On Σ-products of Σ-spaces

by

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Dedicated to
the late Mr. Mitsuo Morishita

Abstract. In the present paper, we prove the following result: Let Σ be a Σ-product of paracompact Σ-spaces. If Σ has countable tightness, then it is collectionwise normal.

1. Introduction. First, H. H. Corson [3] introduced the concept of Σ-products, which are quite important subspaces of product spaces, and began to study the normality of them. Subsequently, A. P. Kombarov [6] proved

(A) A Σ-product of Čech-complete paracompact spaces is normal if it has countable tightness.

An affirmative answer of the Corson's problem in [3], which had been unanswered for a long time, were given by S. P. Gul'ko [5] and M. E. Rudin [15] independently. That is, they proved

(B) A Σ-product of metric spaces is normal.

After that, Kombarov [8] obtained a nice generalization of (A) and (B) as follows:

(C) A Σ-product of paracompact p-spaces is (collectionwise) normal if and only if it has countable tightness.

So it seems to be natural from the results (A), (B) and (C) to ask

(Q) Is a Σ-product of paracompact σ-spaces normal if it has countable tightness?

Such a question has not been answered for even other generalized metric spaces such as stratifiable ones etc.

On the other hand, K. Nagami [10] introduced and studied the concept of Σ-spaces whose class includes both ones of σ-spaces and paracompact p-spaces. In particular, the class of paracompact Σ-spaces is a fairly broad one which is countable productive, and it is well known that this class plays an important role in product theory.

The main purpose of this paper is to give a further generalization of the "if" part of (C) and affirmative answer to (Q) simultaneously, considering Σ-products of paracompact Σ-spaces. Next, we also study the dimension of such Σ-products. Consequently, we obtain a generalization of the results of E. Pol [14] for the dimension of infinite products.

Since Σ-products and Σ-spaces have not been studied together but separately all this while, Σ-products have been occasionally called Σ-spaces without confusion (cf. [16] etc.). However, this is not appropriate in this paper.

Our terminology follows [4] unless otherwise stated. All spaces considered here are assumed to be Hausdorff. The set of all natural numbers is denoted by N and natural numbers are denoted by i, j, k, m and n. Let $\mathfrak{m}$ be an infinite cardinal.

2. Definitions and theorems. Recall the following definitions which are quite well known.

Let $X = \prod_{\lambda\in\Lambda} X_\lambda$ be an infinite product space and take a point $s = \{s_\lambda\} \in X$. Consider the dense subspace Σ ($\Sigma_{\mathfrak{m}}$) of X consisting of all points $x = \{x_\lambda\} \in X$ for which the set $\mathrm{Supp}(x) = \{\lambda \in \Lambda |\ x_\lambda \neq s_\lambda\}$ is at most countable ($|\mathrm{Supp}(x)| \leqslant \mathfrak{m}$). This subspace Σ ($\Sigma_{\mathfrak{m}}$) of X is called a *Σ-product* [3] (*$\Sigma_{\mathfrak{m}}$-product* [7]) of spaces X_λ, $\lambda \in \Lambda$, and such a point $s \in X$ is called the *base point* of Σ ($\Sigma_{\mathfrak{m}}$).

A space X is called a *Σ-space* [10] if there exists a sequence $\{\mathscr{F}_n\}_{n=1}^\infty$ of locally finite closed covers of X satisfying the following condition:

If $\{K_n\}_{n=1}^\infty$ is a decreasing sequence of non-empty closed sets in X such that $K_n \subset \bigcap \{F |\ x \in F \in \mathscr{F}_n\}$ for each $n \in N$ and some $x \in X$, then $\bigcap_{n=1}^\infty K_n \neq \emptyset$.

A space X is called to have *countable tightness* (*tightness not exceeding* $\mathfrak{m}$) [1] if for any subset Y of X and any point x of $\overline{Y}$ there exists an at most countable set M (a set M with $|M| \leqslant \mathfrak{m}$) such that $M \subset Y$ and $x \in \overline{M}$.

Now, we state three theorems as follows:

THEOREM 1. *Let Σ be a Σ-product of paracompact Σ-spaces. If Σ has countable tightness, then it is collectionwise normal.*

THEOREM 2. *Let $\Sigma_{\mathfrak{m}}$ be a $\Sigma_{\mathfrak{m}}$-product of paracompact Σ-spaces X_λ, $\lambda \in \Lambda$. If $\Sigma_{\mathfrak{m}}$ has tightness not exceeding $\mathfrak{m}$, then the following are equivalent.*

(a) *$\prod_{\lambda\in\Gamma} X_\lambda$ is normal for each $\Gamma \subset \Lambda$ with $|\Gamma| \leqslant \mathfrak{m}$.*

(b) *$\Sigma_{\mathfrak{m}}$ is (collectionwise) normal.*

THEOREM 3. *Let Σ be a Σ-product of paracompact Σ-spaces X_λ, $\lambda \in \Lambda$, such that $\dim(X_{\lambda_1} \times \ldots \times X_{\lambda_k}) \leqslant n$ for each $\lambda_1, \ldots, \lambda_k \in \Lambda$. If Σ has countable tightness, then $\dim \Sigma \leqslant n$.*

Remark. Theorem 2 is a slight generalization of Theorem 1 and [7, Theorems 1 and 2]. It should be noted by [9, Proposition 1] that the condition of Σ having countable tightness can be replaced by the following one: Each finite product of factors of Σ has countable tightness. Of course, Theorem 1 is not true without this condition (cf. [6]).

3. Corollaries. We state some corollaries which follow from Theorems 1 and 3.

COROLLARY 1. *A Σ-product of paracompact first-countable Σ-spaces is collectionwise normal.*

This is an immediate consequence of Theorem 1 and Remark.

COROLLARY 2. *If $\{X_\lambda\}_{\lambda\in\Lambda}$ is a collection of paracompact first-countable Σ-spaces such that $\dim(X_{\lambda_1} \times \ldots \times X_{\lambda_k}) \leqslant n$ for each $\lambda_1, \ldots, \lambda_k \in \Lambda$, then $\dim \prod_{\lambda\in\Lambda} X_\lambda \leqslant n$.*

Proof. Let Σ be a Σ-product of the spaces X_λ, $\lambda \in \Lambda$. By Theorem 3, we have $\dim \Sigma \leqslant n$. Since Σ is C-embedded in $\prod_{\lambda\in\Lambda} X_\lambda$ (cf. [16, Theorem 2.2]), $\dim \Sigma \leqslant n$ implies $\dim \prod_{\lambda\in\Lambda} X_\lambda \leqslant n$ (cf. [4, Corollary 7.1.8]).

COROLLARY 3. *If $\{X_\lambda\}_{\lambda\in\Lambda}$ is a collection of paracompact first-countable Σ-spaces such that $\dim X_\lambda = 0$ for each $\lambda \in \Lambda$, then $\dim \prod_{\lambda\in\Lambda} X_\lambda = 0$.*

Proof. Since a finite product of paracompact Σ-spaces is rectangular (cf. [13, Proposition 1]), $\dim(X_{\lambda_1} \times \ldots \times X_{\lambda_k}) = 0$ for each $\lambda_1, \ldots, \lambda_k \in \Lambda$ (cf. [13, Theorem 1]). Hence this is a special case of Corollary 2.

Our Corollaries 2 and 3 are generalizations of the results of E. Pol [14].

4. Proofs of Theorems. First, we prepare some lemmas and notations for the proof of Theorem 1.

LEMMA 1. *Let X be a Σ-space. Then there exists a sequence $\{\mathscr{F}_n\}_{n=1}^\infty$ of locally finite closed covers of X, satisfying the following conditions:*

(1) *$\mathscr{F}_n = \{F(\alpha_1 \ldots \alpha_n) |\ \alpha_1, \ldots, \alpha_n \in \Omega\}$ for each $n \in N$.*

(2) *Each $F(\alpha_1 \ldots \alpha_n)$ is the sum of all $F(\alpha_1 \ldots \alpha_n \alpha_{n+1})$, $\alpha_{n+1} \in \Omega$.*

(3) *For each $x \in X$ there exists a sequence $\alpha_1, \alpha_2, \ldots \in \Omega$, satisfying*

(i) *$\bigcap_{n=1}^\infty F(\alpha_1 \ldots \alpha_n)$ contains x,*

(ii) *if $\{K_n\}_{n=1}^\infty$ is a decreasing sequence of non-empty closed sets in X such that $K_n \subset F(\alpha_1 \ldots \alpha_n)$ for each $n \in N$, then $\bigcap_{n=1}^\infty K_n \neq \emptyset$.*

Lemma 1 is due to Nagami [10] and the sequence $\{\mathscr{F}_n\}_{n=1}^\infty$ is called a *spectral Σ-net* of X. Moreover, we say that the sequence $\{F(\alpha_1 \ldots \alpha_n)\}_{n=1}^\infty$ in (3) of Lemma 1 is a *local Σ-net* of x. Note that the intersection $\bigcap_{n=1}^\infty K_n$ is countably compact.

For a space X, let $\mathscr{D}$ be a collection of subsets of X and M a subset of X. By $\mathscr{D}|M$ we denote $\{D \cap M |\ D \in \mathscr{D}\}$. We say that $\mathscr{D}$ is *non-discrete* at $x \in X$ if each open neighborhood of x intersects at least two members of $\mathscr{D}$. For a continuous map p of X, $p(\mathscr{D})$ denotes $\{p(D) |\ D \in \mathscr{D}\}$.

LEMMA 2. *Let X be a space which has countable tightness. Let $\mathcal{D}$ be a collection of subsets of a space Y. Let p be a continuous map of Y into X. If $p(\mathcal{D})$ is non-discrete at $x \in X$, then there exists a countable subset M of $\bigcup\mathcal{D}$ such that $p(\mathcal{D}|M)$ is non-discrete at x.*

Proof. In the case of $x \in \bigcup\{\overline{p(D)}|\ D \in \mathcal{D}\}$: Take some $D_0 \in \mathcal{D}$ such that $x \in \overline{p(D_0)}$. Note $x \in (\bigcup\{p(D)|\ D \in \mathcal{D}$ and $D \neq D_0\})^-$. So we can choose two countable sets M_0 and M_1 such that $M_0 \subset D_0$, $M_1 \subset \bigcup\{D \in \mathcal{D}|\ D \neq D_0\}$ and $x \in \overline{p(M_0)} \cap \overline{p(M_1)}$. Put $M = M_0 \cup M_1$. In the case of $x \notin \bigcup\{\overline{p(D)}|\ D \in \mathcal{D}\}$: Since $x \in (\bigcup\{\overline{p(D)}|\ D \in \mathcal{D}\})^-$, we can choose a countable set M such that $M \subset \bigcup\mathcal{D}$ and $x \in \overline{p(M)}$. In any cases, one can easily verify that the set M is a desired one. The proof is complete.

Notation for Σ (cf. [8]): Let Σ be a Σ-product of spaces X_λ, $\lambda \in \Lambda$. Let Ξ be an index set such that for each $\xi \in \Xi$ R_ξ is a countable subset of Λ. Then $X_\xi = \prod_{\lambda \in R_\xi} X_\lambda$ and p_ξ is the projection of Σ onto X_ξ for each $\xi \in \Xi$. Moreover, p^ξ_μ is the projection of X_ξ onto X_μ for each ξ, $\mu \in \Xi$ with $R_\mu \subset R_\xi$.

Notation for a $n \times n$ matrix $\xi = (\alpha_{ij})_{i,j \leqslant n}$: By ξ_k we denote the $k \times k$ matrix $(\alpha_{ij})_{i,j \leqslant k}$ for $1 \leqslant k \leqslant n$. In particular, ξ_{n-1} is often abbreviated by ξ_- and ξ_0 denotes the empty $(\emptyset)$.

Proof of Theorem 1. Let Σ be a Σ-product of paracompact Σ-spaces X_λ, $\lambda \in \Lambda$, with a base point $s = \{s_\lambda\} \in \Sigma$. Let $\mathcal{D}$ be a discrete collection of closed sets in Σ. Let $\Xi_0 = \{\xi_0\}$, where $\xi_0 = (\emptyset)$, and take an arbitrary non-empty countable subset R_{ξ_0} of Λ.

Now, for each $n \in N$ we construct a collection $\mathcal{G}_n$ of open sets in Σ and an index set Ξ_n of $n \times n$ matrices such that for each $\xi \in \Xi_n$ R_ξ, $\Omega(\xi)$, $E(\xi)$, $H(\xi)$, x_ξ and M_ξ are given, satisfying the following conditions (1)–(7):

(1) Each $\mathcal{G}_n$ is locally finite in Σ such that for each $G \in \mathcal{G}_n$ $\overline{G}$ intersects at most one member of $\mathcal{D}$.

(2) For each $\xi \in \Xi_n$, R_ξ is a countable subset of Λ such that $R_{\xi_-} \subset R_\xi$.

(3) For each $\xi \in \Xi_n$, $\{F(\alpha_1 \dots \alpha_k)|\ \alpha_1, \dots, \alpha_k \in \Omega(\xi)\}$, $k \in N$, is a spectral Σ-net of X_ξ.

(4) For each $\xi = (\alpha_{ij})_{i,j \leqslant n} \in \Xi_n$ and $1 \leqslant i \leqslant n$, we have

$$\alpha_{i1}, \dots, \alpha_{in} \in \Omega(\xi_{i-1}) \quad \text{and so} \quad E(\xi) = \bigcap_{i=1}^{n} (p^{\xi_-}_{\xi_{i-1}})^{-1}(F(\alpha_{i1} \dots \alpha_{in})).$$

(5) $\{p_{\xi_-}^{-1}(H(\xi))|\ \xi \in \Xi_n\}$ is locally finite in Σ such that $H(\xi)$ is open in X_{ξ_-} and contains $E(\xi)$ for each $\xi \in \Xi_n$.

(6) Let $\mu = (\alpha_{ij})_{i,j \leqslant n-1} \in \Xi_{n-1}$, $\alpha_{nj} \in \Omega(\mu)$ and $\alpha_{in} \in \Omega(\mu_{i-1})$ for $1 \leqslant i,\ j \leqslant n$. Then

$$\bigcap_{i=1}^{n} p_{\mu_{i-1}}^{-1}(F(\alpha_{i1} \dots \alpha_{in})) \cap (\Sigma \setminus \bigcup \mathcal{G}_n) \neq \emptyset$$

implies $\xi = (\alpha_{ij})_{i,j \leqslant n} \in \Xi_n$.

(7) For each $\xi \in \Xi_n$, x_ξ is a point of $E(\xi)$ and M_ξ is a countable subset of $\bigcup\mathcal{D}$ such that $p_{\xi_-}(\mathcal{D}|M_\xi)$ is non-discrete at x_ξ and $\bigcup\{\mathrm{Supp}(x)|\ x \in M_\xi\} \subset R_\xi$.

Assume that the above construction has been already performed for no greater than n. Take a $\xi \in \Xi_n$. Set

$$\Phi(\xi) = \{x \in X_\xi|\ p_\xi(\mathcal{D}) \text{ is non-discrete at } x\}$$

and

$$\Xi(\xi) = \{(\alpha_{ij})_{i,j \leqslant n+1}|\ \xi = (\alpha_{ij})_{i,j \leqslant n},\ \alpha_{i,n+1} \in \Omega(\xi_{i-1}) \text{ and } \alpha_{n+1,j} \in \Omega(\xi) \text{ for } 1 \leqslant i \leqslant n \text{ and } 1 \leqslant j \leqslant n+1\}.$$

For each $\eta = (\alpha_{ij})_{i,j \leqslant n+1} \in \Xi(\xi)$, we can set

$$E(\eta) = \bigcap_{i=0}^{n} (p^\xi_{\xi_i})^{-1}(F(\alpha_{i+1,1} \dots \alpha_{i+1,n+1})).$$

Moreover, we set

$$\Xi_+(\xi) = \{\eta \in \Xi(\xi)|\ E(\eta) \cap \Phi(\xi) = \emptyset\} \quad \text{and} \quad \Xi_-(\xi) = \Xi(\xi) \setminus \Xi_+(\xi).$$

Since X_ξ is paracompact (cf. [10, Theorem 3.13]) and $\{E(\eta)|\ \eta \in \Xi(\xi)\}$ is a locally finite collection of closed sets in X_ξ such that $E(\eta) \subset (p^\xi_{\xi_-})^{-1}(H(\xi))$ for each $\eta \in \Xi(\xi)$, there exists a locally finite collection $\mathcal{G}(\xi)$ of open sets in Σ such that

(i) $G = p_\xi^{-1}p_\xi(G) \subset p_{\xi_-}^{-1}(H(\xi))$ for each $G \in \mathcal{G}(\xi)$,

(ii) $\overline{G}$ intersects at most one member of $\mathcal{D}$ for each $G \in \mathcal{G}(\xi)$,

(iii) $p_\xi(\mathcal{G}(\xi))$ covers $\bigcup\{E(\eta)|\ \eta \in \Xi_+(\xi)\}$.

Moreover, there exists a locally finite collection $\{H(\eta)|\ \eta \in \Xi_-(\xi)\}$ of open sets in X_ξ such that $E(\eta) \subset H(\eta) \subset (p^\xi_{\xi_-})^{-1}(H(\xi))$ for each $\eta \in \Xi_-(\xi)$. Here, running $\xi \in \Xi_n$, we set

$$\mathcal{G}_{n+1} = \bigcup\{\mathcal{G}(\xi)|\ \xi \in \Xi_n\} \quad \text{and} \quad \Xi_{n+1} = \bigcup\{\Xi_-(\xi)|\ \xi \in \Xi_n\}.$$

Then it follows from the inductive assumption (5) that $\mathcal{G}_{n+1}$ and $\{p_{\eta_-}^{-1}(H(\eta))|\ \eta \in \Xi_{n+1}\}$ is locally finite in Σ. For each $\eta \in \Xi_{n+1}$ with $\eta_- = \xi$, we can choose some $x_\eta \in E(\eta) \cap \Phi(\xi)$. Since X_ξ has countable tightness, it follows from Lemma 2 that there exists a countable subset M_η of $\bigcup\mathcal{D}$ such that $p_\xi(\mathcal{D}|M_\eta)$ is non-discrete at x_η. Set $R_\eta = \bigcup\{\mathrm{Supp}(x)|\ x \in M_\eta\} \cup R_\xi$. Since each X_η is a Σ-space (cf. [10, Theorem 3.13]), it follows from Lemma 1 that there exists a spectral Σ-net

$$\{F(\alpha_1 \dots \alpha_k)|\ \alpha_1, \dots, \alpha_k \in \Omega(\eta)\},\ k \in N,$$

of X_η for each $\eta \in \Xi_{n+1}$. This construction for $n+1$ satisfies all the conditions (1)–(7). Here, we check only (6). Pick any $\xi \in \Xi_n$ and $\eta \in \Xi(\xi)$. By the above (i) and (iii), $\eta \in \Xi_+(\xi)$ implies

$$p_\xi^{-1}(E(\eta)) \subset p_\xi^{-1}(\bigcup p_\xi(\mathcal{G}(\xi))) = \bigcup\mathcal{G}(\xi) \subset \bigcup\mathcal{G}_{n+1}.$$

Hence, if $p_{\xi}^{-1}(E(\eta))\setminus\bigcup\mathscr{G}_{n+1} \neq \emptyset$, then $\eta \in \Xi_{-}(\xi)\subset\Xi_{n+1}$. For the first step of induction, we can construct $\mathscr{G}_1$, Ξ_1 and others from R_{ξ_0} and $\Omega(\xi_0)$, in the same way as the above. Thus, we have inductively accomplished the desired construction.

Set $\mathscr{G} = \bigcup_{n=1}^{\infty} \mathscr{G}_n$. Now, assume $\mathscr{G}$ is not a cover of Σ. Pick a point $y \in \Sigma\setminus\bigcup\mathscr{G}$. By (3) and (6), we can inductively choose a sequence $(\alpha_{ij})_{i,j=1,2,\ldots}$ such that for each $n \in N$ $\xi^n = (\alpha_{ij})_{i,j\leqslant n} \in \Xi_n$ and $\{F(\alpha_{n1}\ldots\alpha_{nk})|\ k \in N\}$ is a local Σ-net of $p_{\xi^{n-1}}(y)$ in $X_{\xi^{n-1}}$. Since $\xi^n_- = \xi^{n-1}$ for each $n \in N$, by (2), we have $R_{\xi^n_-}\subset R_{\xi^k_-}$ for each n, $k \in N$ with $n\leqslant k$. For each $m\geqslant n$, we set $L_{nm} = \{p^{\xi^m}_{\xi^n_-}(x_{\xi^m}), p^{\xi^{m+1}}_{\xi^n_-}(x_{\xi^{m+1}})\ldots\}$. Then $\{\bar{L}_{nm}\}_{m=n}^{\infty}$ is a decreasing sequence of non-empty closed sets in $X_{\xi^n_-}$. Moreover, by (4), we have $\bar{L}_{nm}\subset F(\alpha_{n1}\ldots\alpha_{nm})$ for each $m\geqslant n$. It follows from the choice of $F(\alpha_{n1}\ldots\alpha_{nm})$ that $K_n = \bigcap_{m=n}^{\infty}\bar{L}_{nm}$ is a non-empty compact subset of $X_{\xi^n_-}$. Since $p^{\xi^n}_{\xi^n_-}(K_{n+1})\subset K_n$, we can choose some $z_n \in K_n$ such that $p^{\xi^n}_{\xi^n_-}(z_{n+1}) = z_n$ for each $n \in N$. Set $R_\infty = \bigcup_{n=1}^{\infty} R_{\xi^n}$. Then we can pick the point $x_\infty = \{x_\lambda\} \in \Sigma$ defined by $p_{\xi^n_-}(x_\infty) = z_n$ for each $n \in N$ and $x_\lambda = s_\lambda$ for each $\lambda \in \Lambda\setminus R_\infty$. By (7), it is easily seen, in the same way as the proof of [6, Theorem 1], that the collection $\mathscr{D}$ is non-discrete at x_∞. This is a contradiction. Hence, by (1), $\mathscr{G}$ is a σ-locally finite open cover of Σ such that for each $G \in \mathscr{G}$ $\bar{G}$ intersects at most one member of $\mathscr{D}$. This implies that Σ is collectionwise normal. The proof of Theorem 1 is complete.

Under the assumption (a) of Theorem 2, it follows from [11, Theorem 2.7] that $\prod_{\lambda\in\Gamma} X_\lambda$ is a paracompact Σ-space for each $\Gamma\subset\Lambda$ with $|\Gamma|\leqslant\mathfrak{m}$. So the proof of (a) $\rightarrow$ (b) in Theorem 2 is quite parallel to that of Theorem 1. The detail is left to the reader. The converse of it is clear.

Next, we show Theorem 3. The proof of it is essentially due to the idea in [14].

LEMMA 3. *Let X be a subparacompact Σ-space. Then there exists a sequence $\{\mathscr{F}_n\}_{n=1}^{\infty}$ of σ-discrete closed covers of X, satisfying the same conditions as* (1)–(3) *of Lemma* 1.

Proof. The space X has a Σ-net $\{\mathscr{F}''_n\}_{n=1}^{\infty}$ such that each $\mathscr{F}''_n$ is finitely multiplicative (cf. [10]). Since X is subparacompact and each $\mathscr{F}''_n$ is locally finite in X, for each $n \in N$ there exists a σ-discrete closed cover $\mathscr{E}_n$ of X such that each member of it intersects at most finitely many members of $\mathscr{F}''_n$. Here, we put $\mathscr{F}'_n = \{E\cap F|\ E \in \mathscr{E}_n$ and $F \in \mathscr{F}''_n\}$ for each $n \in N$. Then each $\mathscr{F}'_n$ is a σ-discrete closed cover of X. So we can obtain a desired sequence $\{\mathscr{F}_n\}_{n=1}^{\infty}$ in the same way as in the proof of [10, Lemma 1.4].

The following lemma is obtained by modifying the proof of Theorem 1, where Lemma 3 is used instead of Lemma 1.

LEMMA 4. *Let Σ be a Σ-product of paracompact Σ-spaces X_λ, $\lambda \in \Lambda$, such that it has countable tightness. If $(A_1, B_1), \ldots, (A_k, B_k)$ are pairs of disjoint closed sets in Σ, then there exists a σ-discrete cover $\mathscr{G}$ of Σ, satisfying for each $G \in \mathscr{G}$*

(1) *there exists a countable subset R of Λ such that $p_R(G)$ is functionally open in X_R and $p_R^{-1}p_R(G) = G$, where $X_R = \prod_{\lambda\in R} X_\lambda$ and p_R is the projection of Σ onto X_R,*

(2) *G is disjoint from A_j or B_j for $1\leqslant j\leqslant k$.*

Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a collection of spaces. Let V_λ be a functionally open in X_λ for each $\lambda \in \Lambda$. We consider the quotient space of $\bigoplus_{\lambda\in\Lambda} X_\lambda$ obtained by collapsing $\bigoplus_{\lambda\in\Lambda}(X_\lambda\setminus V_\lambda)$ to one point. This space and its collapsed point are denoted by $B(X_\lambda, V_\lambda, \Lambda)$ and $b(X_\lambda, V_\lambda, \Lambda)$, respectively.

LEMMA 5. *Let Λ be an index set which is disjoint union of Λ_m, $m \in N$. Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a collection of paracompact Σ-spaces such that for each $\lambda \in \Lambda$ a functionally open set V_λ of X_λ is given. Let $V = \prod_{m=1}^{\infty} B_m\setminus\{\{b_m\}\}$, where $B_m = B(X_\lambda, V_\lambda, \Lambda_m)$ and $b_m = b(X_\lambda, V_\lambda, \Lambda_m)$ for each $m \in N$. If $\dim(X_{\lambda_1}\times\ldots\times X_{\lambda_k})\leqslant n$ for each $\lambda_1, \ldots, \lambda_k \in \Lambda$, then $\dim V\leqslant n$.*

Proof. First, we show $\dim\prod_{i=1}^{k} B_i\leqslant n$ for each $k \in N$. For each $\lambda \in \Lambda$ we can choose a sequence $\{F_\lambda^j\}_{j=1}^{\infty}$ of closed sets in X_λ such that $V_\lambda = \bigcup_{j=1}^{\infty} F_\lambda^j$ and $F_\lambda^j\subset F_\lambda^{j+1}$ for each $j \in N$. Here we set for each $j \in N$

$$\mathscr{E}_j = \{\prod_{i=1}^{k} E_{\lambda_i}^j|\ (\lambda_1, \ldots, \lambda_k) \in \prod_{i=1}^{k}\Lambda_i \text{ and } E_{\lambda_i}^j = F_{\lambda_i}^j \text{ or } \{b_i\} \text{ for } 1\leqslant i\leqslant k\}.$$

Then $\mathscr{E} = \bigcup_{j=1}^{\infty}\mathscr{E}_j$ is a σ-discrete closed cover of $\prod_{i=1}^{k} B_i$. Since each $E \in \mathscr{E}$ is homeomorphic to a closed set of $\prod_{i=1}^{k} X_{\lambda_i}$ for some $(\lambda_1, \ldots, \lambda_k) \in \prod_{i=1}^{k}\Lambda_i$, we have $\dim E\leqslant n$. On the other hand, it is seen by [10, Theorem 3.2] that each B_m is a paracompact Σ-space. Hence $\prod_{i=1}^{k} B_i$ is normal. By [4, Theorems 7.2.1 and 7.2.3], we have $\dim\prod_{i=1}^{k} B_i\leqslant n$. Next, $\prod_{m=1}^{\infty} B_m$ is the limit of the inverse system $\{\prod_{i=1}^{k} B_i, p_j^k\}$, where p_j^k is the projection of $\prod_{i=1}^{k} B_i$ onto $\prod_{i=1}^{j} B_i$ for $k\geqslant j$. Moreover, $\prod_{m=1}^{\infty} B_m$ is paracompact. Since each p_j^k is an open map, it follows from [11, Theorem 1.7] that $\dim\prod_{m=1}^{\infty} B_m\leqslant n$ holds. Since each $B_m\setminus\{b_m\}$ is a functionally open set in B_m, V is a functionally open in $\prod_{m=1}^{\infty} B_m$. By [4, Problem 7.4.12], we have $\dim V\leqslant\dim\prod_{m=1}^{\infty} B_m\leqslant n$. The proof is complete.

Using Lemmas 4 and 5, we can show Theorem 3 in the same way as in the proof in [14, Added in proof]. The detail is left to the reader.

5. Examples. In connection with Theorem 1, H. Ohta has pointed out the following two examples:

EXAMPLE 1. There exists a collectionwise normal Σ-product of M_1-spaces which has no countable tightness.

Let Λ be an uncountable set containing 0. Let $\Lambda' = \Lambda\setminus\{0\}$ and $\varkappa = |\Lambda|$. For each $\lambda \in \Lambda'$, let X_λ be a discrete two-point space $\{0_\lambda, 1_\lambda\}$. Let X_0 be the space Y described in the proof of [12, Theorem 1]. It is easily seen that X_0 is a M_1-space. Let Σ be a Σ-product of X_λ, $\lambda \in \Lambda$, and Σ' a Σ-product of X_λ, $\lambda \in \Lambda'$. Then Σ' is a collectionwise normal space with the weight $\varkappa$ and $\Sigma = X_0 \times \Sigma'$. Hence it follows from Claim 2 of [12, p. 342] that Σ is collectionwise normal. On the other hand, X_0 has tightness $\varkappa$ $(>\omega)$. So Σ has no countable tightness.

EXAMPLE 2. There exists a non-normal Σ-product of M_1-spaces.

Let Λ, Λ' and X_λ, $\lambda \in \Lambda'$, be the same ones as the above. Let X_0 be the M_1-space described in [2, Example 2]. Again, let Σ and Σ' be the same ones as the above. Then $[0, \omega_1)$ can be embedded as a closed subspace of Σ', where $[0, \omega_1)$ denotes the space of all countable ordinals with the ordered topology. Moreover, it follows from [2, Example 2] that $X_0 \times [0, \omega_1)$ is non-normal. Since $\Sigma = X_0 \times \Sigma'$ contains a non-normal closed subspace, it is non-normal.

Unfortunately, by Example 1, the converse implication of Theorem 1 is not true. Moreover, by Example 2, one cannot obtain a positive answer to the question (Q) in the introduction without the assumption of the "if" part of it. Finally, the author thanks H. Ohta for his kind information of these useful examples.

Added in proof. Recently, the author has obtained the following related result: Let Σ be a Σ-product of paracompact Σ-spaces. If Σ is normal, then it is collectionwise normal.

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