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On metrizability of continuous images of compact ordered spaces

by

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Abstract. We prove here the following generalization of Treybig's Product Theorem: if a Hausdorff space X is a continuous image of a compact ordered space, then for every open map from X into a Hausdorff space Y the set of all points of Y having infinite preimages is metrizable.

1. Introduction. It is shown in [1] that if a Hausdorff space X is a continuous image of a compact ordered space then each Hausdorff space which can be obtained as an open infinite-to-one continuous image of X is metrizable. This result generalizes the Theorem of Treybig [6].

If an open continuous map from X onto a Hausdorff space Y has at least one finite fibre, then the space Y need not be metrizable. However, as shown in Section 3, the set of all points of Y having infinite preimages is metrizable. To prove this, we will need some technical lemmas, given in Section 2, and concerning the behaviour of long decreasing sequences of closed subsets of continuous images of compact ordered spaces.

By a (*compact*) *ordered space* we mean a linearly ordered set which is a (compact) space when equipped with the usual open-interval topology. If some convex sets are added to the topology of an ordered space, the resulting space is called a *GO-space*.

Let K be an ordered space and let A be a subset of K. A set $C\subset A$ is called a *convex component* of A if it is the maximal subset of A with respect to the property of being convex. A sequence $\{A_n\colon n = 1, 2, \ldots\}$ of subsets of X is said to be *increasing* (*decreasing*) if any element of A_n is less (greater) than any element of A_m for $n<m$.

All ordinals below are regarded as ordered spaces. A subset of a regular uncountable cardinal $\varkappa$ is called *stationary* if it meets all closed unbounded subsets of $\varkappa$.

Let S be a subset of a cardinal $\varkappa$. A map $f\colon S\to\varkappa$ will be called *regressive* if $f(\alpha)<\alpha$ for $\alpha\in S-\{0\}$. The following theorem will be used below.

Pressing Down Lemma (G. Fodor, see [5], Theorem 8, p. 347). *Let S be a stationary subset of a regular uncountable cardinal $\varkappa$. If $f\colon S\to\varkappa$ is regressive, then there exists an ordinal $\alpha<\varkappa$ such that $f^{-1}(\{\alpha\})$ is stationary.*

The following theorem, which is an extension of the well-known result due to R. Engelking and D. Lutzer [4], is proved in [2].

THEOREM 1. *Let a space X be an image of a GO-space under a closed continuous map. Then X is not paracompact if and only if it contains a closed subset which is homeomorphic to a stationary subset of a regular uncountable cardinal.*

We will use the above theorem only in the case when X is an image of a GO-space under a perfect map. In this case the theorem has the easy proof; let us give here a sketch of it.

Proof. Let f be a perfect map from a GO-space G onto a space X. Assume that X is not paracompact. Then G is not paracompact, either. Thus, by R. Engelking and D. Lutzer Theorem [4], there exists a closed set $S\subset G$ which is homeomorphic to a stationary subset of a regular uncountable cardinal $\varkappa$. Denote

$$\mathscr{R} = \{f^{-1}(x)\cap S\colon x\in f(S)\}.$$

Observe that the elements of $\mathscr{R}$ are compact, and so they are nonstationary subsets of $\varkappa$. Thus, by Pressing Down Lemma, the set $\{\min_\varkappa R\colon R\in\mathscr{R}\}$ contains an unbounded subset F which is closed relatively to S. Observe that F is a closed subset of G homeomorphic to a stationary subset of $\varkappa$ and $f|F$ is a homeomorphism.

2. Long sequences of subsets of compact ordered spaces. The easy proof of the following lemma is omitted.

LEMMA 1. *Let K be a compact ordered space, γ a limit ordinal and n a positive integer. Let $\{F_\alpha\colon \alpha<\gamma\}$ be a sequence of closed subsets of K such that $F_\beta\subset F_\alpha$ for $\alpha<\beta<\gamma$. If each F_α is the union of at most n intervals for $\alpha<\gamma$, then so is the set $F = \bigcap\{F_\alpha\colon \alpha<\gamma\}$.*

LEMMA 2. *Let K be a compact ordered space and $\varkappa$ a regular uncountable cardinal. If $\{F_\alpha\colon \alpha<\varkappa\}$ is a sequence of closed subsets of K such that*

(1) $F_{\alpha+1}\subset \operatorname{int} F_\alpha$ *for* $\alpha<\varkappa$, *and*

(2) $F_\gamma = \bigcap\{F_\alpha\colon \alpha<\gamma\}$ *for every limit* γ, $\gamma<\varkappa$,

then there exists a positive integer n and a closed unbounded subset C of $\varkappa$ such that F_α has at most n convex components for each $\alpha\in C$.

Proof. Put $\mathscr{B}_\alpha = \{B\colon B$ is a convex component of $\operatorname{int}F_\alpha$ which meets $F_{\alpha+1}\}$. Let $f(\alpha)$ be the number of elements of $\mathscr{B}_\alpha$. Since $F_{\alpha+1}$ is compact, $f(\alpha)$ is finite for each $\alpha<\varkappa$. Hence, $f|(\omega,\varkappa)$ is regressive, and so, by Pressing Down Lemma, there exists a positive integer n such that the set $S = f^{-1}(n)$ is stationary. Observe that $F_\gamma = \bigcap\{\operatorname{cl}\bigcup\mathscr{B}_\alpha\colon \alpha<\gamma\}$ for every limit γ, $\gamma<\varkappa$. Thus, by Lemma 1, the set F_γ is the union of at most n intervals for every $\gamma\in S^d$, where S^d is the set of all cluster points of S.

THEOREM 2. *Let X be a continuous image of a compact ordered space and $\varkappa$ a regular uncountable cardinal. If $\{F_\alpha\colon \alpha<\varkappa\}$ is a sequence of closed subsets of X such that*

(1) $F_{\alpha+1}\subset \operatorname{int}F_\alpha$ *for* $\alpha<\varkappa$, *and*

(2) $F_\gamma = \bigcap\{F_\alpha\colon \alpha<\gamma\}$ *for every limit* γ, $\gamma<\varkappa$,

then there exists a positive integer n and a closed unbounded subset C of $\varkappa$ such that the set $\operatorname{Bd}F_\alpha$ has at most n points, for every $\alpha\in C$.

Proof. Let f be a continuous map from a compact ordered space K onto X. We may assume that f is irreducible. Denote $H_\alpha = f^{-1}(F_\alpha)$. Observe that the sequence $\{H_\alpha\colon \alpha<\varkappa\}$ satisfies the assumptions of Lemma 2, and so there exists a positive integer m and a closed unbounded subset C of $\varkappa$ such that the set H_α has at most m convex components for $\alpha\in C$. Thus, $\operatorname{Bd}H_\alpha$ has at most $2m$ points for $\alpha\in C$. Since the map f is irreducible, $\operatorname{Bd}F_\alpha = f(\operatorname{Bd}H_\alpha)$, and so $\operatorname{Bd}F_\alpha$ has at most $2m$ points as well.

LEMMA 3. *Let f be an open continuous map from a Hausdorff space X onto a space Y. For every positive integer n the set $F_n = \{y\in Y\colon f^{-1}(y)$ has at most n points$\}$ is closed.*

Proof. Let y be a point of $Y-F_n$. Choose distinct points $p_1,\ldots,p_{n+1}\in f^{-1}(y)$ and their open disjoint neighbourhoods $U_1,\ldots,U_{n+1}$. The set $W = f(U_1)\cap\ldots$ $\ldots\cap f(U_{n+1})$ is an open neighbourhood of y such that $f^{-1}(z)\cap U_k\neq\emptyset$ for each $z\in W$ and $k = 1,\ldots,n+1$, and so $y\in W\subset Y-F_n$. Hence the set F_n is closed.

LEMMA 4. *Let a Hausdorff space X be a continuous image of a compact ordered space. For every open continuous map f from X into a Hausdorff space Y, the union of all infinite fibres of f is paracompact.*

Proof. Suppose that the union of all infinite fibres of f is not paracompact. Then the set $Z = \{y\in Y\colon f^{-1}(y)$ is infinite$\}$ is not paracompact, either. Observe that Z is an image of a GO-space under a perfect map. Thus, by Theorem 1, Z contains a closed subset S homeomorphic to a stationary subset of a regular uncountable cardinal $\varkappa$. We will use the same notation for $S\subset Z$ and the copy of S in $\varkappa$. Without loss of generality we may assume that S is embedded into $\varkappa$ as a dense subset.

Denote $X' = f^{-1}(\operatorname{cl}_Y S)$. Observe that $f|X'\colon X'\xrightarrow{\text{onto}}\operatorname{cl}_Y S$ is an open continuous map and S is the set of all points of $\operatorname{cl}_Y S$ having infinite preimages. Thus, by Lemma 3, S is a G_δ-subset of $\operatorname{cl}_Y S$. Hence, it is a G_δ-subset of the compact set $\operatorname{cl}_{\varkappa+1}S = \varkappa+1$ as well. But $\operatorname{cf}\varkappa>\omega$, and so there exists an ordinal $\gamma<\varkappa$ such that $[\gamma,\varkappa)\subset S$. Hence Z contains a closed copy of the cardinal $\varkappa$ and by the pseudocompactness of $\varkappa$, $\operatorname{cl}_Y Z$ contains a copy of $\varkappa+1$ embedded in such a way that $(\varkappa+1)\cap Z = \varkappa$. Put $X'' = f^{-1}(\varkappa+1)$ and $g = f|X''\colon X''\xrightarrow{\text{onto}}\varkappa+1$. Observe that the sequence $\{F_\alpha\colon \alpha<\varkappa\}$, where $F_\alpha = g^{-1}([\alpha,\varkappa])$ for $\alpha<\varkappa$, satisfies the assumptions of Theorem 2, and so there exists a closed unbounded subset C of $\varkappa$ such that $\operatorname{Bd}_{X''}F_\alpha$ is finite for every $\alpha\in C$. But the map g is open, and so, for every limit α where $\alpha\in\varkappa$, $\operatorname{Bd}_{X''}F_\alpha = g^{-1}(\alpha)$ and, since $\alpha\in Z$, the set $\operatorname{Bd}_{X''}F_\alpha$ is infinite. This leads to a contradiction.

3. Open maps and metrizability.

LEMMA 5. *Let g be a continuous map from a compact ordered space K onto a Hausdorff space X, f an open continuous map from X onto a space Y and p a point of Y. Let $\{\langle V_n, U_n\rangle : n = 1, 2, \ldots\}$ be a sequence of pairs of open subsets of X such that $\mathrm{cl}\, V_n \subset U_n$, the set $f^{-1}(p) \cap V_n$ is nonempty and the sets $U_1, U_2, \ldots$ are disjoint. If $\mathscr{C}(n)$ is the family of all convex components of $g^{-1}(U_n)$ which meet the set $g^{-1}(\mathrm{cl}\, V_n \cap f^{-1}(p))$, then the family $\mathscr{B}(p) = \{f(V_n \cap \mathrm{int}\, g(\bigcup \mathscr{C}(n))) : n = 1, 2, \ldots\}$ is a neighbourhood base at the point p.*

Proof. First we prove that the elements of $\mathscr{B}(p)$ are neighbourhoods of p. Fix a positive integer n and choose a point $x \in V_n \cap f^{-1}(p)$. Observe that $g^{-1}(x) \subset g^{-1}(V_n \cap f^{-1}(p)) \subset \bigcup \mathscr{C}(n)$, and so $x \in \mathrm{int}\, g(\bigcup \mathscr{C}(n))$. Thus, the set $V = V_n \cap \mathrm{int}\, g(\bigcup \mathscr{C}(n))$ is an open neighbourhood of x, and so $f(V)$ is an open neighbourhood of $f(x) = p$ since the map f is open.

To prove that $\mathscr{B}(p)$ is a base at p, fix an open neighbourhood W of p and denote $H = Y - W$. Suppose that $g^{-1}f^{-1}(H) \cap \bigcup \mathscr{C}(n) \neq \emptyset$ for each n. Choose $C_n \in \mathscr{C}(n)$ so that $g^{-1}f^{-1}(H) \cap C_n \neq \emptyset$. The sets $C_1, C_2, \ldots$ are convex and disjoint, and so there exists an infinite sequence $\{D_n : n = 1, 2, \ldots\} \subset \{C_n : n = 1, 2, \ldots\}$ which is either increasing or decreasing. Say it is increasing. Choose points $h_n \in D_n \cap g^{-1}f^{-1}(H)$ and $p_n \in D_n \cap g^{-1}f^{-1}(p)$, for $n = 1, 2, \ldots$ Let x be the least upper bound of the sequence $\{h_n : n = 1, 2, \ldots\}$. Observe that x is the least upper bound of the sequence $\{p_n : n = 1, 2, \ldots\}$ as well. But $f(g(h_n)) \in H$ for $n = 1, 2, \ldots$, and so $f(g(x)) \in H$ and $f(g(p_n)) = p$ for $n = 1, 2, \ldots$, whence $f(g(x)) = p$. This contradicts the assumption $p \notin H$.

Hence there is a positive integer n such that $g^{-1}f^{-1}(H) \cap \bigcup \mathscr{C}(n) = \emptyset$. Thus, $f(V_n \cap \mathrm{int}\, g(\bigcup \mathscr{C}(n)))$ is an element of $\mathscr{B}(p)$ which is contained in W, and so $\mathscr{B}(p)$ is a base at p.

Let $\mathscr{U}$ be a family of open subsets of a space X and let H be a subset of X. The family $\mathscr{U}$ is called a *strictly irreducible open cover of* H if $H \subset \bigcup \mathscr{U}$ and $H - \mathrm{cl} \bigcup (\mathscr{U} - \{U\}) \neq \emptyset$ for each $U \in \mathscr{U}$. By $\mathscr{U}|H$ we denote a family $\{U \cap H : U \in \mathscr{U}\}$.

Let p be a point of a space X. A family $\mathscr{U}$ of subsets of X is said to be *locally finite at* p if there is an open neighbourhood W of p such that the family $\{U \in \mathscr{U} : W \cap U \neq \emptyset\}$ is finite. If $\mathscr{U}$ is locally finite at each $p \in X$, then it is called *locally finite*.

Let $\mathscr{U}$ and $\mathscr{V}$ be families of open subsets of X. The family $\mathscr{V}$ will be called a (*star*) [*strict*] *refinement of* $\mathscr{U}$ if for each $V \in \mathscr{V}$ there exists a $U \in \mathscr{U}$ such that $V \subset U$ $(\bigcup \{W \in \mathscr{V} : V \cap W \neq \emptyset\} \subset U)$ $[\mathrm{cl}\, V \cup \bigcup \{W \in \mathscr{V} : V \cap W \neq \emptyset\} \subset U]$.

LEMMA 6. *Let f be an open continuous map from a compact Hausdorff space X onto a Hausdorff space Y. If the set Z of all points of Y having infinite preimages is paracompact, then there exist sequences $\{\langle \mathscr{U}_n, \mathscr{W}_n\rangle : n = 1, 2, \ldots\}$ and $\{\varphi_n : n = 1, 2, \ldots\}$ such that*

(1) *$\mathscr{U}_n$ is a family of open subsets of Y such that $Z \subset \bigcup \mathscr{U}_n$,*

(2) *$\mathscr{U}_n|Z$ is locally finite (in Z), and $\{U \in \mathscr{U}_n : z \in U\}$ is finite for each $z \in Z$,*

(3) *$\mathscr{W}_n = \{\mathscr{W}_n(U) : U \in \mathscr{U}_n\}$ is a family such that*

  (a) *$\mathscr{W}_n(U)$ is a finite family of open subsets of X such that $\bigcup \mathscr{W}_n(U) = f^{-1}(U)$, and for each $y \in U$ the family $\mathscr{W}_n(U)$ is a strictly irreducible open cover of $f^{-1}(y)$, where $U \in \mathscr{U}_n$,*

  (b) *$\mathscr{W}_n(U)$ contains at least n disjoint elements, for every $U \in \mathscr{U}_n$, and*

(4) *$\varphi_n : \mathscr{U}_{n+1} \to \mathscr{U}_n$ is a function such that $\mathscr{W}_{n+1}(V)$ is a strict refinement of $\mathscr{W}_n(\varphi_n(V))$ for each $V \in \mathscr{U}_{n+1}$; consequently, $\mathrm{cl}\, V \subset \varphi_n(V)$.*

Proof. Put $\mathscr{U}_1 = \{Y\}$, $\mathscr{W}_1(Y) = \{X\}$ and $\mathscr{W}_1 = \{\mathscr{W}_1(Y)\}$. Fix a positive integer n. Suppose that we have already defined sequences $\{\langle \mathscr{U}_j, \mathscr{W}_j\rangle : j = 1, \ldots, n\}$ and $\{\varphi_j : j = 1, \ldots, n-1\}$ satisfying conditions (1)–(4).

Fix a point $z \in Z$ and an element $U(z) \in \mathscr{U}_n$ such that $z \in U(z)$. The family $\mathscr{W}_n(U(z))$ is a finite open cover of $f^{-1}(z)$, and so we can find a strictly irreducible finite open cover $\mathscr{N}(z)$ of $f^{-1}(y)$ which is a strict refinement of $\mathscr{W}_n(U(z))$. The set $f^{-1}(z)$ is infinite, and so we may assume that $\mathscr{N}(z)$ contains at least $n+1$ disjoint elements. Denote

$$M(z) = \bigcap \{f(V - \mathrm{cl} \bigcup (\mathscr{N}(z) - \{V\})) : V \in \mathscr{N}(z)\} - f(X - \bigcup \mathscr{N}(z)).$$

Observe that $M(z)$ is an open neighbourhood of z such that $\mathrm{cl}\, M(z) \subset U(z)$, $f^{-1}(M(z)) \subset \bigcup \mathscr{N}(z)$, and $\mathscr{N}(z)$ is a strictly irreducible open cover of each set $f^{-1}(y)$, where $y \in M(z)$.

The family $\mathscr{M} = \{M(z) : z \in Z\}$ is an open cover of Z, and so there exists a family $\mathscr{M}'$ of open subsets of Z which is a locally finite refinement of $\mathscr{M}|Z$ since Z is paracompact. For each $V \in \mathscr{M}'$ choose a point $y(V) \in Z$ such that $V \subset M(y(V))$. Let $\tilde{V}$ be an open subset of Y such that $\tilde{V} \cap Z = V$ and $\tilde{V} \subset M(y(V))$. Let us define

$\mathscr{U}_{n+1} = \{\tilde{V} : V \in \mathscr{M}'\}$,
$\mathscr{W}_{n+1}(\tilde{V}) = \mathscr{N}(y(V))|f^{-1}(V)$ for each $V \in \mathscr{M}'$,
$\mathscr{W}_{n+1} = \{\mathscr{W}_{n+1}(\tilde{V}) : \tilde{V} \in \mathscr{U}_{n+1}\}$ and
$\varphi_n(\tilde{V}) = U(y(V))$ for each $\tilde{V} \in \mathscr{U}_{n+1}$.

Notice that the sequences $\{\langle \mathscr{U}_j, \mathscr{W}_j\rangle : j = 1, \ldots, n+1\}$ and $\{\varphi_j : j = 1, \ldots, n\}$ satisfy the inductive assumptions (1)–(4).

The following lemma is proved in [1].

LEMMA 7. *Let X be a compact Hausdorff space and H a closed infinite subset of X. For every sequence $\{\mathscr{U}_n : n = 1, 2, \ldots\}$ such that $\mathscr{U}_n$ is a strictly irreducible finite open cover of H which contains at least n disjoint elements and $\mathscr{U}_{n+1}$ is a strict refinement of $\mathscr{U}_n$ the family $\bigcup \{\mathscr{U}_n : n = 1, 2, \ldots\}$ contains an infinite subfamily consisting of disjoint elements.*

THEOREM 3. *Let a Hausdorff space X be a continuous image of a compact ordered space. If a Hausdorff space Y is an image of X under an open continuous map, then the set of all points of Y having infinite preimages is completely metrizable.*

Proof. Let K be a compact ordered space, g a continuous map from K onto X and f an open continuous map from X onto Y.

Denote $Z = \{y \in Y: f^{-1}(y) \text{ is infinite}\}$. In view of Lemma 4, the set Z is paracompact. By Lemma 6, there exist sequences $\{\langle \mathcal{U}_n, \mathcal{W}_n\rangle: n = 1, 2, \ldots\}$ and $\{\varphi_n: n = 1, 2, \ldots\}$ satisfying conditions (1)–(4).

Fix a positive integer n and an element V of $\mathcal{U}_{n+1}$. In view of condition (4), the family $\mathcal{W}_{n+1}(V)$ is a strict refinement of $\mathcal{W}_n(\varphi_n(V))$. Denote

$$\mathcal{G}(V) = \{\langle P, Q\rangle: P \in \mathcal{W}_{n+1}(V),\ Q \in \mathcal{W}_n(\varphi_n(V)),\ \mathrm{cl}P \subset Q\}.$$

By (3a), the family $\mathcal{G}(V)$ is finite. Let $\mathcal{E}(P, Q)$ be a family of all convex components of $g^{-1}(Q)$ which meet $g^{-1}(\mathrm{cl}P)$ for $\langle P, Q\rangle \in \mathcal{G}(V)$. Observe that the family $\mathcal{E}(P, Q)$ is finite. Put $\mathcal{M}(P, Q) = \{P \cap \mathrm{int}\, g(\bigcup \mathcal{R}): \mathcal{R} \subset \mathcal{E}(P, Q)\}$ for $\langle P, Q\rangle \in \mathcal{G}(V)$. Observe that $\mathcal{M}(P, Q)$ is a finite family of open sets and $\bigcup \mathcal{M}(P, Q) = P$.

Put $\mathcal{B}_n = \{f(U): U \in \mathcal{M}(P, Q), \langle P, Q\rangle \in \mathcal{G}(V) \text{ and } V \in \mathcal{U}_{n+1}\}$. Observe that the family $\mathcal{B}_n|Z$ is locally finite since $\mathcal{U}_{n+1}|Z$ is locally finite and $\mathcal{B}_n$ is obtained from $\mathcal{U}_{n+1}$ by replacing each element $V \in \mathcal{U}_{n+1}$ by a finite family $\mathcal{R}(V) = \{f(U): U \in \mathcal{M}(P, Q), \langle P, Q\rangle \in \mathcal{G}(V)\}$ such that $\bigcup \mathcal{R}(V) = V$.

Put $\mathcal{B} = \bigcup\{\mathcal{B}_n: n = 1, 2, \ldots\}$. We will show that the family $\mathcal{B}|Z$ is a base for Z.

Fix a point $z \in Z$. By conditions (1) and (2), the family $\{U \in \mathcal{U}_n: z \in U\}$ is nonempty and finite for each n, and so, in view of (4), there exists a sequence $\{U_n: n = 1, 2, \ldots\}$ such that $z \in U_n \in \mathcal{U}_n$ and $\varphi_n(U_{n+1}) = U_n$ for $n = 1, 2, \ldots$ In view of (3a), each family $\mathcal{W}_n(U_n)$ is a strictly irreducible open cover of $f^{-1}(z)$. Hence, by (3) and (4), the set $f^{-1}(z)$ and the sequence $\{\mathcal{W}_n(U_n): n = 1, 2, \ldots\}$ satisfy the assumptions of Lemma 7; so there exists an infinite sequence $\{W_{n(k)}: k = 1, 2, \ldots\}$ consisting of disjoint elements and such that $W_{n(k)} \in \mathcal{W}_{n(k)}(U_{n(k)})$ for $k = 1, 2, \ldots$ Moreover, there exists a sequence $\{\tilde{W}_k: k = 1, 2, \ldots\}$ such that $\tilde{W}_k \in \mathcal{W}_{n(k)+1}(U_{n(k)+1})$ and $\mathrm{cl}\tilde{W}_k \subset W_{n(k)}$ for $k = 1, 2, \ldots$

Let $\mathcal{C}(k)$ be the family of all convex components of $g^{-1}(W_{n(k)})$ which meet $g^{-1}(\mathrm{cl}\tilde{W}_k \cap f^{-1}(z))$. Observe that $\mathcal{C}(k) \subset \mathcal{E}(\tilde{W}_k, W_{n(k)})$. The sequences

$$\{(\tilde{W}_k, W_{n(k)}): k = 1, 2, \ldots\} \quad \text{and} \quad \{\mathcal{C}(k): k = 1, 2, \ldots\}$$

satisfy the assumptions of Lemma 5, and so the family

$$\mathcal{B}(z) = \{f(\tilde{W}_k \cap \mathrm{int}\, g(\bigcup \mathcal{C}(k))): k = 1, 2, \ldots\},$$

which is contained in $\mathcal{B}$, is a base at the point z.

Hence, the family $\mathcal{B}|Z$ is a σ-locally finite base for Z, and so Z is metrizable in view of the Nagata–Smirnov Metrization Theorem (see [3], Theorem 4.4.7).

Moreover, by Lemma 3, the set Z is a G_δ-subset of Y, and so Z is completely metrizable.

Corollary 1 ([1]). *Let a Hausdorff space X be a continuous image of a compact ordered space. If a Hausdorff space Y is an image of X under an open infinite-to-one continuous map, then Y is metrizable.*

Corollary 2 (L. B. Treybig [6]). *If a product $X \times Y$ of two infinite Hausdorff spaces is a continuous image of a compact ordered space, then both X and Y are metrizable.*

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