Minimal complementation and maximal conjugation for partitions, with an application to Blackwell sets

by

R. M. Shortt (Houghton, Mich.)

Abstract. Let $(X, \mathscr{B})$ be a measurable space and let $\mathscr{C}$ and $\mathscr{D}$ be sub-σ-algebras of $\mathscr{B}$; $\mathscr{D}$ is a *conjugate* for $\mathscr{C}$ if $\mathscr{C} \cap \mathscr{D} = \{\emptyset, X\}$; $\mathscr{D}$ is a *complement* for $\mathscr{C}$ if also $\sigma(\mathscr{C}, \mathscr{D}) = \mathscr{B}$. We give a characterisation of minimal complements for structures generated by a finite partition (which fails for countable partitions). An application involves the combinatorial structure of Blackwell sets. Partial results are also obtained for maximal conjugates using 0-1 transition kernels and measurable selectors.

§ 0. Preface. The complementation problem for lattices of σ-algebras over a fixed set seems to have originated in the statistical work of D. Basu [1] and has been studied in papers of B. V. Rao [6], H. Sarbadhikari, K. P. S. Bhaskara Rao, and E. Grzegorek [7], and most recently in the monograph of Bhaskara Rao and Rao [4]. In their approach to the question, the latter two authors provide necessary and sufficient conditions for a complement to be minimal (their Proposition 53) as well as restrictions on a separable space $(X, \mathscr{B})$ which guarantee the existence of complements for all countably generated substructures of $\mathscr{B}$ (their Proposition 52). As formulated, Proposition 53 assumes that:

1. the σ-algebras in question are countably generated, and
2. the parent space $(X, \mathscr{B})$ is strongly Blackwell.

We ask and partly answer to what extent these hypotheses are needed: with minor reservations, the first may be weakened and the second localized to the individual atoms of the structure in question. This is the substance of our Theorem 1. Examples are also given to show that the assumptions of Theorem 1 cannot be eliminated.

Theorem 2 gives a necessary and sufficient combinatorial criterion for a σ-algebra to be a minimal complement of a structure generated by a finite partition, with hardly any other hypothesis on the space. Surprisingly enough, the situation changes radically when one passes to σ-algebra generated by countably infinite partitions; an instance of this behaviour is recorded as Example 3.

Theorem 2 has an interesting reformulation when the partition has but two members, *viz*. Theorem 3: if $(X, \mathscr{B})$ is a separable space, then the σ-algebra generated by a two-fold measurable partition $X = C_1 \cup C_2$ has a minimal complement if and only if one of the spaces $(C_1, \mathscr{B}(C_1))$, $(C_2, \mathscr{B}(C_2))$ embeds inside the other. Earlier work on Borel-density and the Blackwell property in [9] and [10] enables us to produce a partition (Example 4) where no such embeddings exist, and therefore also an example of a countably generated substructure with many complements, but no minimal complement (compare problem P14 in [4]).

Another interesting by-product of Theorem 2 is a sufficient condition for the union of a finite number of Blackwell sets to be Blackwell (Theorem 4 *infra*). As was shown in § 11 of [4], the combinatorial behaviour of these sets is singular and remains somewhat of a mystery.

Section 3 explores the problem of maximal conjugation and in Theorem 5 uses 0-1 transition kernels and the notion of a measurable selector to characterize one type of maximal conjugate. Examples 5 and 6 present some of difficulties involved in generalizing these results.

§ 1. Preliminaries. After a few remarks concerning style, this section summarizes some technical details needed in the development of our findings: the notions of Borel-density, the Blackwell property, separation of sets and finally complementation itself are recalled and reviewed. As mentioned above, Theorem 1 improves certain results found in [4], and with this the section concludes.

By and large, our terminology and notational practice conform to what is found in [4], with several exceptions:

1. In our treatment of measurable (i.e. Borel) spaces $(X, \mathscr{B})$ the notation of a σ-algebra is occasionally suppressed: The space is denoted X, and when needed, its measurable structure is indicated by $\mathscr{B} = \mathscr{B}(X)$. For example, if $\mathscr{B}(X)$ is a separable structure, we say that X is a separable space.

2. If $\mathscr{C}$ is a sub-σ-algebra of $\mathscr{B}$, and $A \subset X$, then we use the notation $\mathscr{C}(A) = \{C \cap A\colon C \in \mathscr{C}\}$.

3. Our definitions for Blackwell properties do not insist on separation of points: Say a measurable space $(X, \mathscr{B})$ is *Blackwell* as long as:
(A) the structure $\mathscr{B}$ is countably generated (c.g.), and
(B) whenever $\mathscr{C} \subset \mathscr{B}$ is a c.g. structure with the same atoms as $\mathscr{B}$, then $\mathscr{C} = \mathscr{B}$.
Say $(X, \mathscr{B})$ is *strongly Blackwell* if (B) may be replaced with
(B′) whenever $\mathscr{C} \subset \mathscr{B}$ is c.g. and $B \in \mathscr{B}$ is a union of $\mathscr{C}$-atoms, then $B \in \mathscr{C}$.

4. If $(X, \mathscr{B})$ and $(Y, \mathscr{A})$ are measurable spaces with X and Y disjoint sets, then the *direct sum* of these spaces is a measurable space over the set $X \cup Y$ with σ-algebra given by $\{B \cup A\colon B \in \mathscr{B}(X), A \in \mathscr{A}(Y)\}$. We use the notation $\mathscr{B}(X) \vee \mathscr{A}(Y)$ to denote the direct sum structure.

5. If $\mathscr{B}$ is a Borel structure on a set X, then $\mathscr{B}$ separates points x and y in X if there is some B in $\mathscr{B}$ such that $x \in B$ and $y \in B^c$. Points x and y belong to the same *$\mathscr{B}$-atom* if they are not separated by $\mathscr{B}$. The $\mathscr{B}$-atoms partition X but we do not insist that $\mathscr{B}$-atoms belong to $\mathscr{B}$.

LEMMA 1. *A c.g. space $(X, \mathscr{B})$ is strongly Blackwell if and only if whenever $\mathscr{C} \subset \mathscr{D}$ are substructures of $\mathscr{B}$ with the same atoms, $\mathscr{C}$ c.g., then $\mathscr{C} = \mathscr{D}$.*

Proof. This is essentially Proposition 8 of [4], noting as superfluous the assumption that $\mathscr{D}$ be c.g.

Let X be a subset of an uncountable standard space S. Say that X is *Borel-dense in S* if $S \setminus X$ contains no uncountable members of $\mathscr{B}(S)$. X is *Borel-dense of order 2 in S* if whenever $B \in \mathscr{B}(S \times S)$ is a subset of $(S \times S) \setminus (X \times X)$, then B is contained in a countable union of sets of the form $\{s\} \times S$ and $S \times \{s\}$, where $s \in S \setminus X$. It is not hard to see that Borel-density of order 2 implies that of order 1. Compare [9].

LEMMA 2. *Let X be a subset of an uncountable standard space S; then the following are equivalent:*
1. *X is Borel-dense of order 2 in S;*
2. *X is a Blackwell space and is Borel-dense in S;*
3. *X is strongly Blackwell and Borel-dense in S.*

Proof. This is the principal result of [10].

For example, it can be proved that if S is uncountable and standard, then the complement of any universally null subset of S is Borel-dense of order 2 in S; each such set thus forms a strongly Blackwell space.

LEMMA 3. *Let S be an uncountable standard space; then there is a subset X of S such that both X and $S \setminus X$ are Borel-dense of order 2 in S.*

Proof. Without loss of generality, S may be taken to be the unit interval $[0, 1]$ under its usual Borel structure. Propositions 9 and 10 of [4] ensure the existence of an $X \subset S$ with both X and $S \setminus X$ Borel-dense in S and strongly Blackwell. Lemma 2 completes the argument.

Example 4 will demonstrate that the simplest c.g. σ-algebras may bear no minimal complement; the construction relies on Borel-density and Lemma 3.

Let $\mathscr{F}$ be any collection of non-empty subsets of a given set X; a subset F of X is a *partial selector for* $\mathscr{F}$ if F intersects each member of $\mathscr{F}$ in at most one point; F is a *full selector for* $\mathscr{F}$ if F meets each member of $\mathscr{F}$ in precisely one point. Suppose that $\mathscr{B}$ is a σ-algebra of subsets of X; then subsets D_1 and D_2 of X are *$\mathscr{B}$-separable* if there is some B in $\mathscr{B}$ with $D_1 \subset B$ and $D_2 \subset B^c = X \setminus B$. In this case we say that B (and $\mathscr{B}$) separate D_1 and D_2.

Let $(X, \mathscr{B})$ be a separable space and suppose that $\mathscr{C}$ and $\mathscr{D}$ are sub-σ-algebras of $\mathscr{B}$. Consider the following conditions:

(*)
0. Each $\mathscr{D}$-atom is a partial selector for the atoms of $\mathscr{C}$.
1. The union of any two $\mathscr{D}$-atoms is not a partial selector for the atoms of $\mathscr{C}$.
2. No two $\mathscr{D}$-atoms are $\mathscr{C}$-separable.
3. $\mathscr{C} \cap \mathscr{D}$ is the trivial σ-algebra $\{\emptyset, X\}$ on X.

LEMMA 4. *The σ-algebra $\sigma(\mathscr{C}, \mathscr{D})$ separates points of X if and only if condition* 0 *obtains.*

Proof. It is not hard to see that the atoms of $\sigma(\mathscr{C}, \mathscr{D})$ are the non-empty intersections of $\mathscr{C}$-atoms with $\mathscr{D}$-atoms; $\sigma(\mathscr{C}, \mathscr{D})$ separates points if and only if its atoms are all singleton sets. ■

LEMMA 5. *Assume that $\sigma(\mathscr{C}, \mathscr{D})$ separates points of X; then the implications $1 \to 2 \to 3$ hold for the conditions $(*)$ supra.*

Proof. $1 \to 2$. Let D_1 and D_2 be atoms of $\mathscr{D}$; since $\sigma(\mathscr{C}, \mathscr{D})$ is separating, D_1 and D_2 are (from Lemma 4) partial selectors for the atoms of $\mathscr{C}$. If D_1 and D_2 are separated by some C in $\mathscr{C}$, then $D_1 \cup D_2$ is still a partial selector: each $\mathscr{C}$-atom lies entirely within one of C or $X \setminus C$.

$2 \to 3$. If C is a non-trivial member of $\mathscr{C} \cap \mathscr{D}$ then $\mathscr{D}$-atoms $D_1 \subset C$ and $D_2 \subset X \setminus C$ are separated by $\mathscr{C}$. ■

The implications in Lemma 5 are generally non-reversible, as evinced by the following:

EXAMPLE 1. Let D_1 and D_2 be subsets of the real line that are not Borel-separable. (It is well-known that D_1 and D_2 can be chosen co-analytic. See, for example, [8].) Put $X = D_1 \cup D_2$, $\mathscr{D} = \sigma(D_1, D_2)$, and $\mathscr{C} = \mathscr{B}(D_1 \cup D_2)$, the relative linear Borel structure. Then $\sigma(\mathscr{C}, \mathscr{D})$ is separable, and $\mathscr{C}$ does not separate the $\mathscr{D}$-atoms D_1 and D_2, but their union is a (full) selector for the (singleton) atoms of $\mathscr{C}$.

EXAMPLE 2. Let X be the planar set $\{(x, y)\colon x < y \leqslant x+1\}$ under the usual Borel structure $\mathscr{B}(X)$; let $\mathscr{C}$ and $\mathscr{D}$ be the sub-σ-algebras of $\mathscr{B}(X)$ generated by projection onto the x- and y-axes, respectively. Then the $\mathscr{D}$-atoms

$$\{(x, 0)\colon -1 \leqslant x < 0\} \quad \text{and} \quad \{(x, 1)\colon 0 \leqslant x < 1\}$$

are separated by $\{(x, y) \in X\colon x < 0\}$ in $\mathscr{C}$, but $\mathscr{C} \cap \mathscr{D}$ is trivial.

If $(X, \mathscr{B})$ is measurable space, then sub-σ-algebras $\mathscr{C}$ and $\mathscr{D}$ of $\mathscr{B}$ are *complements in* $\mathscr{B}$ if $\mathscr{C} \cap \mathscr{D}$ is the trival σ-algebra $\{\emptyset, X\}$ and $\sigma(\mathscr{C}, \mathscr{D}) = \mathscr{B}$. $\mathscr{D}$ is a *minimal complement of* $\mathscr{C}$ *(in* $\mathscr{B}$) if:

1. $\mathscr{C}$ and $\mathscr{D}$ are complements in $\mathscr{B}$, and
2. Whenever $\mathscr{D}' \subset \mathscr{D}$ is also a complement of $\mathscr{C}$, then $\mathscr{D}' = \mathscr{D}$.

LEMMA 6. *Suppose $\mathscr{B}(X)$ is c.g.; if $\mathscr{C}$ has a complement $\mathscr{D}$ in $\mathscr{B}(X)$, then $\mathscr{C}$ has a c.g. complement $\mathscr{D}' \subset \mathscr{D}$. Thus, if $\mathscr{D}$ is a minimal complement of $\mathscr{C}$, then $\mathscr{D}$ is c.g.*

Proof. See Proposition 42 of [4], page 47.

This fact will find frequent application in what follows and in the proof of

THEOREM 1. *Let X be a set, $\mathscr{B} = \mathscr{B}(X)$ a σ-algebra separating points of X, and $\mathscr{C}$ and $\mathscr{D}$ sub-σ-algebras of $\mathscr{B}$. Taking note of the conditions in $(*)$, we have:*

i) *If $\sigma(\mathscr{C}, \mathscr{D})$ is c.g. and $(X, \mathscr{B})$ is a Blackwell space, then conditions 0 and 3 together imply that $\mathscr{C}$ and $\mathscr{D}$ are complements in $\sigma(\mathscr{C}, \mathscr{D})$.*

ii) *If $(X, \mathscr{B})$ is separable and non-Blackwell so that there is a separable weakening $\mathscr{B}_0$ of $\mathscr{B}$, then any two complements in $\mathscr{B}_0$ satisfy conditions 0 and 3, but are not complements in $\mathscr{B}$.*

iii) *If $\mathscr{C}$ and $\mathscr{D}$ are complements in $\mathscr{B}$, then conditions 0 and 3 obtain.*

iv) *Let $\mathscr{D}' \subset \mathscr{D}$ both be complements for $\mathscr{C}$ in $\mathscr{B}(X)$; then condition 1 implies that $\mathscr{D}'$ and $\mathscr{D}$ have the same atoms.*

v) *Let $\mathscr{D}$ be a complement of $\mathscr{C}$ in $\mathscr{B}(X)$ with either $(X, \mathscr{B})$ strongly Blackwell or $(X, \mathscr{D})$ Blackwell; then condition 1 implies that $\mathscr{D}$ is a minimal complement for $\mathscr{C}$ in $\mathscr{B}(X)$.*

vi) *Let $\mathscr{D}$ be a minimal complement for $\mathscr{C}$ in $\mathscr{B}(X)$; suppose that $\mathscr{C}$ is c.g. and that the union of any two $\mathscr{D}$-atoms is Blackwell in the relative $\mathscr{B}$-structure; then condition 1 follows.*

vii) *The conclusion of* vi *does not necessarily hold for $\mathscr{C}$ not c.g. or if it is assumed only that each $\mathscr{D}$-atom is Blackwell.*

Proof. i) Lemma 4 and condition 0 together imply that $\sigma(\mathscr{C}, \mathscr{D})$ is a separable structure; the Blackwell property ensures that $\sigma(\mathscr{C}, \mathscr{D}) = \mathscr{B}(X)$.

ii) Immediate.

iii) Again, Lemma 4 applies.

iv) Let D' be an atom of $\mathscr{D}'$ that is not an atom of $\mathscr{D}$; then there are two distinct atoms D_1 and D_2 of $\mathscr{D}$ contained in D'. Since $\mathscr{C}$ and $\mathscr{D}'$ are complements, iii implies that D' is a partial selector for the atoms of $\mathscr{C}$; the same must be true for $D_1 \cup D_2 \subset D'$, contradicting condition 1.

v) Suppose first that $(X, \mathscr{B})$ is strongly Blackwell. If $\mathscr{D}' \subset \mathscr{D}$ is another complement of $\mathscr{C}$ in $\mathscr{B}$, then by Lemma 6, there is a c.g. complement $\mathscr{D}''$ of $\mathscr{C}$ with $\mathscr{D}'' \subset \mathscr{D}' \subset \mathscr{D}$. Part iv now implies that $\mathscr{D}''$ and $\mathscr{D}$ have the same atoms: Lemma 1 and the strong Blackwell property yield $\mathscr{D}'' = \mathscr{D}$.

The same reasoning applies in the case where $(X, \mathscr{D})$ is assumed to be a Blackwell space.

vi) Under these assumptions, suppose that the union of two $\mathscr{D}$-atoms D_1 and D_2 is a partial selector for the atoms of $\mathscr{C}$. Notice that from Lemma 6, $\mathscr{D}$ is c.g. and so D_1, D_2 are elements of $\mathscr{D}$. Define $\mathscr{D}'$ to be the sub-σ-algebra of $\mathscr{D}$ obtained by "clubbing" D_1 and D_2 together: $(X, \mathscr{D}')$ is the direct sum of the Borel spaces $\big(X \setminus D_1 \setminus D_2, \mathscr{D}(X \setminus D_1 \setminus D_2)\big)$ and $(D_1 \cup D_2, \{\emptyset, D_1 \cup D_2\})$; see pp. 7–8 of [4] for an explication of direct sums. We claim that $\mathscr{D}'$ is a complement for $\mathscr{C}$: since $\mathscr{D}$ is a complement of $\mathscr{C}$, and since $\mathscr{D}'(X \setminus D_1 \setminus D_2) = \mathscr{D}(X \setminus D_1 \setminus D_2)$, it remains to check only that $\mathscr{C}(D_1 \cup D_2) = \mathscr{B}(D_1 \cup D_2)$. Now $\mathscr{C}$ is c.g., and since $D_1 \cup D_2$ functions as a partial selector for $\mathscr{C}$-atoms, $\mathscr{C}$ separates points of $D_1 \cup D_2$; the Blackwell property for $\mathscr{B}(D_1 \cup D_2)$ then applies. Thus $\mathscr{D}$ cannot be a minimal complement of $\mathscr{C}$.

vii) Let X be the real numbers under the usual Borel structure $\mathscr{B}$; let $\mathscr{C}$ be the sub-σ-algebra of $\mathscr{B}$ generated by all singleton subsets of X and all symmetric linear Borel sets, i.e., those invariant under the map $x \to -x$. Then $\mathscr{D} = \sigma\big((-\infty, 0]\big)$ is a minimal complement of $\mathscr{C}$ in $\mathscr{B}(X)$, and the union of the two $\mathscr{D}$-atoms $(-\infty, 0]$ and $(0, +\infty)$ is Blackwell under the $\mathscr{B}$-structure, but this union X is also a (full) selector for the (singleton) atoms of $\mathscr{C}$. The problem is that $\mathscr{C}$ is not c.g. Compare the example in [4] bottom of p. 15.

Now let X be the real line once more, and let D be a subset of X such that both D and $D^c = S \setminus X$ are Borel-dense of order 2 in X (see Lemma 3). Let $\mathscr{C}$ be the usual Borel structure on X, put $\mathscr{D} = \sigma(D)$ and define $\mathscr{B} = \sigma(\mathscr{C}, D)$. Then $\mathscr{D}$ is a minimal complement for $\mathscr{C}$ in $\mathscr{B}(X)$, $\mathscr{C}$ is c.g., and (Lemma 2) each of the $\mathscr{D}$-atoms D, D^c is Blackwell, but their union X is a (full) selector for the (singleton) atoms of $\mathscr{C}$. The problem here is that $\mathscr{B}(D \cup D^c) = \mathscr{B}(X)$ is not Blackwell. ■

§ 2. Countable partitions, the union of Blackwell sets, and the nonexistence of minimal complements. Let $\mathscr{C}$ and $\mathscr{D}$ be substructures of a separable σ-algebra $\mathscr{B} = \mathscr{B}(X)$. As mentioned in the preface, in the case where $\mathscr{C}$ is generated by a finite partition, there is a simple combinatorial criterion (Theorem 2) whereby to judge $\mathscr{D}$ to be a minimal complement of $\mathscr{C}$; no other hypothesis on $(X, \mathscr{B})$ is required. If the partition is allowed to become infinite, only part of the result is retained.

THEOREM 2. *Let $(X, \mathscr{B})$ be a separable space and suppose that $C_1, \ldots, C_n$ is a partition of X into n (non-empty) members of $\mathscr{B} = \mathscr{B}(X)$; put $\mathscr{C} = \sigma(C_1, \ldots, C_n)$; then a sub-$\sigma$-algebra $\mathscr{D}$ of $\mathscr{B}$ is a minimal complement of $\mathscr{C}$ in $\mathscr{B}$ if and only if*:

1. *$\mathscr{D}(C_i) = \mathscr{B}(C_i)$ for $i = 1, \ldots, n$, and*
2. *the union of no two $\mathscr{D}$-atoms is a partial selector for the partition $C_1, \ldots, C_n$.*

If the partition $C_1, C_2, \ldots$ is countably infinite, then the "only if" direction of the analogous statement is true.

Proof. If $\mathscr{D}$ is any complement of $\mathscr{C}$ in $\mathscr{B}(X)$, then $\mathscr{B}(C_i) = \sigma(\mathscr{C}, \mathscr{D})(C_i) = \sigma(\mathscr{C}(C_i), \mathscr{D}(C_i)) = \mathscr{D}(C_i)$ for each i. In this case, the atoms of $\mathscr{D}$ contain no more than n elements (or if $C_1, C_2, \ldots$ is infinite, they are countable), and so the union of any two of these atoms is Blackwell in the relative (discrete) $\mathscr{B}$-structure. Now Theorem 1, part vi, implies that if $\mathscr{D}$ is minimal, then 2 above will obtain.

Conversely, suppose that 1 and 2 hold for a certain $\mathscr{D}$. Then given B in $\mathscr{B}(X)$, write

$$B = \bigcup_{i=1}^{n} B \cap C_i = \bigcup_{i=1}^{n} D_i \cap C_i$$

for some D_i in $\mathscr{D}$ *via* 1 above; thus $\sigma(\mathscr{C}, \mathscr{D}) = \mathscr{B}(X)$. Lemma 5 implies that $\mathscr{D}$ is a complement of $\mathscr{C}$.

Suppose that $\mathscr{D}' \subset \mathscr{D}$ were also a complement for $\mathscr{C}$; Theorem 1, part iv, implies that $\mathscr{D}$ and $\mathscr{D}'$ have the same atoms. If A is an atom of $\mathscr{D}$, define

$$S(A) = \{i\colon 1 \leqslant i \leqslant n \text{ and } A \cap C_i \neq \emptyset\}.$$

Let $\mathscr{S}$ be the (finite) collection of such $S(A)$ as A ranges over all $\mathscr{D}$-atoms. Notice that for any two $\mathscr{D}$-atoms A and A', the sets $S(A)$ and $S(A')$ have non-empty intersection: this restates 2.

Given D in $\mathscr{D}$, we must show $D \in \mathscr{D}'$ to establish minimality. Since $\mathscr{D}'$ is also a complement of $\mathscr{C}$, it follows from the first part of the proof that $\mathscr{D}'(C_i) = \mathscr{B}(C_i) = \mathscr{D}(C_i)$ for $i = 1, \ldots, n$. So write

$$D = \bigcup_{i=1}^{n} D \cap C_i = \bigcup_{i=1}^{n} D_i' \cap C_i$$

for certain D_i' in $\mathscr{D}'$. We now claim:

$$D = \bigcup_{S \in \mathscr{S}} \bigcap_{i \in S} D_i',$$

which equation implies that $D \in \mathscr{D}'$ as desired.

To establish the claim, note that if $x \in D$ and A is the atom of $\mathscr{D}$ (or $\mathscr{D}'$) containing x, then A meets the sets C_i for $i \in S(A)$. Thus, A meets and is therefore contained in the sets D_i^1 for $i \in S(A)$; we have proved set inclusion in one direction.

Now suppose that for some atom A of $\mathscr{D}$, x is a member of $\bigcap \{D_i'\colon i \in S(A)\}$; let A' be the $\mathscr{D}$-atom containing x. Choose some j from the intersection $S(A) \cap S(A')$. Now A' meets and is therefore contained in D_j', whilst $A' \cap C_j \subset D_j' \cap C_j = D \cap C_j$, so that A' meets and is therefore contained in D. Thus $x \in D$. ■

COROLLARY. *Let $(X, \mathscr{B})$ be a separable space and suppose that $C_1, C_2, \ldots$ is a countable partition of X into sets in $\mathscr{B}(X)$; a sub-σ-algebra $\mathscr{D}$ of $\mathscr{B}$ is a minimal complement of $\mathscr{C} = \sigma(C_1, C_2, \ldots)$ if*

1. *$\mathscr{D}(C_i) = \mathscr{B}(C_i)$ for $i = 1, 2, \ldots$, and*
2. *there is some n such that the union of no two $\mathscr{D}$-atoms is a partial selector for $\{C_1, C_2, \ldots, C_n\}$, i.e., the union of any two $\mathscr{D}$-atoms meets one of the sets $C_1, \ldots, C_n$ twice.*

Proof. Essentially the same as above; note that in this case, each $\mathscr{D}$-atom meets one of the sets $C_1, \ldots, C_n$.

EXAMPLE 3. Consider the Cantor space in its representation as a denumerable product $S = \{0, 1\} \times \{0, 1\} \times \ldots$ and under its standard product Borel structure $\mathscr{A} = \mathscr{A}(S)$. Let U be a free ultrafilter over the positive integers N, regarded as a subset of S in the usual way; then put $\mathscr{U} = \mathscr{U}(S) = \sigma(\mathscr{A}, U)$. For $i \in N$, define subsets S_i of S by

$$S_i = \{x \in S\colon x(i) = 1 \text{ and } x \in U\} \cup \{x \in S\colon x(i) = 0 \text{ and } x \notin U\},$$

where $x(i)$ denotes the ith co-ordinate of x; also define $C_i = S_i \times \{i\}$ as a subset of $S \times N$.

Let X be the union $X = C_1 \cup C_2 \cup \ldots$ and let $f\colon X \to S$ be projection onto the first co-ordinate; f is surjective. Define the σ-algebras

$$\mathscr{C}(X) = \sigma(C_1, C_2, \ldots), \quad \mathscr{D}(X) = f^{-1}(\mathscr{U}), \quad \mathscr{D}'(X) = f^{-1}(\mathscr{A}), \quad \mathscr{B}(X) = \sigma(\mathscr{C}, \mathscr{D}).$$

Then we claim that

1. $\mathscr{D}'(C_i) = \mathscr{D}(C_i) = \mathscr{B}(C_i)$ for $i = 1, 2, \ldots$, and
2. the union of no two $\mathscr{D}$-atoms remains a partial selector for the partition $C_1, C_2, \ldots$, but
3. $\mathscr{D}'$ is properly a sub-structure of $\mathscr{D}$, and
4. both $\mathscr{D}'$ and $\mathscr{D}$ are complements of $\mathscr{C}$ in $\mathscr{B}(X)$, so that $\mathscr{D}$ is not a minimal complement of $\mathscr{C}$.

To prove 1, note that the sets in $\mathscr{D}'(C_i)$ are of the form $(A \cap S_i) \times \{i\}$ for $A \in \mathscr{A}$, whilst those in $\mathscr{B}(C_i)$ are of the form $(B \cap S_i) \times \{i\}$ for $B \in \mathscr{U}$. So we need to show only that $U \cap S_i$ is a member of $\mathscr{A}(S_i)$: this is immediate from the equality $U \cap S_i = \{x \in S_i\colon x(i) = 1\}$.

To prove 2, note that the $\mathscr{D}$-atoms are the sets of the form

$$f^{-1}(x) = \begin{cases} \{(x, i)\colon x(i) = 1\} & \text{if } x \in U, \\ \{(x, i)\colon x(i) = 0\} & \text{if } x \notin U, \end{cases}$$

as x ranges over S. To show that $f^{-1}(x) \cup f^{-1}(y)$ meets some C_i twice $(x \neq y)$, there are three cases to consider, according as zero, one, or both of x, y are in U; in each case, the result follows from the closure of U under intersections and the maximality of U as a filter.

With regard to 3, we claim that $f^{-1}(U) = (U \times N) \cap X$ is a member of $\mathscr{D}$ but not of $\mathscr{D}'$: if $f^{-1}(U) \in \mathscr{D}'$, then $U = ff^{-1}(U) \in \mathscr{A}$. This last is an impossibility; in fact,

a. U is not in the P-completion of $\mathscr{A}$, where P is the usual Cantor measure on S (the product of "fair coin-toss" Bernoulli measures), and

b. U does not have the Baire property in S (considered as a compact metric space).

To see this, let $g\colon S \to S$ be defined by

$$g(x)(i) = \begin{cases} 1 & \text{if } x(i) = 0, \\ 0 & \text{if } x(i) = 1. \end{cases}$$

Then g preserves the P-measure and Baire category of subsets of S (g is a homeomorphism of S onto itself); noting that U is a "tail event" and that $g(U) = S \setminus U$, statement a follows from the usual Kolmogoroff 0-1 law (Theorem 21.3 of [5]), whereas b follows from its topological analogon (Theorem 21.4 of [5]).

Part 4 then proceeds from 1, 2, and Lemma 5. ■

The case of two-fold partitions allows a new perspective on Theorem 2, one that will enable us to give relatively easy examples of how minimal complements may fail to exist. If $(X_1, \mathscr{B}(X_1))$ and $(X_2, \mathscr{B}(X_2))$ are separable spaces, say that X_1 is *embedded* in X_2 if there is a measurable isomorphism φ of X_1 into X_2.

THEOREM 3. *Let X be a separable space and suppose that $X = C_1 \cup C_2$ is a partition of X into sets C_1, C_2 in $\mathscr{B}(X)$. Then the σ-algebra $\mathscr{C} = \sigma(C_1, C_2)$ has a minimal complement in $\mathscr{B} = \mathscr{B}(X)$ if and only if either*

1. *C_1 is embedded in C_2, or*
2. *C_2 is embedded in C_1.*

Proof. Suppose that $\varphi\colon C_1 \to C_2$ is an embedding of C_1 into C_2. Then define $f\colon X \to C_2$ by the rule

$$f(x) = \begin{cases} x & \text{if } x \in C_2, \\ \varphi(x) & \text{if } x \in C_1. \end{cases}$$

(Compare the second part of the proof of Proposition 52 in [4].) Then $\mathscr{D} = f^{-1}(\mathscr{B}(C_2))$ is a minimal complement of $\mathscr{C}$ in $\mathscr{B}(X)$: both conditions 1 and 2 of Theorem 2 obtain (since $\mathscr{D}$-atoms are of the form $\varphi^{-1}(x) \cup \{x\}$ as x ranges over C_2, the union of any two such atoms meets C_2 in two points).

Now suppose that $\mathscr{D}$ is a minimal complement (countably generated, from Lemma 6) for $\mathscr{C}$ in $\mathscr{B}(X)$ and that $f\colon X \to T$ is a (Marczewski) function into an uncountable standard space T that generates the σ-algebra $\mathscr{D}$. The atoms of $\mathscr{D}$ are precisely the non-empty sets of the form $f^{-1}(t)$ as t ranges over T. Since $\mathscr{D}$ is a complement of $\mathscr{C}$, these atoms are (Theorem 1, Part III) partial selectors for the partition C_1, C_2, i.e., the cardinalities of $f^{-1}(t) \cap C_1$, and $f^{-1}(t) \cap C_2$, are 0 or 1. Since $\mathscr{D}$ is assumed minimal, Theorem 2 condition 2 implies that for distinct atoms $f^{-1}(t_1)$ and $f^{-1}(t_2)$, at least one of $(f^{-1}(t_1) \cup f^{-1}(t_2)) \cap C_1$ and $(f^{-1}(t_1) \cup f^{-1}(t_2)) \cap C_2$ has two elements.

From this we see that for one of the sets C_1 or C_2 (say C_2), $f^{-1}(t) \cap C_2$ is a singleton set for each atom $f^{-1}(t)$ of $\mathscr{D}$. From Theorem 2, $\mathscr{D}(C_2) = \mathscr{B}(C_2)$, which implies that f_0, the restriction of f to C_2, is an isomorphism of C_2 onto $f(C_2)$; from the previous sentence, $f(C_2) = f(X) \supset f(C_1)$. Define $\varphi\colon C_1 \to C_2$ by $\varphi(x) = f_0^{-1}(f(x))$; φ is an embedding of C_1 into C_2. ■

EXAMPLE 4. There is a separable space $(X, \mathscr{B})$ and a two-fold partition of X into sets C_1 and C_2 in $\mathscr{B}(X)$ such that σ-algebra $\mathscr{C} = \sigma(C_1, C_2)$ has no minimal complement in $\mathscr{B}(X)$.

Let $(X, \mathscr{A})$ be an uncountable standard space and let C_1 be a subset of X such that both C_1 and $C_2 = X \setminus C_1$ are Borel-dense of order 2 in X. Lemma 3 guarantees the existence of such sets. Define $\mathscr{B} = \mathscr{B}(X) = \sigma(\mathscr{A}, C_1)$.

From Theorem 3, our claim will be established once we prove that there is no embedding of C_1 into C_2 or of C_2 into C_1. Suppose that $\varphi\colon C_1 \to C_2$ were an embedding of C_1 into C_2 (the other case is treated symmetrically). Then φ extends to an isomorphism $\bar{\varphi}\colon B_1 \to B_2$, where $C_1 \subset B_1$, $C_2 \subset B_2$, and B_1, B_2 are members of $\mathscr{A}(X)$. If G is the graph of $\bar{\varphi}$, then G is a member of $\mathscr{A}(X \times X)$ not contained in a countable union of sets of the form $\{x\} \times X$ and $X \times \{x\}$. But $G \subset (X \times X) \setminus (C_1 \times C_1)$, contradicting the second-order Borel-density of C_1 in X. ■

We conclude this section with an application of Theorem 2 to the problem of determining whether the union of Blackwell sets is again Blackwell. Compare [4], p. 28, Section 5°.

THEOREM 4. *Let $(Y, \mathscr{A})$ be a separable space and let $H_1, H_2, \ldots, H_n$ be subsets of Y with $(H_i, \mathscr{A}(H_i))$ a Blackwell space for $i = 1, \ldots, n$. Then the following is a sufficient condition for the union $H = H_1 \cup \ldots \cup H_n$ to be itself a Blackwell space:*

(A) *For any two points x and y in H, there is some index j with x and y both in H_j.*

Proof. Suppose that condition (A) obtains and that $\mathscr{A}_0(H)$ is a proposed separable weakening of the structure $\mathscr{A}(H)$. Define subsets $C_i = H_i \times \{i\}$ of the product $H \times \{1, \ldots, n\}$ and put $X = C_1 \cup \ldots \cup C_n$. Define $f\colon X \to H$ to be projection on the first co-ordinate and set $\mathscr{D}(X) = f^{-1}(\mathscr{A}(H))$ and $\mathscr{D}_0(X)$

$= f^{-1}(\mathscr{A}_0(H))$. Then we claim that $\mathscr{D}_0(X) \subset \mathscr{D}(X)$ are both minimal complements for $\mathscr{C} = \sigma(C_1, \ldots, C_n)$ in $\mathscr{B}(X) = \sigma(\mathscr{C}, \mathscr{D})$ and so are equal.

To prove this, apply Theorem 2 and note that f is an isomorphism of $(C_i, \mathscr{B}(C_i))$ onto $(H_i, \mathscr{A}(H_i))$, so that $\mathscr{D}(C_i) = \mathscr{B}(C_i)$ is a Blackwell structure for $i = 1, \ldots, n$. Because $\mathscr{D}_0$ separates points of each C_i, therefore also $\mathscr{D}_0(C_i) = \mathscr{D}(C_i) = \mathscr{B}(C_i)$. Condition 2 in Theorem 2 follows from (A) above, and the claim is proved.

So given A in $\mathscr{A}(H)$, $f^{-1}(A)$ is in $\mathscr{D}(X) = \mathscr{D}_0(X)$, and $ff^{-1}(A) = A$ is a member of $\mathscr{A}_0(H)$. Thus $\mathscr{A}_0(H) = \mathscr{A}(H)$. ■

§ 3. **Maximal conjugation.** Let $\mathscr{C}$ and $\mathscr{D}$ be sub-structures of a σ-algebra $\mathscr{B} = \mathscr{B}(X)$. $\mathscr{C}$ and $\mathscr{D}$ are *conjugate* if $\mathscr{C} \cap \mathscr{D} = \{\emptyset, X\}$. A conjugate (resp. complement) $\mathscr{D}$ of $\mathscr{C}$ is a *maximal conjugate* (resp. *maximal complement*) for $\mathscr{C}$ if no sub-σ-algebra of $\mathscr{B}$ properly containing $\mathscr{D}$ is a conjugate (resp. complement) of $\mathscr{C}$. We begin our investigation of such structures with a characterization of a particular, strong form of maximal conjugation.

THEOREM 5. *Let $\mathscr{C}$ and $\mathscr{D}$ be sub-structures of a measurable space $(X, \mathscr{B})$ and suppose that each $\mathscr{C}$-atom is a member of $\mathscr{C}$. Then the following are equivalent:*

I) *$\mathscr{D}$ is a maximal conjugate for $\mathscr{C}$ that separates no two $\mathscr{C}$-atoms.*

II) *There is a mapping $C \to \nu_C$ associating to each $\mathscr{C}$-atom C a 0-1 measure ν_C on $\mathscr{B}$ such, that*

i) *$\nu_C(C) = 1$ for each $\mathscr{C}$-atom C,*

ii) *for each B in $\mathscr{B}$, the union of those $\mathscr{C}$-atoms C such that $\nu_C(B) = 1$ is a member of $\mathscr{C}$,*

iii) *$\mathscr{D} = \{B \in \mathscr{B}\colon \nu_C(B) = \nu_{C'}(B)$ for any two $\mathscr{C}$-atoms $C, C'\}$.*

III) *$\mathscr{D}$ is a maximal complement for $\mathscr{C}$ and does not separate any two $\mathscr{C}$-atoms.*

Proof. II implies I: Clearly, any $\mathscr{D}$ having the form advertised is a conjugate of $\mathscr{C}$. To prove maximality, suppose that $B \in \mathscr{B} \setminus \mathscr{D}$; then there are $\mathscr{C}$-atoms C_1 and C_2 with $\nu_{C_1}(B) = 1$ and $\nu_{C_2}(B) = 0$. Let C_0 be the union of all $\mathscr{C}$-atoms C with $\nu_C(B) = 1$; by hypothesis, C_0 is a (non-trivial) member of $\mathscr{C}$. Then

$$\mathscr{S} = (B \cap C_0) \cup (B^c \cap C_0^c)$$

is a member of $\mathscr{D}$, and $(B \cap \mathscr{S}) \cup (B^c \cap \mathscr{S}^c) = C_0$ is in $\sigma(\mathscr{D}, B)$.

I implies II: We claim that for each $\mathscr{C}$-atom C, the structure $\mathscr{D}$ is an anti-atom in $\sigma(\mathscr{D}, C)$. If not, then there are $\mathscr{D}$-sets D_1 and D_2 with $(D_1 \cap C) \cup (D_2 \cap C^c)$ a non-trivial $\mathscr{C}$-set other than C or its complement. Thus there are $\mathscr{C}$-atoms C_1 and C_2 with C, C_1, C_2 distinct such that $(D_1 \cap C) \cup (D_2 \cap C^c)$ separates C_1 and C_2; but then D_2 separates C_1 and C_2, a contradiction.

Thus, from Proposition 35 of Rao and Rao [4], there are corresponding to each $\mathscr{C}$-atom C certain 0-1 measures ν_C and $\tilde{\nu}_C$ on $\sigma(\mathscr{D}, C)$ such that:

i) $\nu_C(C) = \tilde{\nu}_C(C^c) = 1$ and

ii) $\mathscr{D} = \{B \in \sigma(\mathscr{D}, C)\colon \nu_C(B) = \tilde{\nu}_C(B)\}$.

Consider the σ-ideal of $\mathscr{D}$ defined by:

$$\mathscr{J} = \{D \in \mathscr{D}\colon B \subset D \text{ and } B \in \mathscr{B} \text{ imply } B \in \mathscr{D}\}.$$

CLAIM 1. The σ-ideal $\mathscr{J}$ is maximal in $\mathscr{D}$. Suppose that for some D in $\mathscr{D}$, neither D nor D^c is a member of $\mathscr{J}$: then $\mathscr{D}(D)$ and $\mathscr{D}(D^c)$ are proper sub-structures of $\mathscr{B}(D)$ and $\mathscr{B}(D^c)$, respectively. By maximality, there are non-trivial $\mathscr{C}$-sets C and C' with $C \in \mathscr{B}(D) \vee \mathscr{D}(D^c)$ and C' in $\mathscr{D}(D) \vee \mathscr{B}(D^c)$. So $C = B_1 \cup D_1$ and $C' = D_2 \cup B_2$ for (unique) sets $B_1 \in \mathscr{B}(D)$, $D_1 \in \mathscr{D}(D^c)$, $D_2 \in \mathscr{B}(D)$, $B_2 \in \mathscr{B}(D^c)$. Replacing one of these sets C, C' with its complement (if needed), we find $\mathscr{C}$-atoms $C_1 \subset (C \setminus C')$ and $C_2 \subset (C' \setminus C)$. But then $(C \setminus C') = (B_1 \setminus D_2) \cup (D_1 \setminus B_2) \subset (D \setminus D_2) \cup D_1$ and $(C' \setminus C) = (D_2 \setminus B_1) \cup (B_2 \setminus D_1) \subset D_2 \cup (D^c \setminus D_1) = [(D \setminus D_2) \cup D_1]^c$, so that $\mathscr{D}$ separates C_1 and C_2, a contradiction. The claim is established.

CLAIM 2. If $D \in \mathscr{J}$, then for each $\mathscr{C}$-atom C one has $\nu_c(D) = \tilde{\nu}_c(D) = 0$ and if $D \in \mathscr{D} \setminus \mathscr{J}$, then $\nu_C(D) = \tilde{\nu}_C(D) = 1$, so that the common restriction to $\mathscr{D}$ of the measures ν_C, $\tilde{\nu}_C$ is the 0-1 measure corresponding to $\mathscr{J}$. To see this, note that if $D \in \mathscr{J}$, then $D \cap C$ and $D \cap C^c$ are in $\mathscr{D}$ for each $\mathscr{C}$-atom C, whence $\nu_C(D) = \nu_C(D \cap C) = \tilde{\nu}_C(D \cap C) = 0$, whilst $\tilde{\nu}_C(D) = \tilde{\nu}_C(D \cap C^c) = \nu_C(D \cap C^c) = 0$, as claimed.

CLAIM 3. For each $\mathscr{C}$-atom C, one has $\mathscr{B}(C) = \mathscr{D}(C)$. Suppose that for some $B \in \mathscr{B}$, the set $B \cap C$ is not in $\mathscr{D}(C)$. Then for some choice of D_1, D_2 in $\mathscr{D}$,

$$[D_1 \cap (B \cap C)] \cup [D_2 \cap (B \cap C)^c] = C_0$$

is a non-trivial $\mathscr{C}$-set. Taking a complement if necessary, we may insist that C_0 not contain C; then $C_0 = D_2 \cap (B \cap C)^c$. If C' is a $\mathscr{C}$-atom contained in C_0 then $C' \subset D_2$ and from Claims 1 and 2, $\nu_{C'}(D_2) = 1$. Now $B^c \cap C \subset D_2^c$; since $B \cap C$ is not in $\mathscr{D}(C)$, certainly $B^c \cap C$ is not in $\mathscr{D}$, so that from Claims 1 and 2, $\nu_{C'}(D_2^c) = 1$, a contradiction. The claim is proved.

Note. We have in fact shown that for each B in $\mathscr{B}$ and each $\mathscr{C}$-atom C, one of the sets $B \cap C$, $B^c \cap C$ is a member of $\mathscr{D}$.

From Claim 3 and $\nu_C(C) = 1$, we see that the measures ν_C have unique extensions to 0-1 measures on $\mathscr{B}$; the notation ν_C is preserved for the extension.

Note. For each pair of $\mathscr{C}$-atoms C and C', the measure $\nu_{C'}$, when restricted to $\sigma(\mathscr{D}, C)$, equals $\tilde{\nu}_C$: for D_1 and D_2 in $\mathscr{D}$, $\nu_{C'}((D_1 \cap C) \cup (D_2 \cap C^c)) = \nu_{C'}(D_2 \cap C^c) = \nu_{C'}(D_2)$ while $\tilde{\nu}_C((D_1 \cap C) \cup (D_2 \cap C^c)) = \tilde{\nu}_C(D_2 \cap C^c) = \tilde{\nu}_C(D_2)$, and as noted in Claim 2, $\nu_{C'}$ and $\tilde{\nu}_C$ coincide on $\mathscr{D}$.

CLAIM 4. For each B in $\mathscr{B}$, the union of those $\mathscr{C}$-atoms C each that $\nu_C(B) = 1$ is a member of $\mathscr{C}$. If $B \in \mathscr{J}$, then from Claim 2, $\nu_C(B) = 0$ for all C so that the union in question is null; similarly, if $B \in \mathscr{D} \setminus \mathscr{J}$, then $\nu_C(B) = 1$ for all C and the union is X. Now suppose that $B \in \mathscr{B} \setminus \mathscr{D}$; then for some $\mathscr{D}$-sets D_1 and D_2, $(D_1 \cap B) \cup (D_2 \cap B^c) = C_0$ is a non-trivial $\mathscr{C}$-set. We prove that whenever C_1 and C_2 are $\mathscr{C}$-atoms with $C_1 \subset C_0$ and $C_2 \subset C_0^c$, then $\nu_{C_1}(B) \neq \nu_{C_2}(B)$. This will imply that $\nu_C(B)$ is constant as C ranges over all $\mathscr{C}$-atoms $C \subset C_0$ or $C \subset C_0^c$; then the union of $\mathscr{C}$-atoms in question will equal either C_0 or C_0^c, establishing the claim. So suppose for the sake of contradiction that $\nu_{C_1}(B) = \nu_{C_2}(B)$. Then $\nu_{C_1}(D_1 \cap B) = \nu_{C_2}(D_1 \cap B)$, $\nu_{C_1}(B^c) = \nu_{C_2}(B^c)$, and $\nu_{C_1}(D_2 \cap B^c) = \nu_{C_2}(D_2 \cap B^c)$; but then $\nu_{C_1}(C_0) = \nu_{C_2}(C_0)$, which is absurd.

To conclude this part of the proof, note that by Claim 2, $\mathcal{D}$ is contained in the σ-algebra $\mathcal{D}_0 = \{B: \nu_C(B) = \nu_{C'}(B)$ all $\mathcal{C}$-atoms $C, C'\}$. Observe, however, that $\mathcal{D}_0$ is a conjugate of $\mathcal{C}$, so that by maximality, $\mathcal{D} = \mathcal{D}_0$.

III $\to$ I: Immediate.

II $\to$ III: We have proved already that a $\mathcal{D}$ having this form is a maximal conjugate for $\mathcal{C}$; it remains only to prove that $\mathcal{B} = \sigma(\mathcal{C}, \mathcal{D})$. Given B in $\mathcal{B}$, let C_0 be the union of all $\mathcal{C}$-atoms C with $\nu_C(B) = 1$. Then $\nu_C(B \setminus C_0) = 0$ for each $\mathcal{C}$-atom C, so that $B \setminus C_0 \in \mathcal{D}$. Also, $\nu_C((B \cap C_0) \cup C_0^c) = 1$ for all $\mathcal{C}$-atoms C, so that $(B \cap C_0) \cup C_0^c$ is a set in $\mathcal{D}$. Now B may be written as $[C_0 \cap [(B \cap C_0) \cup C_0^c]] \cup \cup (B \setminus C_0)$ and so is a member of $\sigma(\mathcal{C}, \mathcal{D})$ as desired. ■

COROLLARY. *Let $(X, \mathcal{B})$ be a separable space and let $\mathcal{C}$ be a countably generated sub-σ-algebra of $\mathcal{B}$. A substructure $\mathcal{D}$ of $\mathcal{B}$ is a maximal conjugate for $\mathcal{C}$ separating no two $\mathcal{C}$-atoms if and only if $\mathcal{D} = \{B \in \mathcal{B}: S \subset B$ or $S \subset B^c\}$ for some set S such that*

i) *$S \in \mathcal{B}$,*

ii) *S is a full selector for the atoms of $\mathcal{C}$, and*

iii) *$\mathcal{B}(S) = \mathcal{C}(S)$.*

Therefore, $\mathcal{C}$ has such a maximal conjugate (necessarily a c.g. complement) precisely when such a full selector exists.

Proof. Suppose that $\mathcal{D}$ is such a maximal conjugate for $\mathcal{C}$ and apply Theorem 5. For each $\mathcal{C}$-atom C, the 0-1 measure ν_C is a unit mass at some point $p(C)$ in C. Define $f\colon X \to X$ by the rule $f(x) = p(C)$ when $x \in C$. Given any B in $\mathcal{B}$, $f^{-1}(B)$ is the union of all $\mathcal{C}$-atoms C such that $p(C) \in B$, i.e., such that $\nu_C(B) = 1$. Since all such unions are in $\mathcal{C}$, f is a Marczewski function for $\mathcal{C}$. Then $S = \{x\colon f(x) = x\}$ is the desired full selector. For any B in $\mathcal{B}$, $f^{-1}(B \cap S)$ is a $\mathcal{C}$-set with $f^{-1}(B \cap S) \cap S = B \cap S$, proving that $\mathcal{B}(S) = \mathcal{C}(S)$. Again using Theorem 5, $\mathcal{D} = \{B: \nu_C(B) = \nu_{C'}(B)$ all $\mathcal{C}$-atoms $C, C'\} = \{B: S \subset B$ or $S \subset B^c\}$.

Given $\mathcal{D}$ with this form, define the 0-1 measure ν_C to be the point mass at $S \cap C$, for each $\mathcal{C}$-atom C; clearly, $\nu_C(C) = 1$. Now given B in $\mathcal{B}$, choose C_0 in $\mathcal{C}$ with $B \cap S = C_0 \cap S$; then for each $\mathcal{C}$-atom C, $\nu_C(B) = \nu_C(B \cap S) = \nu_C(C_0 \cap S) = \nu_C(C_0)$, so that C_0 is the union of all $\mathcal{C}$-atoms C with $\nu_C(B) = 1$. Finally, $\mathcal{D} = \{B: \nu_C(B) = \nu_{C'}(B)$ all $\mathcal{C}$-atoms $C, C'\}$ so that by Theorem 5, $\mathcal{D}$ is a maximal conjugate for $\mathcal{C}$ of the type desired. ■

The following examples will give some idea of the complications involved in an analysis of more general types of maximal conjugate.

EXAMPLE 5. Let $f\colon I \to I$ be a function from the unit interval to itself that is not Borel measurable and let G be the graph of f. Let $\mathcal{B}$ denote the standard product structure on $I \times I$ and $\mathcal{C}$ the substructure generated by projection on the first coordinate. Set $\mathcal{D} = \{B \in \mathcal{B}: G \subset B$ or $G \subset B^c\}$. Then $\mathcal{D}$ does not separate any two $\mathcal{C}$-atoms and is maximal with respect to this property. However, $\mathcal{D}$ is a non-maximal conjugate for $\mathcal{C}$, as is shown by the corollary to Theorem 5.

EXAMPLE 6. Let f be a one-one correspondence of the interval $I_0 = [0, 1]$ onto itself that is not Borel measurable. Put $X = I \times I_0$ ($I = [0, 1]$) and $\mathcal{B}$ denote the standard Borel structure on X. Let $\mathcal{C}$ be the substructure of $\mathcal{B}$ generated by projection onto I and define $\mathcal{D}$ as $\{B \in \mathcal{B}: (0, y) \in B$ if and only if $(f^{-1}(y), y) \in B\}$. Then $\mathcal{D}$ is a complement and a maximal conjugate for $\mathcal{C}$, but $\mathcal{D}$ is not c.g.: $\mathcal{B} = \sigma(\mathcal{C}, \mathcal{D})$ because each $\mathcal{B}$-set of the form $I \times B$ is in $\mathcal{D}$. $\mathcal{D}$ is a conjugate for the reason that if B is non-null in $\mathcal{C} \cap \mathcal{D}$ and contains a $\mathcal{C}$-atom $\{x\} \times I_0$, then $(0, f(x)) \in B$ and so B contains the $\mathcal{C}$-atoms $\{0\} \times I_0$: this constrains B to meet every $\mathcal{C}$-atom, so that $B = X$. To prove maximality, suppose that $B \in \mathcal{B}$ contains $(f^{-1}(y), y)$ but not $(0, y)$. Then $B \cap \{(f^{-1}(y), y), (0, y)\} = \{(f^{-1}(y), y)\}$ and since $(f^{-1}(y) \times I_0) \setminus \{(f^{-1}(y), y)\}$ is in $\mathcal{D}$, the $\mathcal{C}$-atom $f^{-1}(y) \times I_0$ is in $\sigma(\mathcal{D}, B)$.

Finally, assume that $\mathcal{D}$ is countably generated and apply Theorem 1 from III. 39. VII of [3] to a Marczewski function for $\mathcal{D}$ to conclude that the union of all non-singleton $\mathcal{D}$-atoms, namely the union of $\{0\} \times I_0$ with the graph of f, is an analytic set. Thus the graph of f is analytic and by a well-known result e.g. III. 4. 1 of [2] it follows that f is a measurable function, a contradiction. So $\mathcal{D}$ is not c.g.

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DEPARTMENT OF MATHEMATICAL AND COMPUTER SCIENCES
MICHIGAN TECHNOLOGICAL UNIVERSITY
Houghton, Michigan 49931

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