Relative expressiveness of the edge/adjacency language for graph theory

by

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Abstract. A strengthening of Whitney's edge-isomorphism theorem characterizes those graph-theoretic properties which can be expressed in a formal language based on edges and edge adjacency. The connection with line graphs is also considered.

The similarity (or analogy or duality) between vertices and edges has been frequently mentioned in the graph-theoretic literature, often in connection with edge-isomorphism or line graphs. The lack of similarity is also evident: edge adjacency is clearly a more awkward notion than vertex adjacency. As a contribution to the foundations of graph theory, we take a "linguistic" approach and determine how much of graph theory can be done in languages based on edges and their adjacency, without mentioning vertices. This study is significantly different from, yet closely related to Whitney's "everything except K_3 and $K_{1,3}$" result [6, Thm. 1] on edge-isomorphism. It is also related to the popular line graph approach.

Because of our interest in languages and their relative expressiveness, we must pay attention to several logical matters. For our limited purposes, however, we presuppose no particular logical background beyond that reasonably expected of a mathematician. In fact, we shall remain intentionally vauge about exactly what a language is until we need to be specific to prove our theorem. We consider only finite graphs without loops or multiple edges, conforming to the terminology and notation of [2]. As a convenience, we consider only connected graphs. (This allows our results to be stated in terms of K_3 and $K_{1,3}$ rather than graphs all of whose components are either K_3 or $K_{1,3}$.) The excellent survey of line graphs by Hemminger and Beineke [4] is our primary reference.

Suppose $\mathscr{L}_0$ is any language based on vertices and vertex adjacency, with $\mathscr{L}_1$ an equally sophisticated language, except built from edges and edge adjacency instead. It is easy to see that anything expressible in $\mathscr{L}_1$ (or, for that matter, in any simple sort of graph-theoretic language) can be mechanically translated into $\mathscr{L}_0$ by replacing all (mentions of) edges by pairs of adjacent vertices. Our question involves going in the other direction, from $\mathscr{L}_0$ to $\mathscr{L}_1$, and is partially motivated by various applications where edges are more naturally interpreted than vertices.

It is easy to see that certain $\mathscr{L}_0$ notions cannot be expressed at all in $\mathscr{L}_1$. A very simple example is isomorphism with K_3; this is because every description of K_3 in terms of edges (for instance, the $\mathscr{L}_1$-expressible sentence σ_0 stating "there are exactly three edges, every two of which are adjacent") describes $K_{1,3}$ just as well, since K_3 and $K_{1,3}$ are edge-isomorphic. As another example, two-regularity cannot be expressed as a sentence σ within $\mathscr{L}_1$ since if it could, then the conjunction $\sigma_0 \& \sigma$ would express being isomorphic with K_3. The same argument can be used to prove the following lemma. We say that a property *differentiates* between K_3 and $K_{1,3}$ if and only if it is true of exactly one of them.

LEMMA. *Every $\mathscr{L}_0$ property which differentiates between K_3 and $K_{1,3}$ cannot be expressed within $\mathscr{L}_1$.*

Thus properties such as Eulerian, regular, complete, nonseparable, and bipartite are automatically inexpressible within an edge/adjacency language. Properties such as three-regular and diameter $\leqslant 2$ are left open, and the reader is encouraged to try expressing them using just edges and edge adjacency. As an example of a property which easily is so expressible, consider "every 4-cycle has a diagonal."

To be specific, our language $\mathscr{L}_0$ will consist of lower-case variables (to be interpreted as vertices), binary relation symbols for equality ($=$) and adjacency ($-$), propositional connectives for negation ($\neg$), conjunction (&), disjunction ($=$ "or") ($\vee$) and implication ($\rightarrow$), and the universal ($\forall$) and existential ($\exists$) quantifiers. Formulas are built in the natural manner using parantheses. An *$\mathscr{L}_0$ sentence* is any formula in which all variables are quantified, and so sentences correspond to graph-theoretic properties which are expressible within the language. The language $\mathscr{L}_1$ is defined in the same way except that upper-case variables are used and are to be interpreted as edges.

As we have defined them so far, $\mathscr{L}_0$ and $\mathscr{L}_1$ are what are called "first-order languages" and so are inherently incapable of expressing many common graph-theoretic properties such as connected, regular, or Eulerian. (Much of the problem with first-order languages for graph theory is shown in [3]; [1] is an excellent general survey.) To remedy this, our languages should be strengthened; for instance, allowing conjunction and disjunction over infinite sets of formulas will produce a very expressive language for which our theorem (below) still holds. Other possible strengthenings are described in [1]. In any case, the theorem can be interpreted as expressing the relative expressibility of $\mathscr{L}_0$ and $\mathscr{L}_1$ with respect to a strengthening of our basic first-order formulation.

THEOREM. *A sentence of $\mathscr{L}_0$ can be equivalently expressed within $\mathscr{L}_1$ if and only if it does not differentiate between K_3 and $K_{1,3}$.*

Proof. The easy direction was observed as the lemma. For the hard direction, suppose σ is an $\mathscr{L}_0$ sentence which does not differentiate K_3 from $K_{1,3}$. To translate σ into $\mathscr{L}_1$ we need to be able to specify vertices in terms of edges. (Note that even a vertex of degree three cannot be specified as three mutually adjacent edges, since a triangle will fit that description as well.) Also, we somehow have to use the assumption that σ is "nondifferentiating."

To begin, each universal quantifier in σ can be replaced with an existential quantifier and negations, since $\forall v\varphi$ is equivalent to $\neg\exists v\neg\varphi$ for each formula φ of $\mathscr{L}_0$. Let $A(X, Y, Z)$ abbreviate the $\mathscr{L}_1$ formula asserting that edges X, Y and Z are pairwise adjacent. Let $T(X, Y, Z)$ similarly abbreviate that every edge is adjacent to an even number of the edges X, Y and Z. For the time being we shall restrict our attention to (connected) graphs of order greater than four. For such graphs, $A(X, Y, Z) \& T(X, Y, Z)$ says that X, Y and Z form a triangle. (K_4 illustrates the need for the restriction on order.)

Replace each subformula of σ of the form $\exists v\varphi$ by the disjunction of the following three formulas, each again involving φ.

(1) $(\exists X)(\exists Y)(\exists Z)[A(X, Y, Z) \& \neg T(X, Y, Z) \& \varphi]$.

(2) $(\exists X)(\exists Y)[X - Y \& (\forall Z)[A(X, Y, Z) \rightarrow T(X, Y, Z)] \& \varphi]$.

(3) $(\exists X)[(\forall Y)(\forall Z)[(X - Y \& X - Z) \rightarrow A(X, Y, Z) \& \neg T(X, Y, Z)] \& \varphi]$.

These three formulas correspond to v having degree greater than two and equal to two and one, respectively.

These replacements produce a confused version of σ having no remaining vertex quantifiers, but lots of vertex variables within statements of equality and adjacency. Replace each occurring formula of the sort $u - v$ by the equivalent formula $u - v \& \neg(u = v)$. We can now replace these occurrences of variables by expressions involving the edge variables introduced using the formulas (1), (2) and (3) above.

Consider any subformula $v = v'$, where $\exists v$ was replaced using formula (i) ($i \in \{1, 2, 3\}$) with edge variables X, Y, Z and $\exists v'$ using formula (j) with X', Y', Z'. If $i = j = 1$, then replace $v = v'$ by an $\mathscr{L}_1$ expression stating that each of X, Y and Z is adjacent or equal to each of X', Y' and Z'; this is equivalent to $v = v'$ in this case. Similarly, if $i = j = 2$, replace $v = v'$ by an $\mathscr{L}_1$ statement that $\{X, Y\} = \{X', Y'\}$; if $i = j = 3$, by $X = X'$. Since equality demands equal degrees, replace any $v = v'$ by $X \neq X$ whenever $i \neq j$.

The only remaining occurrences of vertex variables are in subformulas of the sort $v - v'$. Again, assume $\exists v$ was replaced as in (i) with X, Y and Z and $\exists v'$ as in (j) with X', Y' and Z'. Replace $v - v'$ by $\mathscr{L}_1$ expressions as follows: if $i = 1, j = 1$, by the existence of an edge adjacent or equal to each of X, X', Y, Y', Z, Z'; if $i = 1, j = 2$, by one of X' or Y' being adjacent or equal to each of X, Y and Z; if $i = 1, j = 3$, by X' being adjacent or equal to each of X, Y and Z; if $i = 2, j = 2$, by $\{X, Y\}$ and $\{X', Y'\}$ having a unique edge in common; if $i = 2$, $j = 3$, by X' being X or Y; if $i = 3$, $j = 3$, by $X \neq X$; etc.

This process replaces σ by an $\mathscr{L}_1$ sentence σ^* which is true in exactly the same (connected) graphs (of order greater than four) as σ. Let σ_0 be the $\mathscr{L}_1$ sentence

$$(\exists X)(\exists Y)(\exists Z)[A(X, Y, Z) \& (\forall W)(W = X \vee W = Y \vee W = Z)],$$

so that K_3 and $K_{1,3}$ are the only graphs which satisfy σ_0. Let $\sigma_1, \ldots, \sigma_8$ be $\mathscr{L}_1$ sentences which similarly characterize the remaining eight connected graphs of order $\leqslant 4$. (These can be easily found in each case; for instance the path of length three is characterized by the sentence having the same form as σ_0 except with $A(X, Y, Z)$ replaced by $[X-Y \& Y-Z \& \neg(X-Z) \& \neg(X=Z)]$.) Since σ does not differentiate K_3 from $K_{1,3}$, we can relabel the σ_i's as τ_i's in such a way that, for some $k \leqslant 8$, $\tau_i \to \sigma$ for $i \leqslant k$, while $\tau_i \to \neg\sigma$ for $i > k$. Then σ is equivalent to the $\mathscr{L}_1$ sentence

$$(\sigma^* \vee \tau_0 \vee \ldots \vee \tau_k) \& \neg\tau_{k+1} \& \ldots \& \neg\tau_8,$$

with the obvious modifications of $k = 0$ or $k = 8$. This completes the proof of the theorem.

We can now deduce Whitney's Theorem [6, Thm. 1] (or [4, Cor. 3.3]). Note that since our graphs are assumed to be finite, two are vertex [edge] isomorphic if and only if they satisfy exactly the same $\mathscr{L}_0$ [respectively, $\mathscr{L}_1$] sentences.

COROLLARY 1. *K_3 and $K_{1,3}$ are the only connected edge-isomorphic graphs which are not vertex isomorphic.*

Suppose, towards a contradiction, that G and H are two other such graphs. Since they are finite, each can be characterized up to isomorphism by an $\mathscr{L}_0$ sentence that asserts the existence of exactly the right number of vertices, with each pair explicitly made adjacent or nonadjacent, as appropriate. One of these sentences will be an $\mathscr{L}_0$ sentence σ which differentiates between G and H but not between K_3 and $K_{1,3}$ (since it is false for both). By the theorem, σ is equivalent to an $\mathscr{L}_1$ sentence which differentiates G from H, contradicting their being edge isomorphic and so satisfying the same $\mathscr{L}_1$ sentences.

In somewhat the same sense as edge/cycle properties correspond to the "matroidal" properties of graph theory, the edge/adjacency properties are those which can be characterized in a particular way using line graphs. If an $\mathscr{L}_0$ sentence σ can be equivalently expressed as an $\mathscr{L}_1$ sentence σ', then replacing all (mentions of) edges in σ' with vertices produces an $\mathscr{L}_0$ sentence σ'' such that, for all graphs G, σ holds for G if and only if σ'' holds for the line graph $L(G)$. The theorem then yields the following.

COROLLARY 2. *For precisely those $\mathscr{L}_0$ sentences σ that do not differentiate between K_3 and $K_{1,3}$, there exist an $\mathscr{L}_0$ sentence σ'' such that σ holds for a connected graph G if and only if σ'' holds for $L(G)$.*

Section 6 of [4] consists of examples of such σ, σ'' pairs in which (contrary to what is suggested by our approach) σ'' is simpler than σ; indeed, each example starts with a natural σ'' and exhibits a corresponding σ (which, of course, cannot differentiate K_3 from $K_{1,3}$). Section 5 of [4] illustrates the reverse procedure, starting with a nice σ and exhibiting a suitable σ''. Because, strangely enough, each σ there does differentiate K_3 from $K_{1,3}$, these examples cannot exhibit the phenomenon of Corollary 2, but rather a weakened version of it: σ holds for some G having line graph H if and only if σ'' holds in H.

The most basic property which does not differentiate K_3 from $K_{1,3}$ is simply "being a graph." But our approach does not produce a corresponding sentence completely axiomatizing being a graph in terms of edges and adjacency. This is because all of our work has been within the context of being a graph, and so the equivalent $\mathscr{L}_1$ sentence in the theorem (or σ' in the proof of Corollary 2) could be any $\mathscr{L}_1$ expression which is true for all graphs; "equivalent" here means "valid," not necessarily a "complete axiomatization." The van Rooij–Wilf characterization of line graphs [5] (or [4, Thm. 4.3 (iii)]) fills this gap. By taking the conjunction of their two conditions as σ'' in Corollary 2, we can translate this into $\mathscr{L}_1$ to obtain a complete $\mathscr{L}_1$ axiomatization of graphs: If edge A is adjacent to each of the edges B_1, B_2 and B_3, then at least two of the B_i's are adjacent to each other; and, if A_1 and A_2 are adjacent edges with each adjacent to both edges B_1 and B_2, then either B_1 is adjacent to B_2 or, for $i = 1$ or $i = 2$, each edge is adjacent to an even number of A_1, A_2 and B_i.

In conclusion, it is interesting to note how close $\mathscr{L}_1$ comes to being as expressive as $\mathscr{L}_0$.

COROLLARY 3. *Every $\mathscr{L}_0$ sentence which cannot be expressed within $\mathscr{L}_1$ can be written in the form $\sigma_1 \vee \sigma_2$ where σ_1 is in $\mathscr{L}_1$ and σ_2 has a very specific, restricted mention of vertices.*

Specifically, suppose σ is such an $\mathscr{L}_0$ sentence. By the theorem, we can assume (without loss of generality) that σ holds in K_3 but not $K_{1,3}$. Take σ_0 as in the proof of the theorem. Since $\sigma \& \neg\sigma_0$ does not differentiate K_3 from $K_{1,3}$, take it as σ_1. Take σ_2 to state $(\forall v)$ (v is incidence with an even number of edges) so as to use only a single vertex variable (plus edges and vertex/edge incidence). (This shows we cannot form intermediate languages between $\mathscr{L}_1$ and $\mathscr{L}_0 \cup \mathscr{L}_1$ by allowing varying numbers of vertex quantifiers — a single one allows full expressiveness.) Alternatively, we could avoid incidence and make σ_2 a "pure" vertex sentence by having it assert that the order is exactly three.

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Received 2 September 1982;
in revised form 22 February 1983