Mapping theorems for compacta with an arbitrary involution

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Abstract. Compacta on which acts an arbitrary involution with fixed points are considered. Two invariants (δ-index and δh-index) of such a space are introduced. They indicate its "proximity" to some Euclidean ball and are analogous to Yang's B-index and to the Smith index, respectively. (The last two concepts concern spaces with a fixed-point free involution.) Mapping theorems for δ and δh are proved. As a corollary, estimates of the mapping set for maps from a Euclidean ball into another one are obtained.

Introduction. Properties of compacta on which acts some fixed-point free involution have been studied by many authors. An instrument for obtaining Borsuk–Ulam type theorems in such spaces is the concept of B-index introduced by C. T. Yang in [5], and also the Smith index, which appears in the theory of homeomorphisms of finite period developed by P. A. Smith in [3].

We shall study in this paper compacta on which acts an arbitrary involution with fixed points. In order to obtain mapping theorems in this case, we introduce in part I the concept of the δ-index of such a space, which indicates, in a manner, its "resemblance" to some Euclidean ball (considered together with the central symmetry). In part II another index is introduced — this is the homological δ-index (or δh-index), which appears after consideration of the invariant homological structure modulo Z_2 of a space with involution. Our investigations in part II make use of Smith's theory. We establish the relations between δ, δh and dim and prove mapping theorems for δ and δh. As a corollary to these theorems we obtain a mapping theorem for maps from a Euclidean ball into another one, as well as several generalizations of a theorem due to K. Borsuk.

I. The concept of δ-index

1. Definition of the δ-index of a pair. Throughout this and the remaining sections by a space we mean a compact metric space (compactum), by a map we mean a continuous one.

A *T-space* is a space X on which acts the involution $T\colon X \to X$. The notation $(X; T)$ means "X is a T-space".

A subset Y of the T-space X is said to be *invariant* if $TY = Y$.

Recall first the concept of B-index of a T-space.

DEFINITION 1.1 (Yang [5]). Let X be a T-space, where T is a fixed-point free involution. We say that the *B-index of X is not greater than n* if there exists a map $f\colon X \to R^{n+1}$ such that

$$f(Tx) \neq f(x) \quad \text{for all } x \in X.$$

We then write $B(X; T) \leqslant n$.

The B-index of X is equal to n iff $B(X; T) \leqslant n$ and $B(X; T) \nleqslant n-1$. In the case $X = \emptyset$ we set for convenience $B(X; T) = -1$.

It is easy to prove that $B(X; T) < \infty$ for any T-space X. Indeed, let $X = \bigcup_{i=0}^{n} F_i$ be a decomposition of X into closed subsets such that $TF_i \cap F_i = \emptyset$ for $i = 0, 1, \ldots, n$. Put $f_i(x) = \varrho(x, F_i)$ and $f = f_0 \times f_1 \times \ldots \times f_n$ (where ϱ is the metric on X). Then $f\colon X \to R^{n+1}$ is a map such that $f(Tx) \neq f(x)$ for any $x \in X$; thus $B(X; T) \leqslant n$.

We shall make use of some other well-known equivalent definitions of this concept, which can be found for example in [5] or [6].

Note that the Borsuk–Ulam theorem is equivalent to $B(S^n; T_0) = n$ where T_0 is the central symmetry with respect to the origin $T_0(x) = -x$. It is not difficult to prove that this equality holds for any other involution acting on S^n.

It has been shown in [6] that

$$B(X; T) \leqslant \dim X$$

for any T-space X.

Let X be a T-space. We shall denote by $\Theta(T)$ the set of all fixed points of T

$$\Theta(T) = \{x \in X \mid Tx = x\}.$$

Apparently, $\Theta(T)$ is an invariant compact subset of X.

A pair (X, Y) is said to be a *T-pair* if

i) is a T-space,

ii) Y is a closed invariant subset of X nonintersecting $\Theta(T)$.

We shall now introduce the concept of δ index of a T-pair.

DEFINITION 1.2. We say, that the *δ index of the T-pair (X, Y) is not greater than n* if there exists in X an invariant partition C between $\Theta(T)$ and Y such that $B(C; T) \leqslant n-1$. Then we shall write

$$\delta(X, Y; T) \leqslant n.$$

The δ-index of a T-pair (X, Y) is equal to n iff $\delta(X, Y; T) \leqslant n$ and $\delta(X, Y; T) \nleqslant n-1$. If at least one of the sets $\Theta(T)$, X, Y is empty, we define for the sake of convenience $\delta(X, Y; T) = -1$.

Note that the equality

$$\delta(B^n, S^{n-1}; T_0) = n$$

has been established by K. Borsuk. (Here B^n is the n-dimensional unit ball, S^{n-1} is its boundary and $T_0(x) = -x$.) He has proved that if C is an invariant partition in R^n between the origin and ∞, then for any map $f\colon C \to R^{n-1}$ there exists an $x_0 \in C$ such that $f(-x_0) = f(x_0)$, which is equivalent to $\delta(B^n, S^{n-1}; T_0) \geqslant n$ (the converse inequality is trivial).

It is clear that each inequality of type $\delta(X, Y; T) \geqslant n$ may be formulated as a mapping theorem. We shall give in part II several generalizations of Borsuk's theorem.

2. Two elementary propositions. A closed invariant subset F of the T-space X is said to be an *antipodal partition* in X if for any $x \in X \setminus F$ it is a partition in X between x and Tx. Evidently, F is an antipodal partition in X iff $X \setminus F = U_+ \cup U_-$, where U_+ and U_- are disjoint open subsets of X such that $TU_+ = U_-$. Note that every antipodal partition in X contains $\Theta(T)$.

Let X be a T-space and let X' be a T'-space. A map $\varphi\colon X \to X'$ is called *equivariant* if

$$\varphi(Tx) = T'\varphi(x) \quad \text{for any } x \in X.$$

The notation $\varphi\colon (X; T) \to (X'; T')$ means "φ is an equivariant map from X into X'". If (X, Y) is a T-pair and (X', Y') is a T'-pair, a map $\varphi\colon (X, Y) \to (X', Y')$ is an equivariant map $\varphi\colon X \to X'$ such that $\varphi(Y) \subset Y'$.

The following two lemmas are elementary.

LEMMA 1.1. *Let F be a closed invariant subset of the T-space X and let $\varphi\colon F \to R^n$ be an equivariant map (R^n is considered with the involution $T_0(x) = -x$). Then φ admits an equivariant extension $\tilde{\varphi}\colon X \to R^n$.*

Proof. Let $\varphi_1\colon X \to R^n$ be an arbitrary extension of φ. Then

$$\tilde{\varphi}(x) = \tfrac{1}{2}(\varphi_1(x) - \varphi_1(Tx))$$

is an equivariant extension of φ.

LEMMA 1.2. *Let Y be a closed invariant subset of the T-space X and let F be an antipodal partition in Y. Then there exists such an antipodal partition $\tilde{F}$ in X that $\tilde{F} \cap Y = F$.*

Proof. Let $Y \setminus F = V_+ \cup V_-$ where V_+ and V_- are open disjoint subsets of Y such that $TV_+ = V_-$. Set

$$\varphi(x) = \begin{cases} \varrho(x, F) & \text{for } x \in V_+ \cup F, \\ -\varrho(Tx, F) & \text{for } x \in V_- \end{cases}$$

(ϱ is the metric in X).

Then $\varphi\colon Y \to R^1$ is an equivariant map with $\varphi^{-1}(0) = F$. According to Lemma 1.1 φ admits an equivariant extension $\tilde{\varphi}\colon X \to R^1$. Clearly, the antipodal partition $\tilde{F} = \tilde{\varphi}^{-1}(0)$ meets the case.

3. Several properties of δ, the relationship with dim.

LEMMA 1.3. *Let $\varphi\colon (X, Y) \to (X', Y')$ maps the T-pair (X, Y) into the T'-pair (X', Y'). Then*

$$\delta(X, Y; T) \leqslant \delta(X', Y'; T').$$

Proof. Indeed, if C' is an invariant partition in X' between $\Theta(T')$ and Y', then $C = \varphi^{-1}(C')$ is an invariant partition in X between $\Theta(T)$ and Y; hence $B(C;T) \leqslant B(C';T')$ since there exists an equivariant map $\varphi|_C\colon C \to C'$. The required inequality holds by the definition of δ.

LEMMA 1.4. *For any T-pair (X, Y) the following inequalities hold*:

a) $\delta(X, Y; T) \leqslant \dim X$,

b) $\delta(X, Y; T) \leqslant B(Y; T)+1$.

Proof. a) Suppose that $\dim X \leqslant n$. One can find in X an invariant partition C between $\Theta(T)$ and Y with $\dim C \leqslant n-1$. Then $B(C;T) \leqslant \dim C \leqslant n-1$; thus $\delta(X, Y; T) \leqslant n$.

b) Suppose that $B(Y;T) \leqslant n$, i.e., that there exists an equivariant map $\varphi\colon Y \to S^n$ (see for ex. [6]). According to Lemma 1.1, φ admits an equivariant extension $\tilde{\varphi}\colon X \to B^{n+1}$. Denote by Σ the sphere $\Sigma = \{x \in B^{n+1} |\ \|x\| = \frac{1}{2}\}$. Then $C = \tilde{\varphi}^{+1}(\Sigma)$ is an invariant partition in X between $\Theta(T)$ and Y with $B(C;T) \leqslant n$. Therefore $\delta(X, Y; T) \leqslant n+1$.

Remark. As a corollary to b) we get

$$\dim Y \geqslant \delta(X, Y; T)-1\,,$$

b) implies also that $\delta(X, Y; T) < \infty$ for any T-pair (X, Y).

LEMMA 1.5. *The following two conditions are equivalent*:

a) $\delta(X, Y; T) \leqslant n$.

b) *There exists in X such an antipodal partition F that $\delta(F, F \cap Y; T) \leqslant n-1$.*

Proof. a) $\Rightarrow$ b) Let C be an invariant partition in X between $\Theta(T)$ and Y with $B(C;T) \leqslant n-1$. Then we can find in C an antipodal partition C_1 with $B(C_1;T) \leqslant n-2$ (see [6]). According to Lemma 1.2, there exists in X an antipodal partition F such that $F \cap C = C_1$. Obviously $\delta(F, F \cap Y; T) \leqslant n-1$.

b) $\Rightarrow$ a) Denote by C an arbitrary invariant partition in F between $\Theta(T)$ and $F \cap Y$ with $B(C;T) \leqslant n-2$. We can find in X such a partition C_1 between $\Theta(T)$ and Y that $C_1 \cap F = C$. Set $\tilde{C} = C_1 \cup TC_1$. Then $\tilde{C}$ is an invariant partition in X between $\Theta(T)$ and Y such that $\tilde{C} \cap F = C$. Evidently, C is an antipodal partition in $\tilde{C}$; hence $B(\tilde{C};T) \leqslant n-1$. Therefore $\delta(X, Y; T) \leqslant n$.

4. The mapping theorem for δ. Let X be a T-space. Given a map $f\colon X \to M$, denote as usual

$$A(f) = \{x \in X |\ f(Tx) = f(x)\}\,.$$

The following theorem states that if the pair (X, Y) is of a great δ-index and M is a low dimensional Euclidean space, then the δ-index of the pair $(A(f), A(f) \cap Y)$ is large enough for any f.

THEOREM 1.1. *Let $\delta(X, Y; T) \geqslant n$. Then for any map $f\colon X \to R^k$ the inequality*

$$\delta(A(f), A(f) \cap Y; T) \geqslant n-k$$

holds.

Proof. We shall carry out the proof by induction on k. The case $k = 0$ is trivial. Assume the inequality to be true for $k = s$ and consider the case $k = s+1$. Let $f\colon X \to R^{s+1}$ be an arbitrary map. Denote by $\pi_s\colon R^{s+1} \to R^s$ the orthogonal projection defined by

$$\pi_s(x_1, x_2, \ldots, x_{s+1}) = (x_2, \ldots, x_{s+1})\,.$$

Consider the composition $f_1 = \pi_s f\colon X \to R^s$. Then the inequality

$$\delta(A(f_1), A(f_1) \cap Y; T) \geqslant n-s$$

holds by the induction hypothesis. Clearly, $A(f) \subset A(f_1)$. We shall prove that $A(f)$ is an antipodal partition in $A(f_1)$. Denote by $\pi_1\colon R^{s+1} \to R^1$ the projection $\pi_1(x_1, x_2, \ldots, x_{s+1}) = x_1$ and put $\lambda(x) = \pi_1 f(x) - \pi_1 f(Tx)$ for any $x \in A(f_1)$. Then $\lambda\colon A(f_1) \to R^1$ is an equivariant map and one can easily check that $\lambda^{-1}(0) = A(f)$. Therefore $A(f)$ is an antipodal partition in $A(f_1)$. According to Lemma 1.5

$$\delta(A(f), A(f) \cap Y; T) \geqslant n-s-1\,.$$

The theorem is proved.

COROLLARY. *Let $\delta(X, Y; T) \geqslant n$. Then for any map $f\colon X \to R^k$*

$$\dim A(f) \geqslant n-k\,.$$

This inequality holds by Theorem 1.1 and Lemma 1.4.

In the case $X = B^n$, $Y = S^{n-1}$ the corollary together with Borsuk's theorem gives the following

THEOREM 1.2. *Let $f\colon B^n \to R^k$ be an arbitrary map and*

$$A(f) = \{x \in B^n |\ f(x) = f(-x)\}\,.$$

Then $\dim A(f) \geqslant n-k$.

5. Definition of δ-index of a single space, the mapping theorem.

DEFINITION 1.3. We say that the *δ-index of the T-space X is not less than n* if there exists in X such a closed invariant subset Y nonintersecting $\Theta(T)$ that $\delta(X, Y; T) \geqslant n$. Then we write

$$\delta(X; T) \geqslant n\,.$$

The equality $\delta(X;T) = n$ means as usual that $\delta(X;T) \geqslant n$ and $\delta(X;T) \not\geqslant n+1$. Evidently, $\delta(X;T) = -1$ iff $\Theta(T) = \varnothing$ or $\Theta(T) = X$.

Note that $\delta(B^n; T_0) = n$, where $T_0(x) = -x$. It is not difficult to give an example of a T-space X with $\delta(X;T) = \infty$.

LEMMA 1.6. *Let $\varphi\colon X \to X'$ be an equivariant map such that $\varphi^{-1}(\Theta(T')) = \Theta(T)$. Then $\delta(X;T) \leqslant \delta(X', T')$.*

Proof. Suppose that (X, Y) is a T-pair and $Y' = \varphi(Y)$. Then $Y' \cap \Theta(T') = \varnothing$, so that (X', Y') is a T'-pair. According to Lemma 1.3 $\delta(X, Y; T) \leqslant \delta(X', Y'; T')$. whereby $\delta(X;T) \leqslant \delta(X';T')$.

LEMMA 1.7. $\delta(X;T)\leqslant\dim X$ *for any* T*-space* X.

This inequality follows immediately from Lemma 1.4.

THEOREM 1.3. *Let* $\delta(X;T)\geqslant n$. *Then for any map* $f\colon X\to R^k$ *the inequality* $\delta(A(f);T)\geqslant n-k$ *holds.*

Proof. There exists a $Y\subset X$ such that $\delta(X, Y; T)\geqslant n$. According to Theorem 1.1 $\delta(A(f), A(f)\cap Y; T)\geqslant n-k$, which implies $\delta(A(f);T)\geqslant n-k$.

EXAMPLE. Denote by T the involution $T\colon B^n\to B^n$ defined by

$$T(x_1,\dots,x_k,x_{k+1},\dots,x_n)=(-x_1,\dots,-x_k,x_{k+1},\dots,x_n)\,.$$

Then $\delta(B^n;T)=k$. Indeed, consider B^k with the involution $T_0(x)=-x$. There exists an equivariant projection $\pi\colon B^n\to B^k$ such that $\pi^{-1}(\Theta(T_0))=\Theta(T)$; hence $\delta(B^n;T)\leqslant\delta(B^k;T_0)=k$ (see Lemma 1.6). On the other hand, there exists an equivariant embedding $j^n\colon B^k\to B^n$ such that $j^{-1}(\Theta(T))=\Theta(T_0)$; hence $\delta(B^n;T)\geqslant\delta(B^k;T_0)=k$.

II. Homological approach to compacta with an arbitrary involution — the concept of δh-index.

1. Preliminaries. In the second part we shall study the special homological structure of a space on which acts an involution with fixed points, considering only the chains modulo $\boldsymbol{Z}_2$, whose simplexes are permuted with one another by the involution. We introduce in Section 3 the concept of the homological δ-index of such a space (δh-index). Its definition is based on the theory of the index of a periodical transformation acting on a topological space, developed by P. A. Smith in [3]. A detailed exposition of Smith's theory applied to involutions is given by Yang [4]. We shall list only those definitions and propositions which will be directly used here, and refer the reader to [4] for details.

Let P be a simplicial space which is the body of a finite simplicial complex, and let $T\colon P\to P$ be a fixed-point free involution. Then P is said to be a *simplicial* T*-space* if the simplexes of P are permuted with one another by T, i.e., if T is a simplicial map. It is easy to see that $T\tau\cap\tau=\emptyset$ for any simplex τ of P. T induces a chain mapping of the chains modulo $\boldsymbol{Z}_2$ of P into themselves, which we also denote by T. An n-chain $\varkappa$ is called a T*-invariant* n*-chain*, or simply a (T,n)-chain, if $T\varkappa=\varkappa$. All the (T,n)-chains in P form a group $C_n(P;T)$. An n-chain $\varkappa$ is a (T,n)-chain iff $\varkappa=\lambda+T\lambda$ for some n-chain λ.

Define as usual

$$Z_n(P;T)=\{\varkappa\in C_n(P;T)\mid\partial\varkappa=0\}\,,$$
$$B_n(P;T)=\partial C_{n+1}(P;T)\,,$$
$$H_n(P;T)=Z_n(P;T)/B_n(P;T)\,.$$

An equivariant simplicial map $\varphi\colon P\to P'$ defines a chain mapping

$$\Phi\colon C_n(P;T)\to C_n(P';T')$$

and then induces a homomorphism

$$\varphi_*\colon H_n(P;T)\to H_n(P';T')\,.$$

For any simplicial T-space P there exists a homomorphism

$$\nu\colon Z_n(P;T)\to\boldsymbol{Z}_2$$

defined by recurrence as follows:

Let $z=\varkappa+T\varkappa$ be a (T,n)-cycle. Then

$$\nu(z)=\begin{cases}I(\varkappa) & \text{if } n=0;\\ \nu(\partial\varkappa) & \text{if } n>0\end{cases}$$

where $I(\varkappa)$ is the index of the 0-chain $\varkappa$ (in our case $I(\varkappa)=1$ iff the number of all simplexes of $\varkappa$ is odd.)

A) ν is independent of the choice of $\varkappa$ and $\nu B_n(P;T)=0$. Then ν induces the so-called *index homomorphism*

$$\nu\colon H_n(P;T)\to\boldsymbol{Z}_2$$

such that, if $\zeta\in H_n(P;T)$ and z is a (T,n)-cycle in ζ,

$$\nu(\zeta)=\nu(z)\,.$$

B) If $\varphi\colon P\to P'$ is an equivariant simplicial map, then

$$\nu(\varphi_*(\zeta))=\nu(\zeta)\quad\text{for any }\zeta\in H_n(P;T)\,.$$

C) For any simplicial T-space P there is an integer n such that

$$\nu H_s(P;T)=\begin{cases}\boldsymbol{Z}_2 & \text{for } 0\leqslant s\leqslant n;\\ 0 & \text{for } s>n\,.\end{cases}$$

The integer n is called the *Smith index* of P and it is written $\operatorname{in}(P;T)$.

D) $\operatorname{in}(S^n;T_0)=n$ (where $T_0(x)=-x$) for any invariant simplicial subdivision of S^n.

2. The simplicial case. We shall extend the concept of simplicial T-space to the case of an arbitrary T.

Let P be a simplicial space which is the body of a finite simplicial complex and let $T\colon P\to P$ be an arbitrary involution. We say that P is a *simplicial* T*-space* if the following two conditions are satisfied:

i) T is simplicial,

ii) if $\tau=[a_0,\dots,a_s]$ is a simplex of P such that $T\tau=\tau$, then $Ta_i=a_i$ for $i=0,\dots,s$.

It is easy to see that the set $\Theta(T)$ of all fixed points of T is a simplicial subspace of P. Indeed, if $x\in\Theta(T)$ and τ is the minimal simplex (with respect to inclusion) of P containing x, then $\tau\cap T\tau$ is a simplex containing x; hence $T\tau=\tau$, so that $\tau\subset\Theta(T)$ (as follows from i)).

By an invariant n-chain in P, or (T, n)-chain, we mean an n-chain $\varkappa$ modulo Z_2 in P such that $T\varkappa = \varkappa$. A (T, n)-chain $\varkappa$ is a (T, n)-cycle if $\partial\varkappa = 0$. Two (T, n)-chains $\varkappa_1$ and $\varkappa_2$ are homologous if there exists a $(T, n+1)$-chain $\varkappa$ such that $\partial\varkappa = \varkappa_1 - \varkappa_2$.

The pair (P, Q) is said to be a *simplicial T-pair* if P is a simplicial T-space and Q is its invariant simplicial subspace nonintersecting $\Theta(T)$.

Let F be a closed invariant subset of the T-space X. We say that F is a *weak antipodal partition* in X if $X \setminus F = U_+ \cup U_- \cup U_0$ where U_+, U_- and U_0 are disjoint open subsets of X such that $TU_+ = U_-$ and $U_0 \subset \Theta(T)$.

The following lemma is the key tool for our investigations in part II.

LEMMA 2.1. *Let (P, Q) be a simplicial T-pair,* $\dim P \leqslant n$, *and let z be a $(T, n-1)$-cycle in Q with $\nu(z) = 1$ homologous to zero in P. Suppose that F is an $(n-1)$-dimensional simplicial subspace of P which is a weak antipodal partition in P. Then there is a $(T, n-2)$-cycle ζ in $F \cap Q$ with $\nu(\zeta) = 1$ homologous to zero in F.*

Proof. Since F is a weak antipodal partition in P, $P \setminus F = U_+ \cup U_- \cup U_0$ where U_+, U_- and U_0 are disjoint open subsets of P such that $TU_+ = U_-$ and $U_0 \subset \Theta(T)$. There exists in P a (T, n)-chain $\varkappa$ such that $\partial\varkappa = z$. Then $\varkappa = \varkappa_+ + \varkappa_- + \varkappa_0$, where $\varkappa_+$ (resp. $\varkappa_-$, $\varkappa_0$) consists of all simplexes of $\varkappa$ contained in $\overline{U}_+$ (resp. $\overline{U}_-$, $\overline{U}_0$). For an arbitrary simplex τ of P denote by $b_+(\tau)$ (resp. $b_-(\tau)$, $b_0(\tau)$) the number of all simplexes of $\varkappa_+$ (resp. $\varkappa_-$, $\varkappa_0$) containing τ.

Denote by λ the $(n-1)$-dimensional chain containing all $(n-1)$-dimensional simplexes τ^{n-1} of P such that

$$b_+(\tau^{n-1}) \equiv 1 (\mathrm{mod}\, 2) \quad \text{and} \quad b_-(\tau^{n-1}) \equiv 1 (\mathrm{mod}\, 2).$$

We are going to establish several properties of λ.

i) $T\lambda = \lambda$. This equality holds by $T\varkappa_+ = \varkappa_-$ and $b_+(T\tau) = b_-(\tau)$.

ii) All simplexes of λ lie in F. Indeed, if τ^{n-1} is a simplex of λ, then $b_+(\tau^{n-1}) \neq 0$ and $b_-(\tau^{n-1}) \neq 0$, whence $\tau^{n-1} \subset F$.

iii) Denote by z_+ (resp. z_-) the chain containing all simplexes τ^{n-1} of z with $b_+(\tau^{n-1}) \equiv 1 (\mathrm{mod}\, 2)$ (resp. $b_-(\tau^{n-1}) \equiv 1 (\mathrm{mod}\, 2)$). Then $z = z_+ + z_-$ since $b_0(\tau^{n-1}) = 0$ for any simplex τ^{n-1} of z, so that the number of all n-dimensional simplexes of $\varkappa$ containing τ^{n-1} is $b_+(\tau^{n-1}) + b_-(\tau^{n-1}) \equiv 1 (\mathrm{mod}\, 2)$. Obviously, $Tz_+ = z_-$. Consequently $\nu(\partial z_+) = 1$ (see A) in the previous section). We shall prove that

$$\partial\lambda = \partial z_+ .$$

Let τ^{n-2} be an arbitrary $(n-2)$-dimensional simplex of P. Set

$p =$ the number of all simplexes of λ containing τ^{n-2},

$q =$ the number of all simplexes of z_+ containing τ^{n-2}.

It is enough to prove that $p \equiv q (\mathrm{mod}\, 2)$. Let $\tau_1^{n-1}, \ldots, \tau_s^{n-1}$ be all $(n-1)$-dimensional simplexes of P containing τ^{n-2}. Consider the sum

$$N = \sum_{i=1}^{s} b_+(\tau_i^{n-1}) .$$

Clearly, N is even since every n-dimensional simplex τ^n of $\varkappa_+$ containing τ^{n-2} has exactly two $(n-1)$-dimensional faces, τ_i^{n-1} and τ_j^{n-1}, containing τ^{n-2}, so that τ^n is counted in N exactly twice. Consider the set A of all $(n-1)$-dimensional simplexes τ^{n-1} containing τ^{n-2} and such that $b_+(\tau^{n-1}) \equiv 1 (\mathrm{mod} 2)$. Then $A = A' \cup A''$, where

$$A' = \{\tau^{n-1} \in A |\ b_-(\tau^{n-1}) \equiv 1 (\mathrm{mod}\, 2)\} .$$
$$A'' = \{\tau^{n-1} \in A |\ b_-(\tau^{n-1}) \equiv 0 (\mathrm{mod}\, 2)\} ,$$

It is easy to check that

$$|A'| = p , \quad |A''| = q .$$

The first equality holds by the fact that $\tau^{n-1} \in A'$ iff both $b_+(\tau^{n-1})$ and $b_-(\tau^{n-1})$ are odd numbers, i.e., iff τ^{n-1} takes part in λ. Pass to the second equality. Let $\tau^{n-1} \in A''$, i.e., $b_+(\tau^{n-1})$ is odd and $b_-(\tau^{n-1})$ is even. We shall prove that τ^{n-1} takes part in z. Note that $b_0(\tau^{n-1}) = 0$. Indeed, suppose that $b_0(\tau^{n-1}) \neq 0$. Then $T\tau^{n-1} = \tau^{n-1}$ and if τ^n is a simplex of $\varkappa_+$ containing τ^{n-1}, $T\tau^n$ is a simplex of $\varkappa_-$ containing τ^{n-1}, so that $b_+(\tau^{n-1}) = b_-(\tau^{n-1})$, which is a contradiction. Since $b_0(\tau^{n-1}) = 0$, the number of all simplexes of $\varkappa$ containing τ^{n-1} is $b_+(\tau^{n-1}) + b_-(\tau^{n-1}) \equiv 1 (\mathrm{mod}\, 2)$; hence τ^{n-1} is a simplex of $z = \partial\varkappa$. It is clear that $\tau^{n-1} \in z_+$ ($b_+(\tau^{n-1})$ is odd). Conversely, if τ^{n-1} is a simplex of z_+, then $b_0(\tau^{n-1}) = 0$, $b_+(\tau^{n-1}) + b_-(\tau^{n-1}) \equiv 1 (\mathrm{mod}\, 2)$ and $b_+(\tau^{n-1}) \equiv 1 (\mathrm{mod}\, 2)$, whence $b_-(\tau^{n-1}) \equiv 0 (\mathrm{mod}\, 2)$, i.e., $\tau^{n-1} \in A''$. Consequently $|A''| = q$.

Obviously, $N \equiv (p+q) (\mathrm{mod}\, 2)$, so that $p \equiv q (\mathrm{mod}\, 2)$ whereby $\partial\lambda = \partial z_+$. Set $\zeta = \partial\lambda = \partial z_+$. Then ζ is a $(T, n-2)$-cycle in $F \cap Q$ homologous to zero in F (λ lies in F) and $\nu(\zeta) = \nu(\partial z_+) = 1$.

The lemma is proved.

3. Definition of the δh-index. Let X be a compact metric space with involution T. Fix $\varepsilon > 0$. An n-dimensional ε-chain $\varkappa$ in X modulo Z_2 in the sense of Vietoris is a linear form $\varkappa = \tau_1 + \ldots + \tau_s$ where the simplexes $\tau_i = (a_0, \ldots, a_n)$ are systems of $n+1$ points of X with $\operatorname{diam} \tau_i < \varepsilon$. The vertices of a simplex are not assumed to be different points of X. The vertices of $\varkappa$ are the vertices of all of its simplexes. For a simplex $\tau = (a_0, \ldots, a_s)$ put $T\tau = (Ta_0, \ldots, Ta_s)$. An n-dimensional ε-chain $\varkappa$ is said to be *invariant* (or (T, n, ε)-chain) if

i) a simplex τ takes part in $\varkappa$ iff so does $T\tau$,

ii) if $T\tau = \tau$, where $\tau = (a_0, \ldots, a_s)$ is a face of some n-dimensional simplex of $\varkappa$, then $Ta_i = a_i$ for $i = 0, \ldots, s$.

Evidently, if $\varkappa$ is a (T, n, ε)-chain, then $\partial\varkappa$ is a $(T, n-1, \varepsilon)$-chain.

By an n-dimensional *invariant true chain*, or a (T, n)-true chain, we understand a sequence $\varkappa = \{\varkappa_i\}$ of (T, n, ε_i)-chains $\varkappa_i$ such that $\lim_{i\to\infty} \varepsilon_i = 0$. Define as usual $\varkappa + \varkappa' = \{\varkappa_i + \varkappa_i'\}$ and $\partial\varkappa = \{\partial\varkappa_i\}$. Two invariant true chains $\varkappa$ and $\varkappa'$ are said to be *homologous* if there exists an invariant true chain λ such that $\partial\lambda = \varkappa - \varkappa'$. An invariant true chain z is an *invariant true cycle* if $\partial z = 0$. An invariant closed subset F of X is a *carrier* of the invariant true chain $\varkappa$ if all vertices of $\varkappa_i$ lie in F for any i.

A carrier of an invariant true chain is always assumed to be invariant. For a (T, n, ε)-chain $\varkappa$ denote by $\tilde{\varkappa}$ the body of the simplicial complex containing all simplexes of $\varkappa$, as well as all their faces. Clearly, T induces an involution $\tilde{T}$: $\tilde{\varkappa} \to \tilde{\varkappa}$ and $\tilde{\varkappa}$ is a simplicial $\tilde{T}$-space. Then $\varkappa$ may be regarded as a $(\tilde{T}, n)$-chain in $\tilde{\varkappa}$. Suppose now that $\varkappa$ is a cycle and $\Theta(\tilde{T}) = \emptyset$. Then the number $\nu(\varkappa)$ is defined.

Let $z = \{z_i\}$ be a (T, n)-true cycle with a carrier Y such that $Y \cap \Theta(T) = \emptyset$. Then $\nu(z_i)$ is defined for every i. We shall write $\nu(z) = 1$ if $\nu(z_i) = 1$ for i large enough; otherwise $\nu(z) = 0$. Recall that the Smith index of a T-space Y with $\Theta(T) = \emptyset$ is not less than n $(\operatorname{in}(Y; T) \geqslant n)$ if there exists in Y a (T, n)-true cycle z with $\nu(z) = 1$.

We are ready to introduce the concept of the homological δ-index (δh-index) of a T-space. As in part I, we shall do this for T-pairs first.

Definition 2.1. Let (X, Y) be a T-pair (so that $Y \cap \Theta(T) = \emptyset$). We say that the *homological δ-index of the pair (X, Y) is not less than n* if there exists an $(n-1)$-dimensional invariant true cycle z in Y with $\nu(z) = 1$ homologous to zero in X. Then we shall write

$$\delta h(X, Y; T) \geqslant n .$$

The equality $\delta h(X, Y; T) = n$ is equivalent to $\delta h(X, Y; T) \geqslant n$ and $\delta h(X, Y; T) \ngeqslant n+1$. If at least one of the sets X, Y, $\Theta(T)$ is empty, set $\delta h(X, Y; T) = -1$. It follows from the definition that $\delta h(X, Y; T) \geqslant n$ implies $\operatorname{in}(Y; T) \geqslant n-1$.

Definition 2.2. Let X be a T-space. We say that the *homological δ-index of X is not less than n* if there exists in X such a closed invariant subset Y non-intersecting $\Theta(T)$ that $\delta h(X, Y; T) \geqslant n$. Then we write

$$\delta h(X; T) \geqslant n .$$

As usual, $\delta h(X; T) = n$ iff $\delta h(X; T) \geqslant n$ and $\delta h(X; T) \ngeqslant n+1$.

Example. $\delta h(B^n, S^{n-1}; T_0) = n$, where $T_0(x) = -x$. It is enough to take for every natural i an invariant subdivision of B^n of mesh $< 1/i$ and to denote by $\varkappa_i$ the invariant $1/i$-chain containing all of its n-dimensional simplexes. Then $z = \{\partial \varkappa_i\}$ is an invariant true $(n-1)$-cycle in S^{n-1} with $\nu(z) = 1$ (see D)). Therefore $\delta h(B^n, S^{n-1}; T_0) = n$.

4. Several properties of δh. The next two lemmas are similar to Lemmas 1.3 and 1.6:

Lemma 2.2. *Let φ: $X \to X'$ be an equivariant map which maps the T-pair (X, Y) into the T'-pair (X', Y'). Then $\delta h(X, Y; T) \leqslant \delta h(X', Y'; T')$.*

Proof. Suppose that $\delta h(X, Y; T) \geqslant n$ and let z be a $(T, n-1)$-true cycle in Y with $\nu(z) = 1$ homologous to zero in X. Then $\varphi(z)$ is a $(T', n-1)$-true cycle in Y' homologous to zero in X' and $\nu(\varphi(z)) = \nu(z) = 1$ (see B)). Hence $\delta h(X', Y'; T') \geqslant n$.

Lemma 2.3. *Let φ: $X \to X'$ be an equivariant map such that $\varphi^{-1}(\Theta(T')) = \Theta(T)$. Then $\delta h(X; T) \leqslant \delta h(X'; T')$.*

Proof. Suppose $\delta h(X; T) \geqslant n$. There exists such a closed invariant $Y \subset X$ that $\delta h(X, Y; T) \geqslant n$. Then $Y' = \varphi(Y)$ is a closed invariant subset of X' such that $Y' \cap \Theta(T') = \emptyset$, and according to Lemma 2.2 $\delta h(X', Y'; T') \geqslant n$, so that $\delta h(X'; T') \geqslant n$.

Denote by ϱ the metric in X and by $\overline{O_\delta F} = \{x \in X |\ \varrho(x, F) \leqslant \delta\}$ — the closed δ-neighbourhood of F in X.

Lemma 2.4. *Let (X, Y) be a T-pair and let F be a closed invariant subset of X. If for any closed invariant neighbourhood $\overline{OF}$ of F in X we have $\delta h(\overline{OF}, \overline{OF} \cap Y; T) \geqslant n$, then the inequality $\delta h(F, F \cap Y; T) \geqslant n$ holds.*

Proof. Fix $\varepsilon > 0$. Since X is compact, there exists such a positive $\gamma < \frac{1}{3}\varepsilon$ that $\varrho(x, y) < \gamma$ implies $\varrho(Tx, Ty) < \frac{1}{3}\varepsilon$ for any $x, y \in X$. We shall prove that there exists a $\delta > 0$ such that $x \in \overline{O_\delta F} \cap Y$ implies $\varrho(x, F \cap Y) \leqslant \gamma$. Suppose the contrary. Then for any $\delta = 1/i$ there exists an $x_i \in \overline{O_{1/i} F} \cap Y$ with $\varrho(x_i, F \cap Y) \geqslant \gamma$. We may assume that $x_i \to x_0 \in Y$. Then $\varrho(x_0, F \cap Y) \geqslant \gamma$. Since $x_i \in \overline{O_{1/i} F} \cap Y$, there exists a $y_i \in F$ with $\varrho(x_i, y_i) < 1/i$; hence $y_i \to x_0$, so that $x_0 \in F \cap Y$, which is a contradiction.

Set $\beta = \min(\gamma, \delta)$. Since $\delta h(\overline{O_\beta F}, \overline{O_\beta F} \cap Y; T) \geqslant n$, there exists in $\overline{O_\beta F} \cap Y$ a $(T, n-1)$-true cycle $z = \{z_i\}$ with $\nu(z) = 1$ homologous to zero in $\overline{O_\beta F}$. Let $\partial \varkappa = z$, where $\varkappa = \{\varkappa_i\}$ is a (T, n)-true chain in $\overline{O_\beta F}$. Take i so large that all simplexes of $\varkappa_i$ have a diameter $< \frac{1}{3}\varepsilon$ and $\nu(z_i) = 1$. Let a_s be a vertex of $\varkappa_i$ which is not a vertex to $\partial \varkappa_i$. There exists a point $a'_s \in F$ such that $\varrho(a_s, a'_s) < \beta \leqslant \gamma < \frac{1}{3}\varepsilon$. Then $\varrho(Ta_s, Ta'_s) < \frac{1}{3}\varepsilon$. Set $(Ta_s)' = Ta'_s$. In the case where a_s is a vertex of $\partial \varkappa_i$ we have $a_s \in \overline{O_\beta F} \cap Y$. But $\beta \leqslant \delta$; consequently there exists an $a'_s \in F \cap Y$ with $\varrho(a_s, a'_s) \leqslant \gamma < \frac{1}{3}\varepsilon$. Thus $\varrho(Ta_s, Ta'_s) < \frac{1}{3}\varepsilon$ and we shall set as above $(Ta_s)' = Ta'_s$ (note that $Ta'_s \in F \cap Y$). In this way, to every simplex of $\varkappa_i$ $\tau = (a_0, \ldots, a_n)$ corresponds some $\tau' = (a'_0, \ldots, a'_n)$. We will show that $\operatorname{diam} \tau' < \varepsilon$. Indeed, for any two vertices $a'_i, a'_j \in \tau'$ we have

$$\varrho(a'_i, a'_j) \leqslant \varrho(a'_i, a_i) + \varrho(a_i, a_j) + \varrho(a_j, a'_j) < 3 \cdot \tfrac{1}{3}\varepsilon = \varepsilon .$$

If $\varkappa_i = \tau_1 + \ldots + \tau_p$, set $\varkappa'_i = \tau'_1 + \ldots + \tau'_p$. Then $\varkappa'_i$ is a (T, n, ε)-chain in F. It is not difficult to prove that $(\partial \varkappa_i)' = \partial \varkappa'_i$, so that $z'_i = \partial \varkappa'_i$ is a $(T, n-1, \varepsilon)$-cycle in $F \cap Y$ homologous to zero in F. Since z'_i is the image of z_i under an equivariant map, $\nu(z'_i) = \nu(z_i) = 1$ (see B)).

Clearly, when $\varepsilon \to 0$ we obtain a $(T, n-1)$-true cycle $z' = \{z'_i\}$ in $F \cap Y$ with $\nu(z') = 1$ homologous to zero in F. Hence $\delta h(F, F \cap Y; T) \geqslant n$.

One can prove in the same way next lemma.

Lemma 2.5. *Let (X, Y) be a T-pair. If, for any closed invariant neighbourhood $\overline{OY}$ of Y, $\operatorname{in}(\overline{OY}; T) \geqslant n$, then $\operatorname{in}(Y; T) \geqslant n$.*

If we replace the simplicial T-pair in Lemma 2.1 by an arbitrary T-pair, we obtain the following important

LEMMA 2.6. *Let F be an antipodal partition in the T-space X and $\delta h(X, Y; T) \geqslant n$. Then*

$$\delta h(F, F \cap Y; T) \geqslant n-1 .$$

Proof. Let z be a $(T, n-1)$-true cycle in Y with $\nu(z) = 1$ homologous to zero in X. Denote by OF an arbitrary open invariant neighbourhood of F in X. As follows from Lemma 2.4, it is enough to prove that

$$\delta h(\overline{OF}, \overline{OF} \cap Y; T) \geqslant n-1 .$$

There exists in X a (T, n)-true chain $\varkappa = \{\varkappa_i\}$ with $\partial\varkappa = z$. Let i_0 be so large that $\nu(z_i) = 1$ for $i > i_0$. Consider the simplicial space $P_i = \tilde{\varkappa}_i$. Denote by F_i the union of all simplexes of P_i contained in OF. Then F_i is an invariant subset of P_i which is an antipodal partition in P_i for i large enough. Indeed, $X \setminus F = U_+ \cup U_-$, where U_+ and U_- are disjoint open subsets of X such that $TU_+ = U_-$; hence $\overline{P_i \setminus F_i} = P_i^+ \cup P_i^-$ where P_i^+ (resp. P_i^-) is the union of all simplexes of P_i intersecting $U_+ \setminus OF$ (resp. $U_- \setminus OF$). But $P_i^+ \cap P_i^- = \emptyset$ for i large enough and $TP_i^+ = P_i^-$, i.e., F_i is an antipodal partition in P_i. It is easy to see then that its $(n-1)$-dimensional skeleton $F_i^{(n-1)}$ is a weak antipodal partition in P_i. Denote by Q_i the body of z_i. Then Q_i is an invariant simplicial subspace of P_i and z_i is a $(T, n-1)$-cycle in Q_i with $\nu(z_i) = 1$ homologous to zero in P_i. According to Lemma 2.1, there exists a $(T, n-2)$-cycle ζ_i in $F_i^{(n-1)} \cap Q_i$ with $\nu(\zeta_i) = 1$ homologous to zero in $F_i^{(n-1)}$. Consider the $(T, n-2)$-true cycle $\zeta = \{\zeta_i\}$. Evidently, ζ lies in $\overline{OF} \cap Y$, $\nu(\zeta) = 1$, and ζ is homologous to zero in $\overline{OF}$. Hence $\delta h(\overline{OF}, \overline{OF} \cap Y; T) \geqslant n-1$ whereby $\delta h(F, F \cap Y; T) \geqslant n-1$.

We shall now establish the relationship with δ.

LEMMA 2.7. a) *$\delta h(X, Y; T) \leqslant \delta(X, Y; T)$ for any T-pair (X, Y).*

b) *$\delta h(X; T) \leqslant \delta(X; T)$ for any T-space X.*

Proof. a) We shall prove that $\delta h(X, Y; T) \geqslant n$ implies $\delta(X, Y; T) \geqslant n$ by induction on n. The case $n = 0$ is trivial. Assume that the lemma is true for $n = k$ and let $\delta h(X, Y; T) \geqslant k+1$. Then for any antipodal partition F in X we have $\delta h(F, F \cap Y; T) \geqslant k$; hence $\delta(F, F \cap y; T) \geqslant k$. According to Lemma 1.5 $\delta(X, Y; T) \geqslant k+1$.

b) follows immediately from a).

Note that the inverse inequalities are not always valid — see the example in the last section.

COROLLARY. a) *$\delta h(X, Y; T) \leqslant \dim X$ for any T-pair (X, Y).*

b) *$\delta h(X; T) \leqslant \dim X$ for any T-space X.*

5. The mapping theorems. The next two theorems are the main results in part II. They give estimates of the δh-index of the set

$$A(f) = \{x \in X \mid f(Tx) = f(x)\}$$

where f maps X into a Euclidean space.

THEOREM 2.1. *Let (X, Y) be a T-pair with $\delta h(X, Y; T) \geqslant n$ and $f\colon X \to R^k$ maps X into the k-dimensional Euclidean space. Then the following inequality holds:*

$$\delta h(A(f), A(f) \cap Y; T) \geqslant n-k .$$

Proof. The proof is identical with the proof of Theorem 1.1 in part I. The only difference is that we must refer to Lemma 2.6 instead of Lemma 1.5.

As a corollary, we easily obtain a famous theorem due to Yang (see [4]).

COROLLARY. *Let $\operatorname{in}(X; T) \geqslant n$. Then for any map $f\colon X \to R^k$ holds the inequality*

$$\operatorname{in}(A(f); T) \geqslant n-k .$$

Proof. Denote by $X_1 = CX$ the cone over X with a vertex a. Clearly, T may be extended to $T_1\colon X_1 \to X_1$ with $\Theta(T_1) = \{a\}$. There existsi in X a (T, n)-true cycle z with $\nu(z) = 1$. Then $\delta h(X_1, X; T_1) \geqslant n+1$, since z is homologous to zero in X_1. Denote by $f_1\colon X_1 \to R^k$ an arbitrary extension of f. According to Theorem 2.1

$$\delta h(A(f_1), A(f_1) \cap X; T_1) \geqslant n+1-k .$$

But $A(f) = A(f_1) \cap X$, whence $\operatorname{in}(A(f); T) \geqslant n-k$.

THEOREM 2.2. *Let $\delta h(X; T) \geqslant n$. Then for any map $f\colon X \to R^k$ holds the inequality $\delta h(A(f); T) \geqslant n-k$.*

Proof. There exists in X such a closed invariant Y that $\delta h(X, Y; T) \geqslant n$. According to the previous theorem $\delta h(A(f), A(f) \cap Y; T) \geqslant n-k$, whence $\delta h(A(f); T) \geqslant n-k$.

The following theorem is a generalization of Borsuk's theorem mentioned in part I and gives an estimate of the Smith index of a wide class of T-spaces. In the case $X = B^n$, $Y = S^{n-1}$ we get a theorem due to D. G. Bourgin (see [2]).

THEOREM 2.3. *Let (X, Y) be a T-pair with $\delta h(X, Y; T) \geqslant n$ and C be an invariant partition in X between $\Theta(T)$ and Y. Then $\operatorname{in}(C; T) \geqslant n-1$.*

Proof. There exists in Y a $(T, n-1)$-true cycle $z = \{z_i\}$ with $\nu(z) = 1$ homologous to zero in X. Let $\partial\varkappa = z$. As follows from Lemma 2.5, it is enough to prove that $\operatorname{in}(\overline{OC}; T) \geqslant n-1$ for any open invariant neighbourhood OC of C such that $\overline{OC} \cap \Theta(T) = \emptyset$. Consider an arbitrary OC. T induces an involution $T_i\colon P_i \to P_i$ in the simplicial space $P_i = \tilde{\varkappa}_i$. Denote by C_i the subset of P_i containing all of its simplexes lying in OC. Then, for i large enough, C_i is an invariant partition in P_i between $Q_i = \tilde{z}_i$ and $\Theta(T_i)$. Indeed, since C is a partition in X between $\Theta(T)$ and Y, $X \setminus C = U_1 \cup U_2$, where U_1 and U_2 are non-intersecting open invariant subsets of X such that $U_1 \supset \Theta(T)$, $U_2 \supset Y$. Evidently, for i large enough, $\overline{P_i \setminus C_i} = P_i^1 \cup P_i^2$, where P_i^1 and P_i^2 are disjoint invariant simplicial subsets of P_i such that $P_i^1 \supset \Theta(T_i)$ and $P_i^2 \supset Q_i$. P^1 (resp. P_i^2) contains all simplexes of P_i intersecting $U_1 \setminus OC$ (resp. $U_2 \setminus OC$). Let λ_i denote the (T_i, n)-chain containing all simplexes of P_i^1. Note that $\tilde{\lambda}_i \cap \Theta(T_i) = \emptyset$. Set

$$\zeta_i = \partial\lambda_i + z_i .$$

Then ζ_i is a $(T_i, n-1)$-cycle homologous to z_i; thus $\nu(\zeta_i) = \nu(z_i) = 1$ (see A)). We shall prove that all simplexes of ζ_i lie in C_i. Suppose that τ^{n-1} is an $(n-1)$-simplex of P_i^1 which does not lie in C_i. Then all n-simplexes of P_i containing τ^{n-1} take part in λ_i. There are two possibilities:

i) τ^{n-1} does not take part in z_i. Then τ^{n-1} is not a simplex of ζ_i, since the number of all n-simplexes of λ_i containing τ^{n-1} is even.

ii) τ^{n-1} takes part in z_i. Then τ^{n-1} takes part in $\partial\lambda_i$, and therefore it is not a simplex of ζ_i.

Clearly, $\zeta = \{\zeta_i\}$ is a $(T, n-1)$-true cycle in $\overline{OC}$ with $\nu(\zeta) = 1$, i.e., $\mathrm{in}(\overline{OC}; T) \geqslant n-1$, whence $\mathrm{in}(C; T) \geqslant n-1$.

Remark. The inequality $\mathrm{in}(C; T) \geqslant n-1$ for any partition C between $\Theta(T)$ and Y is not sufficient for $\delta h(X, Y; T) \geqslant n$. Let (X, Y) be the 2-dimensional T-pair constructed in Section 6. Then $\delta(X, Y; T) = 2$, so that for any invariant partition C in X between $\Theta(T)$ and Y we have $B(C; T) \geqslant 1$, whence $\mathrm{in}(C; T) \geqslant 1$ (it is not difficult to prove it). On the other hand, $\delta h(X, Y; T) = 1$.

The last two theorems give other generalizations of Borsuk's theorem.

THEOREM 2.4. *Let $\delta h(X, Y; T) \geqslant n$ and C be a closed invariant subset of X nonintersecting Y and $\Theta(T)$. Suppose that no invariant k-cycle z^k in Y with $\nu(z^k) = 1$ is homologous to zero in $X \setminus C$. Then*

$$B(C; T) \geqslant n-k-1 .$$

Proof. We shall proceed by induction on n. Let $n = 1$, $k = 0$. (The case $n = 1$, $k \geqslant 1$ is trivial). Since $\delta h(X, Y; T) \geqslant 1$, there exists in Y an invariant 0-cycle z^0 with $\nu(z^0) = 1$ homologous to zero in X. But z^0 is not homologous to zero in $X \setminus C$, whence $C \neq \emptyset$, i.e., $B(C; T) \geqslant 0$.

Assume the theorem to be valid for $n = s$ and let $\delta h(X, Y; T) \geqslant s+1$. We have to prove $B(C; T) \geqslant s+1-k-1 = s-k$. Suppose $B(C; T) \leqslant s-k-1$. Then C may be represented as the union $C = \bigcup_{i=1}^{s-k} C_{\pm i}$ of its closed cubsets such that $C_{+i} \cap C_{-i} = \emptyset$ and $TC_{+i} = C_{-i}$ (see [6]). Consider the set $C_{+1} \cup C_{-1}$. There exists in X an antipodal partition X_1 nonintersecting $C_{+1} \cup C_{-1}$ (see Lemma 1.2). According to Lemma 2.6 $\delta h(X_1, X_1 \cap Y; T) \geqslant s$. Write $C_1 = \bigcup_{i=2}^{s-k} C_{\pm i}$. Then $B(C_1; T) \leqslant s-k-2$. On the other hand, no invariant k-cycle z^k in $X_1 \cap Y$ with $\nu(z^k) = 1$ is homologous to zero in $X_1 \setminus C_1$. Then the inequality $B(C_1; T) \geqslant s-k-1$ holds by the induction hypothesis, which is a contradiction.

Remark. We cannot replace the inequality $\delta h(X, Y; T) \geqslant n$ by $\delta(X, Y; T) \geqslant n$. Let (X, Y) be the T-pair constructed in the next section and $C = \emptyset$. Then no invariant 1-cycle z^1 in Y with $\nu(z^1) = 1$ is homologous to zero in $X \setminus C = X$, but $B(C; T) = -1 < 2-1-1 = 0$.

QUESTION. Can we replace the inequality $B(C; T) \geqslant n-k-1$ by

$$\mathrm{in}(C; T) \geqslant n-k-1 ?$$

An affirmative answer in the case $k = 0$ is given by Theorem 2.3; the case $k = n-1$ is trivial.

Assume for convenience

$$S^k = \{x \in S^n |\ x_{k+2} = \ldots = x_{n+1} = 0\} ,$$

so that $S^k \subset S^n$ for $k < n$. Denote by $\sigma^n = \{\sigma_i^n\}$ some invariant true n-cycle in S^n, such that σ_i^n is formed of all n-simplexes of some invariant subdivision of S^n of mesh $< 1/i$, which induces an invariant subdivision of S^k for any $k < n$.

THEOREM 2.5. *Let $X \subset R^N$ and (X, S^{n-1}) be a T_0-pair with $\delta h(X, S^{n-1}; T_0) \geqslant n$, where $T_0(x) = -x$. Suppose that C is a closed invariant subset of X such that $C \not\ni \Theta$, $C \cap S^{n-1} = \emptyset$, and the cycle σ^k on S^k is not homologous to zero in $X \setminus C$. Then*

$$B(C; T_0) \geqslant n-k-1 .$$

The proof is identical with the proof of the previous theorem; we must only choose X_1 in such a way that $X_1 \cap S^s = S^{s-1}$.

COROLLARY. *Let C be a closed invariant subset of B^n such that $C \not\ni \Theta$, $C \cap S^{n-1} = \emptyset$ and the cycle σ^k on S^k is not homologous to zero in $B^n \setminus C$. Then $B(C; T_0) \geqslant n-k-1$.*

Clearly, when $k = 0$, C is a partition between Θ and S^{n-1}; hence we again get Borsuk's theorem.

6. An example. We are going to give an example of a 2-dimensional simplicial T-pair (X, Y) such that

$$1 = \delta h(X, Y; T) < \delta(X, Y; T) = 2 .$$

Let

$$D = \{x \in R^3 |\ x_1^2 + x_2^2 \leqslant 16,\ x_3 = 0\} ,$$

$$U_+ = \{x \in R^3 |\ (x_1-2)^2 + x_2^2 < 1,\ x_3 = 0\} , \quad U_- = -U_+ ,$$

$$A = D \setminus (U_+ \cup U_-) .$$

Also denote by B_+ and B_- the cylinders

$$B_+ = \{x \in R^3 |\ (x_1-2)^2 + x_2^2 = 1,\ 0 \leqslant x_3 \leqslant 1\} , \quad B_- = -B_+$$

and put $X_1 = A \cup B_+ \cup B_-$. Obviously, X_1 is symmetric with respect to the origin Θ. Let R be the following relation in X_1: R identifies only the pairs $(x', x'') \in X_1^2$, where $x' \in \mathrm{Fr}\, U_+$ and $x' = (x_1, x_2, 0)$, $x'' = (x_1, -x_2, 1)$, or $x' \in \mathrm{Fr}\, U_-$ and $x' = (x_1, x_2, 0)$, $x'' = (x_1, -x_2, -1)$. Denote by $\psi\colon X_1 \to X_1/R$ the canonical map and finally put

$$X = X_1/R .$$

Clearly, the central symmetry induces an involution T in X with $\Theta(T) = \{\Theta\}$. Then ψ is an equivariant map. The sets $C_+ = \psi(B_+)$ and $C_- = \psi(B_-)$ are Klein bottles such that $TC_+ = C_-$. Let

$$Y = \{x \in R^3 |\ x_1^2 + x_2^2 = 16,\ x_3 = 0\}$$

be the boundary of D. We shall prove that the T-pair (X, Y) meets the case.

a) $\delta h(X, Y; T) = 1$. We may consider (X, Y) as a simplicial T-pair for some simplicial T-invariant subdivision of X of a small mesh. Then the sets $\Sigma_+ = \psi(\operatorname{Fr} U_+)$ and $\Sigma_- = \psi(\operatorname{Fr} U_-)$ are 1-dimensional simplicial subspaces of X homeomorphic with S^1. The inequality $\delta h(X, Y; T) \geqslant 1$ is obvious. Suppose now that

$$\delta h(X, Y; T) \geqslant 2,$$

i.e., that there exists in Y an invariant 1-cycle z with $\nu(z) = 1$ homologous to zero in X. Let $\partial\varkappa = z$. Evidently, all 2-simplexes lying in A take part in $\varkappa$. They form a chain $\varkappa_1$. Since $\partial\varkappa_1 \neq z$, there is a simplex of $\varkappa$ lying in C_+. Then all 2-simplexes of C_+ take part in $\varkappa$. Consequently, $\varkappa$ contains all 2-simplexes of X. But every 1-simplex τ of Σ_+ is contained in exactly 3 2-simplexes, and thus τ is a simplex of $\partial\varkappa$, which contradicts the equality $\partial\varkappa = z$.

b) $\delta(X, Y; T) = 2$. The inequality $\delta(X, Y; T) \leqslant 2$ holds by $\delta(X, Y; T) \leqslant B(Y; T)+1 = 2$ (see Lemma 1.4).

Suppose that $\delta(X, Y; T) \leqslant 1$. According to Lemma 1.5 there exists in X an antipodal partition F with $\delta(F, F \cap Y; T) \leqslant 0$, i.e., $F = F_1 \cup F_2$, where F_1 and F_2 are closed invariant sets such that $F_1 \ni \Theta$, $F_2 \supset F \cap Y$ and $F_1 \cap F_2 = \emptyset$. Let $X \setminus F = U_+ \cup U_-$, where U_+ and U_- are disjoint open subsets of X such that $TU_+ = U_-$. Denote by L the space

$$L = S^2 \cup \{x \in R^3 \mid -1 \leqslant x_1 \leqslant 1,\ x_2 = x_3 = 0\}.$$

We shall find an equivariant map $\varphi\colon X \to L$ such that $\varphi(Y) \subset S$, where $S = \{x \in S^2 \mid x_1 = 0\}$. Since $B(F_2 \cup Y; T) \leqslant 1$, there exists an equivariant map $\lambda\colon F_2 \cup Y \to S$. Let $\lambda_1\colon F \cup Y \to S \cup \{\Theta\}$ be the following equivariant extension of λ:

$$\lambda_1(x) = \begin{cases} \lambda(x) & \text{for } x \in F_2 \cup Y, \\ \Theta & \text{for } x \in F_1. \end{cases}$$

Consider the spaces $L_+ = \{x \in L \mid x_1 \geqslant 0\}$ and $L_- = -L_+$. Clearly, L_+ and L_- are contractible and $L_+ \cup L_- = L$, $L_+ \cap L_- = S \cup \{\Theta\}$. The map λ_1 admits an arbitrary extension $\tilde{\lambda}_1\colon F \cup Y \cup U_+ \to L_+$. Define $\varphi\colon X \to L$ by

$$\varphi(x) = \begin{cases} \tilde{\lambda}_1(x) & \text{for } x \in F \cup Y \cup U_+, \\ -\tilde{\lambda}_1(Tx) & \text{for } x \in U_-. \end{cases}$$

Then φ is the required equivariant map.

Consider the commutative triangle

$$\begin{array}{ccc} \Sigma_+ & \xrightarrow{\varphi_1} & L \\ {\scriptstyle j}\downarrow & \nearrow{\scriptstyle \varphi_2} & \\ C_+ & & \end{array}$$

where $\varphi_1 = \varphi|\Sigma_+$, $\varphi_2 = \varphi|C_+$ and j is the inclusion map. Then the triangle

$$\begin{array}{ccc} \pi_1(\Sigma_+) & \xrightarrow{\varphi_{1*}} & \pi_1(L) \\ {\scriptstyle j_*}\downarrow & \nearrow{\scriptstyle \varphi_{2*}} & \\ \pi_1(C_+) & & \end{array}$$

is also commutative. It is easy to prove that the fundamental group $\pi_1(L)$ is isomorphic to $\boldsymbol{Z}$. Let α be the formant of $\pi_1(\Sigma_+)$. Clearly, $2j_*(\alpha) = 0$ (C_+ is a Klein bottle). Then

$$2\varphi_{1*}(\alpha) = \varphi_{1*}(2\alpha) = \varphi_{2*}j_*(2\alpha) = \varphi_{2*}(2j_*(\alpha)) = 0,$$

whence $\varphi_{1*}(\alpha) = 0$, since $\pi_1(L) \approx \boldsymbol{Z}$. Therefore $\varphi_{1*} \equiv 0$. Consider now $\operatorname{Fr} U_+$ and the map $\mu = \varphi\psi|\operatorname{Fr} U_+\colon \operatorname{Fr} U_+ \to L$. Evidently, $\mu_* \equiv 0$; hence μ admits an extension $\tilde{\mu}\colon \overline{U}_+ \to L$. Define the map $h\colon D \to L$ by

$$h(x) = \begin{cases} \varphi\psi(x) & \text{for } x \in A, \\ \tilde{\mu}(x) & \text{for } x \in \overline{U}_+, \\ -\tilde{\mu}(-x) & \text{for } x \in \overline{U}_-. \end{cases}$$

This is an equivariant map such that $h(Y) \subset S$; hence the inequality

$$\delta(D, Y; T_0) \leqslant \delta(L, S; T_0)$$

holds by Lemma 1.3 ($T_0(x) = -x$)). On the other hand, $\delta(D, Y; T_0) = 2$, $\delta(L, S; T_0) = 1$, which is a contradiction.

Remark. It is not difficult to give an example (based on the same idea) of a 2-dimensional T-space X with

$$1 = \delta h(X; T) < \delta(X; T) = 2.$$

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