Almost-n-fully normal spaces

by

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Abstract. We construct for each finite n a space which is almost-n- but not almost-$n+1$-fully normal, thus answering a question of Mansfield. We also construct a space which is almost-n-fully normal for all n but not almost-finitely-fully normal. We summarize the known results in this area and give an example of a perfect map which does not preserve $\varkappa$-full normality.

0. Introduction. In this paper we study the notions of $\varkappa$-full normality and almost-$\varkappa$-full normality introduced by Mansfield [Ma]. In Section 2 we present a characterization of almost-n-full normality for finite n, which will be usefull in Section 3 where we present our main results. Namely for each $n\in\omega$, $n\geqslant 2$ we construct a topology τ_n on the reals which is almost-n- but not almost-$n+1$-fully normal and we construct a topology τ_∞ on $\boldsymbol{R}$ which is almost-n-fully normal for all n, but not almost-finitely-fully normal.

In Section 4 we summarize the known implications and non-implications between the various full normality concepts. We conclude with an example showing that $\varkappa$-full normality is not preserved by perfect mappings. The technique of adding limit points to sets which should have them, used in Section 3, was invented by Ostaszewski [Os] and used, modified and exploited by many others, see e.g. [JuKuRu], [vD] and [Pr]. The trick, which is used to get nice normal open covers, is essentially due to Charamboulos [Ch].

1. Definitions and notation. All standard topological notions can be found in [En]. We start with the definitions of the covering properties which concern us in this paper.

1.0. Definition. Let $\mathscr{U}$ and $\mathscr{V}$ be covers of a set X and $\varkappa\geqslant 2$ a cardinal number. We say that $\mathscr{V}$ is an *almost-$\varkappa$-star refinement of* $\mathscr{U}$ if $\mathscr{V}$ refines $\mathscr{U}$ and for all $x\in X$ if $A\subseteq \mathrm{St}(x,\mathscr{V})$ has cardinality $\leqslant\varkappa$ then $A\subseteq U$ for some $U\in\mathscr{U}$. $\mathscr{V}$ is said to be a *$\varkappa$-star refinement of* $\mathscr{U}$ if whenever $\mathscr{V}'\subseteq\mathscr{V}$ has cardinality $\leqslant\varkappa$ and $\bigcap\mathscr{V}'\neq\emptyset$ then $\bigcup\mathscr{V}'\subseteq U$ for some $U\in\mathscr{U}$. If we replace "has cardinality $\leqslant\varkappa$" by "is finite" then we get the definitions of (*almost-*) *finite-* and *finite-star refinements.* ■

1.1. Definition. Let X be a topological space and $\varkappa\geqslant 2$ a cardinal number.

a. X is called *(almost-)$\varkappa$-fully normal* iff every open cover of X has an open (almost-)$\varkappa$-star refinement.

b. X is called *(almost-)finitely-fully normal* iff every open cover has an open (almost-)finite-star refinement. ■

The notions of (almost-)$\varkappa$-full normality and finite-full normality are due to Mansfield [Ma].

We introduced almost-finite-full normality to complete the analogy between the almost- and the not-almost case. In Section 4 we will summarize the known relations between these properties.

In the construction of our examples we need some notation which we introduce now:

1.2. Let X be a set and $\varkappa$ a cardinal. $[X]^{\leqslant\varkappa}$ denotes the set $\{A\subseteq X\colon |A|\leqslant\varkappa\}$, similar for $[X]^{\varkappa}$ and $[X]^{<\varkappa}$.

1.3. If X is a set and $n\in\omega$ then $\Delta^n X = \{\langle x, \ldots, x\rangle\colon x\in X\}$ is the diagonal of X^n. As usual $\Delta X = \Delta^2 X$.

1.4. If $\mathscr{U}$ and $\mathscr{V}$ are covers of a set X then $\mathscr{U}\wedge\mathscr{V} = \{U\cap V\colon U\in\mathscr{U},\ V\in\mathscr{V}\}$, the "largest" common refinement of $\mathscr{U}$ and $\mathscr{V}$.

1.5. If $\langle X, \tau\rangle$ is a topological space then we let τ also denote the topology of X^2, X^3, etc. For instance $\bar{A}^\tau$ may denote closure in X, X^{25}, X^{1000}, etc.

1.6. $\boldsymbol{R}$ denotes the set of real numbers, d denotes the usual metric on each of $\boldsymbol{R}, \boldsymbol{R}^2, \boldsymbol{R}^3, \ldots$ etc. and $\bar{A}^d$ denotes closure w.r.t. the d-topology.

1.7. To abbreviate some formulas we let $(x\pm\varepsilon)$ be shorthand for the interval $(x-\varepsilon, x+\varepsilon)$ when $x, \varepsilon\in\boldsymbol{R}$, $\varepsilon>0$.

1.8. If X is a set and $R\subseteq X\times X$, then $R[x] = \{y\in X\colon \langle x, y\rangle\in R\}$. Note that if R is open in $X\times X$ (w.r.t some topology) then $R[x]$ is open in X.

1.9. A topological space X is called *divisible* iff for every open $U\supseteq\Delta X$ there is an open $V\supseteq\Delta X$ such that $V\circ V\subseteq U$.

It is known [Coh] that almost-2-fully normal = divisible → collectionwise normal.

2. A characterization of almost-n- and almost-finite-full normality. In this section we characterize almost-n- and almost-finite-full normality using normal open covers.

2.0. THEOREM. *Let X be a topological space and $n\geqslant 2$. Then the following are equivalent*:

i) *X is almost-n-fully normal,*

ii) *For every open cover $\mathscr{U}$ of X there exists a normal open cover $\mathscr{Q}$ of X such that*:

$$\forall O\in\mathscr{Q}\ \ \forall F\in[O]^{\leqslant n}\ \ \exists U\in\mathscr{U}\colon F\subseteq U.$$

iii) *For every open $U\supseteq\Delta^n X$ there exists a normal open cover $\mathscr{Q}$ of X such that*:

$$\bigcup\{O^n\colon O\in\mathscr{Q}\}\subseteq U.$$

Proof. ii) ↔ iii) is easy: Given $U\supseteq\Delta^n X$ open, let $\mathscr{V} = \{V\subseteq X\colon V$ is open and $V^n\subseteq U\}$. If $\mathscr{Q}$ is associated to $\mathscr{V}$ as in ii), then it is also associated to U as required by iii).

Given $\mathscr{U}$ let $V = \bigcup\{U^n\colon U\in\mathscr{U}\}$. If $\mathscr{Q}$ is associated to V as in iii), then it is also associated to $\mathscr{U}$ as required by iii).

ii) → i). Let $\mathscr{U}$ be an open cover of X and let $\mathscr{Q}$ be as in ii). Let $\mathscr{W}$ be a star refinement of $\mathscr{Q}$. Then, since for all x $\mathrm{St}(x, \mathscr{W}\wedge\mathscr{U})\subseteq\mathrm{St}(x, \mathscr{W})\subseteq$ some $O\in\mathscr{Q}$, it follows that $\mathscr{W}\wedge\mathscr{U}$ is an almost-n-star refinement of $\mathscr{U}$.

i) → ii). Let $\mathscr{U}$ be an open cover of X and let $\mathscr{V}$ be an almost-n-star refinement of $\mathscr{U}$. Let $O = \bigcup\{V^2\colon V\in\mathscr{V}\}$. Since X is divisible, we can find a sequence $\langle O_p\rangle_{p\in\omega}$ of symmetric open sets containing ΔX such that $O_0 = O$ and $O_{p+1}\circ O_{p+1}\subseteq O_p$ for all $p\in\omega$. Let $\mathscr{O}_p = \{O_p[x]\colon x\in X\}$. It is easy to show that $\mathrm{St}(x, \mathscr{O}_{p+1})\subseteq O_p[x]$ for all x and all p, so that $\mathscr{O}_0$ is a normal open cover of X. Furthermore $O_0[x]\subset\mathrm{St}(x, \mathscr{V})$ for all x, so it easily follows that $\mathscr{O}_0$ is as required. ■

Similar to i) ↔ ii) in Theorem 2.0 we can prove:

2.1. THEOREM. *Let X be a topological space. Then X is almost-finitely-fully normal iff for every open cover $\mathscr{U}$ of X there exists a normal open cover $\mathscr{Q}$ of X such that*:

$$\forall O\in\mathscr{Q}\ \ \forall F\in[O]^{<\omega}\ \ \exists U\in\mathscr{U}\colon F\subseteq U. \ ■$$

These characterizations are not particularly striking but they show nicely how almost-n-full normality relates to the product structure of X^n. Moreover, the examples which will be constructed in § 3 have easy-to-handle normal open covers, which is how we came to the above characterizations in the first place.

3. The examples. In this section we will construct for each $n\in N$ with $n\geqslant 2$ a topology τ_n on $\boldsymbol{R}$ and a topology τ_∞ such that

i) $\langle\boldsymbol{R}, \tau_n\rangle$ is almost-n-fully normal but not almost-$n+1$-fully normal and

ii) $\langle\boldsymbol{R}, \tau_\infty\rangle$ is almost-n-fully normal for all n but not almost-finitely-fully normal.

3.0. EXAMPLES. Fix $n\in N$, $n\geqslant 2$. Let $\{B_p\}_{p\in\omega}$ be the collection of all open intervals of $\boldsymbol{R}$ with rational endpoints.

For every $k\in\omega$ choose

$$q_0^k<\ldots<q_n^k \quad \text{in } B_k\cap\boldsymbol{Q}$$

such that

$$q_n^k-q_0^k<\frac{1}{k+1}$$

and

$$k\neq l\to\{q_0^k, \ldots, q_n^k\}\cap\{q_0^l, \ldots, q_n^l\} = \varnothing$$

we let

$$C_k = \{q_0^k, \ldots, q_n^k\}.$$

Let $\langle A_\alpha\colon \alpha\in 2^\omega\rangle$ enumerate

$$\{A\in[\boldsymbol{R}^n]^\omega\colon A\cap\Delta^n\boldsymbol{R} = \varnothing \text{ and } |\bar{A}^d\cap\Delta^n\boldsymbol{R}| = 2^\omega\}.$$

We will choose, for each $\alpha \in 2^\omega$,

$$x_\alpha \in \boldsymbol{R} \quad \text{and} \quad \langle x^1_{\alpha,p}, \ldots, x^n_{\alpha,p} \rangle \in A_\alpha \quad (p \in \omega),$$

such that

$$\langle x^1_{\alpha,p}, \ldots, x^n_{\alpha,p} \rangle \to \langle x_\alpha, \ldots, x_\alpha \rangle \quad (p \to \omega) \text{ (w.r.t. } d).$$

Furthermore we will define neighborhood systems

$$\{B_k(x)\}_{k\in\omega} \quad (x \in \boldsymbol{R})$$

such that

(1) $$\forall x \in \boldsymbol{R}\ \forall k \in \omega\colon\ C_k \not\subseteq B_0(x)$$

and

(2) $$\forall \alpha \in 2^\omega\colon\ \bigcup_{p>k} \{x^1_{\alpha,p}, \ldots, x^n_{\alpha,p}\} \subseteq B_k(x_\alpha).$$

We then let $\tau = \tau_n$ be the topology generated by these neighborhood systems. Condition (1) will be used to show that $\mathcal{U} = \{B_0(x)\}_{x\in\boldsymbol{R}}$ does not have an almost-$n+1$-star refinement and (2) shows that $\bar{A}^\tau_\alpha \cap \Delta^n\boldsymbol{R} \neq \emptyset$ for all α, which will be used to show that $\langle \boldsymbol{R}, \tau_n \rangle$ is almost-n-fully normal.

We now begin with the construction: Assume that $\alpha \in 2^\omega$ and that $x_\beta \in \boldsymbol{R}$, $\langle x^1_{\beta,p}, \ldots, x^n_{\beta,p} \rangle \in A_\beta$ and $\{B_k(x_\beta)\}_{k\in\omega}$ are found for $p \in \omega$ and $\beta \in \alpha$ such that (1) and (2) are satisfied and such that furthermore:

(3) $$B_k(x_\beta) \subseteq \left(x_\beta \pm \frac{1}{k+1}\right) \quad \text{and} \quad B_{k+1}(x_\beta) \subseteq B_k(x_\beta),$$

(4) $B_k(x_\beta)$ is closed in the natural topology of $\boldsymbol{R}$.

Let x_α be an element of the derived set of $\{x \in \boldsymbol{R}\colon \langle x, \ldots, x \rangle \in \bar{A}^d_\alpha\}$ such that $x_\alpha \notin \boldsymbol{Q}$ (hence $x_\alpha \notin \bigcup_{k\in\omega} C_k$), and

$$\beta \in \alpha \wedge p \in \omega \wedge 1 \leqslant i \leqslant n \to x_\alpha \neq x_\beta \wedge x_\alpha \neq x^i_{\beta,p}.$$

The fact that x_α is in the derived set allows us to go through the picking of the $\langle x^1_{\alpha,p}, \ldots, x^n_{\alpha,p} \rangle \in A_\alpha$ without having to consider (trivial) cases.

Let $p \in \omega$ and assume that we have found $\eta_q > 0$, $\varepsilon_q > 0$, $l_q \in \omega$ and

$$\langle x^1_{\alpha,q}, \ldots, x^n_{\alpha,q} \rangle \in A$$

for $q \in p$ such that

(i)$_q$ $0 < |x^i_{\alpha,q} - x_\alpha| < \eta_q$, $1 \leqslant i \leqslant n$,

(ii)$_q$ if $k \geqslant l_q$ and $x^i_{\alpha,q} \neq x^j_{\alpha,q}$ then

$$C_k \cap \left(x^i_{\alpha,q} \pm \frac{1}{l_q+1}\right) = \emptyset \vee C_k \cap \left(x^j_{\alpha,q} \pm \frac{1}{l_q+1}\right) = \emptyset,$$

(iii)$_q$ $(x^i_{\alpha,q} \pm \varepsilon_q) \subseteq (x_\alpha \pm \eta_q)$,

(iv)$_q$ $(x^i_{\alpha,q} \pm \varepsilon_q) \cap \bigcup_{k\in l_q} C_k \subseteq \{x^i_{\alpha,q}\}$,

(v)$_q$ if $k \geqslant l_q$ and $C_k \cap \bigcup_{i=1}^{m} (x^i_{\alpha,q} \pm \varepsilon_q) \neq \emptyset$ then $C_k \cap (x_\alpha \pm \varepsilon_q) = \emptyset$.

Now pick $\eta_p > 0$ as follows:

if $p = 0$, let $\eta_p = 1$,

if $p > 0$, let $\eta_p \leqslant \varepsilon_{p-1}, \dfrac{1}{p+1}$ and make sure that $(x_\alpha \pm \eta_p) \cap \bigcup_{k\in l_{q-1}} C_k = \emptyset$.

It follows that for all $k \in \omega$ we have

$$C_k \cap \bigcup_{i=1}^{n} (x^i_{\alpha,p-1} \pm \varepsilon_{p-1}) = \emptyset \vee C_k \cap (x_\alpha \pm \eta_p) = \emptyset.$$

Choose $\langle x^1_{\alpha,p}, \ldots, x^n_{\alpha,p} \rangle \in A_\alpha$ such that $0 < |x^i_{\alpha,p} - x_\alpha| < \eta_p$. Let

$$\delta_p = \min\{|x^i_{\alpha,p} - x_\alpha|\colon 1 \leqslant i \leqslant n\} \cup \{|x^i_{\alpha,p} - x^j_{\alpha,p}|\colon 1 \leqslant j \leqslant n \wedge x^i_{\alpha,p} \neq x^j_{\alpha,p}\}.$$

Pick $l_p \in \omega$ such that $\dfrac{1}{l_p+1} \leqslant \dfrac{\delta_p}{3}$, then (ii)$_p$ holds because if $C_k \cap \left(x^i_{\alpha,p} \pm \dfrac{1}{l_p+1}\right) \neq \emptyset$ then $C_k \subseteq \left(x^i_{\alpha,p} \pm \dfrac{2}{l_p+1}\right)$ which is disjoint from $\left(x^j_{\alpha,p} \pm \dfrac{1}{l_p+1}\right)$ by choice of δ_p and l_p.

Furthermore we have that $\bigcup_{k\in l_p} C_k$ is finite so we can pick $\varepsilon_q \leqslant \dfrac{1}{l_p+1}$ such that (iv)$_p$ holds ((iii)$_p$ is easy). The choice of δ_p, l_p and ε_p also guarantees that (v)$_p$ holds.

Now we turn to the construction of $\{B_k(x_\alpha)\}_{k\in\omega}$. If $x^i_{\alpha,p} = x_\beta$ for some $\beta \in \alpha$ then we already have $\{B_k(x^i_{\alpha,p})\}_{k\in\omega}$, for the other $x^i_{\alpha,p}$ we let $B_k(x^i_{\alpha,p}) = \{x^i_{\alpha,p}\}$ $(k \in \omega)$. Then (1), (3) and (4) hold for these $B_k(x^i_{\alpha,p})$.

Let $k(p, i) = \min\{k \in \omega\colon B_k(x^i_{\alpha,p}) \subseteq (x_\alpha \pm \eta_p)\}$. By (3) $k(p, i)$ is well-defined. Let

$$B_k(x_\alpha) = \{x_\alpha\} \cup \bigcup_{k\in p\in\omega} \bigcup_{i=1}^{n} B_{k(p,i)}(x^i_{\alpha,p}).$$

We verify (1), (2), (3) and (4):

(2) holds because $x^i_{\alpha,p} \in B_{k(p,i)}(x^i_{\alpha,p})$.

(3) follows from the fact that

$$B_{k(p,i)}(x^i_{\alpha,p}) \subseteq (x_\alpha \pm \eta_p) \subseteq \left(x_\alpha \pm \frac{1}{p+1}\right),$$

(4) holds because for all $\varepsilon > 0$ $(x_\alpha \pm \varepsilon)$ contains all but finitely many of the $B_{k(p,i)}(x^i_{\alpha,p})$.

Next we turn to (1): Assume $C_k \subseteq B_0(x_\alpha)$ for some $k \in \omega$, since $x_\alpha \notin C_k$ we have that

$$C_k \subseteq \bigcup_{p\in\omega} \bigcup_{i=1}^{n} B_{k(p,i)}(x^i_{\alpha,p}).$$

By the choice of the ε_p and η_p we have that

$$p \neq q \to C_k \cap \bigcup_{i=1}^{n} (x^i_{\alpha,p} \pm \varepsilon_p) = \emptyset \vee C_k \cap \bigcup_{i=1}^{n} (x^i_{\alpha,q} \pm \varepsilon_q) = \emptyset .$$

Hence for some $p \in \omega$ we have

$$C_k \subseteq \bigcup_{i=1}^{n} (x^i_{\alpha,p} \pm \varepsilon_p) .$$

By (iv)$_p$ we have $C_k \subseteq \{x^i_{\alpha,p}\}_{i=1}^n$ if $k \in l_p$, but $|\{x^1_{\alpha,p}, \ldots, x^n_{\alpha,p}\}| \leqslant n < n+1 = |C_k|$, so we must have $k \geqslant l_p$. By (ii)$_p$ we have that $C_k \subseteq (x^i_{\alpha,p} \pm \varepsilon_p)$ for some i, if $k \geqslant l_p$, so that $C_k \subseteq B_{k(p,i)}(x^i_{\alpha,p}) \subseteq B_0(x^i_{\alpha,p})$ which contradicts the inductive assumption. Thus (1) holds.

For all other points $x \in \boldsymbol{R}$ we let $B_k(x) = \{x\}$. Then $\langle \boldsymbol{R}, \tau_n \rangle$ has the desired properties:

3.0.0. *$\langle \boldsymbol{R}, \tau_n \rangle$ is almost-n-fully normal.*

Proof. Let O be a τ_n-open set containing $\Delta^n \boldsymbol{R}$. Let $A = \boldsymbol{R}^n \setminus O$. Then $|\bar{A}^d \cap \Delta^n \boldsymbol{R}| \leqslant \omega$.

For assume $\bar{A}^d \cap \Delta^n \boldsymbol{R}$ is uncountable and hence of cardinality 2^ω. Then, since $\boldsymbol{R}^n$ is hereditarily separable, there is an $\alpha \in 2^\omega$ such that

$$A_\alpha \subseteq A \quad \text{and} \quad \bar{A}^d_\alpha = \bar{A}^d .$$

But then

$$\langle x_\alpha, \ldots, x_\alpha \rangle \in \bar{A}^\tau_\alpha \subseteq \bar{A}^\tau = A ,$$

a contradiction. So $B = \{x \in \boldsymbol{R} \colon \langle x, \ldots, x \rangle \in \bar{A}^d\}$ is countable. Hence, since each $B_k(x)$ is τ_n-open and d-closed, for each $x \in B$ we can find a clopen neighborhood V_x such that $V^n_x \subseteq O$. Furthermore there is a subcollection $\{B_k\}_{k \in H} \subseteq \{B_k\}_{k \in \omega}$ such that $\boldsymbol{R} \setminus B = \bigcup_{k \in H} B_k$ and $B^n_k \subseteq O$ $(k \in H)$. Then $\{B_k\}_{k \in H} \cup \{V_x\}_{x \in B}$ is a countable cover of $\langle \boldsymbol{R}, \tau_n \rangle$, which consists of cozero sets and which is therefore normal [En, Ex. 5.1.J]. By construction $\bigcup \{B^n_k \colon k \in H\} \cup \bigcup \{V^n_x \colon x \in B\} \subseteq O$. So $\langle \boldsymbol{R}, \tau_n \rangle$ is almost-n-fully normal by Theorem 2.0. ■

3.0.1. *$\langle \boldsymbol{R}, \tau_n \rangle$ is not almost-$n+1$-fully normal.*

Proof. Let $\mathscr{U} = \{B_0(x)\}_{x \in \boldsymbol{R}}$. Let $\mathcal{O}$ be a normal open cover of $\langle \boldsymbol{R}, \tau_n \rangle$. Let $\mathcal{O}_1$ be a star-refinement of $\mathcal{O}$. By 3.0.0 we can find a $H \subseteq \omega$ and a countable $B \subseteq \boldsymbol{R}$ such that $\boldsymbol{R} \setminus B \subseteq \bigcup_{k \in H} B_k$ (hence $H \neq \emptyset$) and $\bigcup_{k \in H} B^n_k \subseteq \bigcup \{O^n \colon O \in \mathcal{O}_1\}$, in particular,

$$\bigcup \{B_k \times B_k \colon k \in H\} \subseteq \bigcup \{O \times O \colon O \in \mathcal{O}_1\} .$$

It follows that $\mathrm{St}(x, \{B_k\}_{k \in H}) \subseteq \mathrm{St}(x, \mathcal{O}_1)$ for all $x \in \boldsymbol{R} \setminus B$. Now let $k \in H$. Then $C_k \subseteq B_k$ hence $C_k \subseteq \mathrm{St}(q^k_0, \mathcal{O}_1)$ and so $C_k \subseteq O$ for some $O \in \mathcal{O}$. But by our construction $C_k \nsubseteq B_0(x)$ for all $x \in \boldsymbol{R}$. So by Theorem 2.0 $\langle \boldsymbol{R}, \tau_n \rangle$ is not almost-$n+1$-fully normal. ■

The above construction can be modified to yield a topology τ_∞ on $\boldsymbol{R}$ which is almost-n-fully normal for all $n \geqslant 2$ but not almost-finitely-fully normal. We will not repeat the entire construction but we will indicate the modifications.

3.1. Example. A topology τ_∞ on $\boldsymbol{R}$ which is almost-n-fully normal for all n but not almost-finitely-fully normal. Again we let $\{B_k\}_{k \in \omega}$ be the collection of all open intervals with rational endpoints.

We choose

$$q^k_0 < \ldots < q^k_k \in B_k \cap \boldsymbol{Q}$$

such that $|q^k_k - q^k_0| < \dfrac{1}{k+1}$ and we let

$$C_k = \{q^k_0, \ldots, q^k_k\} \qquad (k \in \omega) .$$

We assume that $C_k \cap C_l = \emptyset$ if $k \neq l$. This time $\langle A_\alpha \colon \alpha \in 2^\omega \rangle$ enumerates the collection of all sets A such that for some $n \geqslant 2$ A is a countable subset of $\boldsymbol{R}^n$ such that

$$|\bar{A}^\alpha \cap \Delta^n \boldsymbol{R}| = 2^\omega .$$

The construction in 3.0 can be repeated with one exception: when constructing $\langle x^1_{\alpha,p}, \ldots, x^n_{\alpha,p} \rangle \in A_\alpha$ we must take n into account. The number n is determined by A_α and to make sure that $C_k \nsubseteq B_0(x_\alpha)$ for all k, we must choose η_0 in such a way that $(x_\alpha \pm \eta_0) \cap \bigcup_{k \leqslant n} C_k = \emptyset$. Then certainly $C_k \nsubseteq B_0(x_\alpha)$ for $k \leqslant n$. For $k > n$, the previous construction suffices (the proof that $C_k \nsubseteq B_0(x_\alpha)$ used the fact that $|\{x^1_{\alpha,p}, \ldots, x^n_{\alpha,p}\}| < |C_k|$ for all p). That $\langle \boldsymbol{R}, \tau_\infty \rangle$ is almost-n-fully normal for all n is established as in 3.0.0, and the fact that it is not almost-finitely-fully normal is proved as in 3.0.1, using Theorem 2.1 and the fact that we can assume that H is infinite. ■

4. A summary of known implications and examples. In this section we summarize the known implications among the properties introduced in Section 1 and the known counterexamples in this area.

4.1. [Ma] Let $\varkappa > \lambda \geqslant 2$ be cardinals. Then

$\varkappa$-fully normal $\to$ λ-fully normal,
almost-$\varkappa$-fully normal $\to$ almost-λ-fully normal and
λ-fully normal $\to$ almost-λ-fully normal.

4.2. [Ma] Let $k \in \omega$, $k \geqslant 2$. Then k-fully normal $\to$ k^2-fully normal (a k-star refinement of a k-star refinement is a k^2-star refinement). Hence 2-full normality is the same as k-full normality for all $k \in \omega$, $k \geqslant 2$.

These are the only implications known to be valid.

We now mention some counterexamples.

4.3. [Ma] Let $\varkappa \geqslant \omega$. Then the ordinal space $\varkappa^+$ is $\varkappa$-fully normal, but not almost-$\varkappa^+$-fully normal.

4.4. [Cor] An uncountable Σ-product of the integers is almost-ω_0-fully normal, but not 2-fully normal.

4.5. [Ju, Ex. 2.3.2] The product $A(\omega_1) \times \omega_1$ is almost-ω_0-fully normal but not

2-fully normal. Here $A(\varkappa)$ denotes the one-point compactification of the discrete space of cardinality $\varkappa$.

The proof used by Junilla can be adapted to show that $A(\varkappa^+)\times\varkappa^+$ is almost-$\varkappa$-fully normal but not 2-fully normal.

4.6. [Ha] Mary Ellen Rudin's Dowker space is finitely-fully normal but not almost-ω_0-fully normal.

4.7. Example(s) 3.0 show(s) that the analogue of 4.2 is not true in the almost case.

4.8. Example 3.1 is almost-k-fully normal for all k but not almost-finitely-fully normal.

After filling in the non-implications resulting from the above examples, we see that the following two questions remain unsettled:

4.9. Questions. a) Must a 2-fully normal space be finitely-fully normal?

b) Must a 2-fully normal space be almost-finitely-fully normal?

A possible approach to a negative answer would be to modify Example 3.1 to make it orthocompact since in that case it would become 2-fully normal by [Ju, Cor. 2.2.11]. However it seems unlikely that this can be done, since 3.1 is very similar to 3.0 which must yield a non-orthocompact space by 4.1, 4.2 and Junnila's result.

Finally we make some remarks concerning closed continuous images.

4.10. Example. $A(\varkappa^+)\times\varkappa^+$ can be the perfect image of a $\varkappa$-fully normal space. Consequently the perfect image of a $\varkappa$-fully normal space need not even be 2-fully normal.

a) The easier (but weaker) result that $A(\varkappa^+)\times\varkappa^+$ is the closed continuous image of a $\varkappa$-fully normal space can be obtained by adapting [Ju; 2.3.1 and 2.3.2]. If we collapse the set of limit ordinals in $\varkappa^+$ to a point, we obtain $A(\varkappa^+)$. Thus we obtain a natural map of $\varkappa^+\times\varkappa^+$ onto $A(\varkappa^+)\times\varkappa^+$. As in [Ju; 2.3.1] $\varkappa^+\times\varkappa^+$ is $\varkappa$-fully normal.

As in [Ju; 2.3.2] (with proper modifications) the map is closed.

b) To get a perfect preimage we adapt Burke's construction [Bu] of a perfect map which does not preserve orthocompactness.

Let $X_\varkappa$ be the Alexandroff-double [En; 3.1.G] of the Cantor cube $2^\varkappa$. Let $Y_\varkappa = X_\varkappa\times\varkappa^+$. The underlying set of $Y_\varkappa$ is $Y_0\cup Y_1$ where $Y_i = 2^\varkappa\times\{i\}\times\varkappa^+$.

A basic nbd of $\langle x,0,\alpha\rangle$ is of the form

$$U(x,\alpha,\beta,O) = O\times\{0\}\times(\beta,\alpha]\cup O\setminus\{x\}\times\{1\}\times(\beta,\alpha],$$

where $\beta<\alpha$ and $O\ni x$ open in $2^\varkappa$.

A basic nbd of $\langle x,1,\alpha\rangle$ is of the form $\{x\}\times\{1\}\times(\beta,\alpha]$.

(1) $Y_\varkappa$ is $\varkappa$-fully normal. Let $\mathcal{O}$ be an open cover of $Y_\varkappa$. Fix $x\in 2^\varkappa$, and for each $\alpha\in\varkappa$ pick $\beta_\alpha\in\alpha$ and $U_\alpha\ni x$ open such that

$$U(x,\alpha,\beta_\alpha,U_\alpha)\subseteq \text{some } O\in\mathcal{O}.$$

Using the pressing-down lemma and the fact that $2^\varkappa$ has weight $\varkappa$ we can find $\beta_x\in\varkappa$ and $U_x\ni x$ open such that

$$\alpha>\beta_x \to U(x,\alpha,\beta_x,U_x)\subseteq \text{some } O\in\mathcal{O}.$$

Since $2^\varkappa$ is compact, we can find $x_1,\dots,x_m\in 2^\varkappa$ such that $2^\varkappa = U_{x_1}\cup\dots\cup U_{x_m}$. Let $V_1,\dots,V_m\subseteq 2^\varkappa$ be disjoint and clopen such that

$$2^\varkappa = V_1\cup\dots\cup V_m \quad\text{and}\quad V_i\subseteq U_{x_i} \quad\text{for each } i.$$

For $1\leqslant i\leqslant m$ let

$$\mathcal{V}_i = \{V_i\times\{0\}\times(\beta,\alpha]\cup V_i\setminus\{x_1,\dots,x_m\}\times\{1\}\times(\beta,\alpha]:\ \beta\in\alpha\in\varkappa^+\},$$

where $\beta = \max(\beta_{x_1},\dots,\beta_{x_m})$.

Then $\mathcal{V}_i$ $\varkappa$-star refines itself and for all $V\in\mathcal{V}$ $V\subseteq O$ for some $O\in\mathcal{O}$. For $1\leqslant i\leqslant m$ let $\mathcal{W}_i$ be a $\varkappa$-star refinement in $\{x_i\}\times\{1\}\times(\beta,\varkappa^+)$ of $\mathcal{O}\upharpoonright\{x_i\}\times\{1\}\times(\beta,\varkappa^+)$.

Finally $Z_\beta = 2^\varkappa\times\{0,1\}\times[0,\beta]$ is compact and clopen in $Y_\varkappa$, so we can find a star refinement $\mathcal{W}$ of $\mathcal{O}\upharpoonright Z_\beta$. Then $\mathcal{W}\cup\bigcup_{i=1}^m\mathcal{V}_i\cup\bigcup_{i=1}^m\mathcal{W}_i$ is a $\varkappa$-star refinement of $\mathcal{O}$.

(2) If we collapse $2^\varkappa\times\{0\}$ to a point in $X_\varkappa$ then we obtain $A(2^\varkappa)$. Thus we obtain a continuous map of $Y_\varkappa$ onto $A(2^\varkappa)\times\varkappa^+$. This map is closed since its restriction to the closed set Y_0 is closed [En; 2.4.13]. The map has compact fibers since $2^\varkappa$ is compact, hence it is perfect.

Since $A(\varkappa^+)\times\varkappa^+$ is a closed subspace of $A(2^\varkappa)\times\varkappa^+$, it is the perfect image of a ($\varkappa$-fully normal) closed subspace of $Y_\varkappa$. ■

We conclude with the following question:

4.11. Question. Is the closed continuous or perfect image of an almost-$\varkappa$-fully normal space almost-λ-fully normal for some $\lambda\leqslant\varkappa$ (preferably $\lambda=\varkappa$ of course)?

References

[Bu] D. K. Burke, *Orthocompactness and perfect mappings*, Proc. Amer. Math. Soc. 79 (1980), pp. 484–486.

[Ch] M. G. Charamboulos, *An example concerning inverse limit sequences of normal spaces*, Proc. Amer. Math. Soc. 78 (1980), pp. 605–607.

[Coh] H. J. Cohen, *Sur une problème de M. Dieudonné*, C. R. Acad. Sci. Paris 234 (1952), pp. 290–292.

[Cor] H. H. Corson, *Examples relating to normality in topological spaces*, Trans. Amer. Math. Soc. 99 (1961), pp. 205–211.

[vD] E. K. van Douwen, *A technique for constructing honest locally compact submetrizable examples* (to appear).

[En] R. Engelking, *General Topology*, Warszawa 1977.

[Ha] K. P. Hart, *More on M. E. Rudin's Dowker space*, Proc. Amer. Math. Soc. 86 (1982), pp. 508–510.

[JuKuRu] I. Juhász, K. Kunen and M. E. Rudin, *Two more hereditarily separable non-Lindelöf spaces*, Canad. J. Math. 28 (1976), pp. 998–1005.
[Ju] H. J. K. Junilla, *Covering properties and quasi-uniformities of topological spaces*, Doctoral dissertation VPI 1978.
[Ma] M. J. Mansfield, *Some generalizations of full normality*, Trans. Amer. Math. Soc. 86 (1957), pp. 489–505.
[Os] A. J. Ostaszewski, *On countably compact, perfectly normal spaces*, J. London Math. Soc. 14 (1976), pp. 505–516.
[Pr] T. C. Przymusiński, *Normality and paracompactness in finite and countable Cartesian products*, Fund. Math. 105 (1980), pp. 89–104.

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Almost maximal ideals

by

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Abstract. An ideal in a distributive lattice is said to be almost maximal if it is prime and satisfies a first-order closure condition which, in the presence of the axiom of choice, is equivalent to saying that it is an intersection of maximal ideals. Assuming the axiom of choice, we show that the almost maximal ideals correspond to points of the soberification of the maximal ideal space of the lattice; in the absence of the axiom of choice, we investigate the strength of the "almost maximal ideal theorem" that every nontrivial distributive lattice has an almost maximal ideal. Our two main results are that this assertion implies the Tychonoff theorem for products of compact sober spaces, and that it does not imply the axiom of choice.

Introduction. It is well known that J. L. Kelley [17] proved that the Tychonoff product theorem is logically equivalent (in any reasonable set theory) to the axiom of choice. However, elsewhere in topology and analysis it is far commoner to encounter theorems which are equivalent not to the full axiom of choice but to the prime ideal theorem, i.e., the assertion that every nontrivial Boolean algebra (or equivalently, every nontrivial distributive lattice) has a prime ideal. Among examples of such theorems, let us cite:

(i) The Stone representation theorem for Boolean algebras (or for distributive lattices).

(ii) The Stone–Čech compactification theorem.

(iii) Tychonoff's theorem for products of compact Hausdorff spaces.

(iv) Alaoglu's theorem on compactness of the unit ball of the dual of a Banach space.

(v) The theorem that the hyperspace of a compact Hausdorff space (i.e., the space of closed subsets with the Vietoris topology) is compact.

There is of course a family resemblance between these theorems: each of them asserts the compactness of some space which may be *constructed* without any use of choice, but which will not have its expected properties unless it is compact. What is more striking is that in each case the space in question occurs naturally as the space of points of a certain locale (or "pointless space"; see [15]), and that the compactness of this locale can be proved constructively. (For the appropriate locale-theoretic construction, see [14] in case (i), [2] or [12] in cases (ii) and (iii), [21] in case (iv)