = 0$.

Proof. Let dim $X \leq n$ and $d \in M_n(X)$. We may consider $(X, d)$ as a totally bounded subset of $I^m$. Then for each $e > 0$ there exists an $e$-push $f: X \rightarrow I^m$ such that $f(X)$ is contained in a polyhedron of dimension $\leq n$. Thus $V_f(X, d) = 0$ for $p > n$.

Conversely assume that $V_{\varepsilon+1}(X, d) = 0$ for every $d \in M_n(X)$. By [7] there exists a $\delta \in M_n(X)$ such that $dim X = dim X$, where $\delta$ denotes the completion of $X$ with respect to the metric $\delta$. Since $V_{\varepsilon+1}(X, \delta) = 0$ we infer that $V_{\varepsilon+1}(\bar{X}, \delta) = 0$. Thus by Theorem 1-12 we have dim $X = dim \bar{X}$.

1-15. COROLLARY (cf. Szpilrajn [12]). If $X$ is an $n$-dimensional separable metrizable space then for each $d \in M_n(X)$ we have $m_n(X, d) > 0$.

Proof. Let $(\bar{X}, \delta)$ denote the completion of $X$ with respect to the metric $\delta$. Since

$m_n(X, \delta) = m_n(\bar{X}, \delta) > V_f(\bar{X}, \delta)$ for every $d \in M_n(X)$

the assertion follows from Theorem 1-12.

$\S$ 2. A metric characterization of dimension of separable metrizable spaces. Let $X$ be a separable metrizable space. For every $d \in M_n(X)$ and $e > 0$, put

$k(e, d) = \inf\{r: \text{there exists an } e\text{-net } \{x_1, \ldots, x_r\} \text{ for } (X, d)\}$.

In this section we prove the following

2-1. THEOREM. Let $X$ be a separable metrizable space. Then

(i) If dim $X > n$ then for every $d \in M_n(X)$ we have

$\liminf_{r \to \infty} k(e, d)^r = 0$.

(ii) If dim $X \leq n$ then there exists a $d \in M_n(X)$ such that

$\lim_{r \to \infty} k(e, d)^r = 0$ for

$p > n$. 


Proof. (i) Assume that \( \dim X \geq n \). By Corollary 1-15 for every \( d \in M_\nu(X) \) we have \( \gamma(d) = m_\nu(X, d) > 0 \). Thus for each \( x \in (0, \gamma(d)) \) there is a \( \delta > 0 \) such that for every \( x \in (0, \delta) \) we have

\[
\inf \left\{ \sum_{i=1}^{m} d(A_i) : X = \bigcup_{i=1}^{m} A_i, d(A_i) \leq \varepsilon \right\} > \varepsilon.
\]

Hence

\[
k(x, d, e) \geq n \quad \text{for every } x \in (0, \delta).
\]

This proves (i).

(ii) Let \( M_{2n+1} \) be the \( n \)-dimensional Menger universal space constructed as follows (see Engelking [7], p. 121):

- For every \( i = 0, 1, \ldots \) divide the interval \([0, 1]\) into \( 3^{-i+1} \) equal intervals.
- One gets a subdivision of the cube \( I^{2n+1} \) into \( 3^{2n+1} \) equal small cubes with the length of the edges \( 3^{-(b+1)/2} \).

Let \( \mathcal{X}_i \) denote the family of all such cubes. For every family \( \mathcal{X} \) of cubes, put

\[
|\mathcal{X}| = \bigcup \{ Q : Q \in \mathcal{X} \}, \quad \mathcal{F}(\mathcal{X}) = \bigcup \{ F(Q) : Q \in \mathcal{X} \}.
\]

where \( F(Q) \) denotes the family of all faces of \( Q \) which have dimension \( \leq n \). Moreover, for each \( \mathcal{X} \in \mathcal{X}_i \), put

\[
\mathcal{X}' = \{ Q \in \mathcal{X}_{i+1} : Q \subset |\mathcal{X}| \}.
\]

Let

\[
\mathcal{F}_0 = \{ I^{2n+1} \}, \quad F_0 = |\mathcal{F}_0| = I^{2n+1}
\]

and for every \( i = 1, 2, \ldots \) define \( \mathcal{F}_i \) and \( F_i \) by induction

\[
\mathcal{F}_i = \{ Q \in \mathcal{F}_{i-1} : Q \cap |\mathcal{F}_{i-1}| \neq \emptyset \}, \quad F_i = |\mathcal{F}_i|.
\]

Then \( \{ \mathcal{F}_i = 0, 1, \ldots \} \) is a sequence of finite collections of cubes, \( \mathcal{F}_i \subset \mathcal{X}_i \) for every \( i = 0, 1, \ldots \) and \( F_0 \supset F_1 \supset F_2 \supset \ldots \) is a decreasing sequence of closed subsets of \( I^{2n+1} \). We define \( M_{2n+1} \) by the formula

\[
M_{2n+1} = \bigcap_{i=0}^{\infty} F_i \subset I^{2n+1}.
\]

By the universal space theorem [7] there exists an embedding of \( X \) into \( M_{2n+1} \).

Thus it suffices to prove the theorem for \( X = M_{2n+1} \) and \( n \) is the metric of \( M_{2n+1} \) induced by the norm of the \((2n+1)\)-dimensional Euclidean space \( R^{2n+1} \).

Let us note that each cube \( Q \) with the length of edges \( 3^{-(b+1)/2} \) contains at most \( A^{2n+1} \) cubes with the length of edges \( 3^{-b+1} \) which intersect the \( n \)-dimensional faces of \( Q \), where \( A \) is the number of the \( n \)-dimensional faces of \( Q \).

There are at most

\[
A^{2n+1} = A^{2n+1} \cdot 3^{-(b+1)/2} = A^{2n+1} \cdot 3^{-(b+1)/2}
\]

cubes with the length of edges \( 3^{-b+1} \) intersecting \( M_{2n+1} \). Since these cubes form a cover of \( M_{2n+1} \), for every \( \varepsilon > 0 \), say

\[
\varepsilon \in \{ (2n+1)^{1/2}, (2n+1)^{1/2} + 2, (2n+1)^{1/2} + 3, (2n+1)^{1/2} + 4 \}
\]

we have

\[
k(x, d, e) \leq A^{2n+1} \cdot 3^{-(b+1)/2} = A^{2n+1} \cdot 3^{-(b+1)/2}.
\]

Since \( p > n \) we infer that

\[
\lim_{i \to \infty} k(x, d, e) = \lim_{i \to \infty} (2n+1)^{1/2} A^{2n+1} 3^{-(b+1)/2} = 0.
\]

This completes the proof of Theorem 2-1.

In [12] Zsigriaj has shown that a separable metrizable space \( X \) is of dimension \( \leq n \) if and only if \( X \) is homeomorphic to a subset \( S \) of \( I^{2n+1} \) with \( m_\nu(S) = 0 \) for every \( p > n \). Let us note that the proof of Theorem 2-1 gives the following

2-2. COROLLARY. A separable metrizable space \( X \) is of dimension \( \leq n \) if and only if \( X \) is homeomorphic to a subset \( S \) of the cube \( I^{2n+1} \) such that

\[
lm(x, |||d|||) = 0 \quad \text{for every } p \geq n
\]

where \( |||| \) denotes the norm of the Euclidean cube \( R^{2n+1} \).

2-3. Remark. Obviously if \( k(x, d, e) = 0 \) then \( m_\nu(X, d) = 0 \).

The following example shows that the converse does not hold true even for compact metric spaces.

2-4. EXAMPLE. For every integer \( n \in \mathbb{N} \) there exists a compact metric space \( (X, d) \) such that \( m_\nu(X, d) = 0 \) for every \( p > 0 \) and

\[
lm(x, |||d|||) = \infty \quad \text{for } p < n.
\]

Proof. In the Euclidean space \( \mathbb{R}^n \) consider the set \( X = A^2 \), where

\[
A = \left\{ 0, \ldots, \frac{1}{2^n} \right\}
\]

Since \( X \) is a countable compact set we have \( m_\nu(X, |||d|||) = 0 \) for every \( p > 0 \).

Now let \( p < n \). For every \( x > 0 \) take \( k \in \mathbb{N} \) such that

\[
\frac{1}{\ln(k+1)} - \frac{1}{\ln(k+2)} < x < \frac{1}{\ln(k)} - \frac{1}{\ln(k+1)}.
\]

Then we have

\[
k(x, |||d|||) = \left( \frac{\ln(1 + \frac{1}{k+1})}{\ln(k+1)(\ln(k+2))} \right)^p = \left( \frac{k^p}{(k+1)^p} \right)^{\frac{1}{\ln(k+1)(\ln(k+2))}}.
\]
Since \( p < n \) we infer that
\[
\lim_{s \to 0} k(s, \| \cdot \|) s^p = 0.
\]
From Theorem 2-1 we get also

2-5. Corollary (Pontryagin–Schnirelman [11], Brujinjiv [5]). For every separable metrizable space \( X \) we have
\[
\dim X = \inf \{ k(d) : d \in M_0(X) \}
\]
where
\[
k(d) = \liminf_{s \to 0}[\log k(s, d) / \log(\epsilon^{-1})].
\]

Proof. Assume that \( p > \dim X \). By Theorem 2-1 there exists a metric \( d \in M_0(X) \) such that \( \lim_{s \to 0} k(s, d) s^p = 0 \). Thus there exists an \( \delta > 0 \) such that
\[
k(s, d) s^p < 1 \quad \text{for every } s \in (0, \delta).
\]
Hence
\[
\log k(s, d) < p \log(\epsilon^{-1}) \quad \text{for every } s \in (0, \delta).
\]
Therefore \( k(d) \not\in \dim X \). Thus \( k(d) \not\in \dim X \).

Conversely assume that \( p > k(d) \) for some metric \( d \in M_0(X) \). Take an \( r \) such that \( p > r > k(d) \). Thus there exists a decreasing sequence of positive numbers \( \{s_n\} \) tending to zero such that
\[
\log k(s_n, d) / \log(\epsilon^{-1}) < r \quad \text{for every } n \in \mathbb{N}.
\]
This implies that
\[
k(s_n, d) < \epsilon^{-r} \quad \text{for every } n \in \mathbb{N}.
\]
Since \( p > r \) we have
\[
\lim_{s \to 0} k(s_n, d) s^p = 0.
\]
Consequently by Theorem 2-1(i) we get \( \dim X \leq p \). Thus
\[
\dim X \leq k(d) \quad \text{for some metric } d \in M_0(X).
\]
This completes the proof of Corollary 2-5.


2-7. Remark. Let us put
\[
K(d) = \limsup_{s \to 0} \{ \log k(s, d) / \log(\epsilon^{-1}) \}.
\]
Brujinjiv [6] has provided an example of a metric space \( (X, d) \) for which \( k(d) = 0 \) whereas \( K(d) = \infty \). He asked whether Corollary 2-5 still holds if \( k(d) \)

is replaced by \( K(d) \)? Is there a metric \( d \in M_0(X) \) for which \( K(d) = \dim X \)? (see [6], p. 45). The following corollary answers affirmatively his questions.

2-8. Corollary. For any separable metrizable space \( X \) we have
\[
\dim X = \inf \{ K(d) : d \in M_0(X) \}.
\]
Moreover this infimum is attained.

Proof. Let \( d \) denote the metric obtained in Theorem 2-1(ii). Then we have
\[
k(d) = K(d) = \lim_{s \to 0} k(s, d) / \log(\epsilon^{-1}) \leq \dim X.
\]
Therefore the result follows from Corollary 2-5.

References


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